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WAVELET ESTIMATION OF SEMIPARAMETRIC ERRORS IN VARIABLES MODEL

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ABSTRACT. Most of the work on wavelet estimation when the variables are measured with errors have centered around nonparametric approaches which cause curse of dimensionality. In this paper it is aimed to avoid this complexity using wavelet semiparametric errors in variables regression model. Using theoretical arguments for nonparametric wavelet estimation a wavelet approach is represented to estimate partially linear errors in variables model which is a semiparametric model when explanatory variable of nonparametric part has measurement error. Assuming that the measurement error has a known distribution we derive an estimator of the linear parts' parameter. In simulation study derived method is compared with no measurement error case.

1. INTRODUCTION

Image processing plays an important role in our daily life. In the process of deblurring and denoising the images are improved to pictorial information of them and with the machine reading the images are processed for automatic machine perception. According to the location and brightness measures of an image, it needs to be digitized with some devices which converts the image into its digital form.

A monochrome image can be expressed by a bivariate function $f(x, y)$, where (x, y) denotes the spatial location in the image and the function value $f(x, y)$ is proportional to the brightness of the image at (x, y). In the computer science literature, the function $f(x, y)$ is often called the image intensity function [1]. An image can be considered as a surface of the image intensity at each pixel. A regression surface from a noisy image is often fitted by local smoothing procedures [1], [2].

In image processing, mathematical operations are used for processing any form of signal processing. Because image processing methods mostly deal with the image as a two-dimensional signals, they use standard signal processing techniques

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to improve it. Signal processing is an enabling technology that encompasses the fundamental theory, applications, algorithms, and implementations of processing or transferring information contained in many different physical, symbolic, or abstract formats broadly designated as signals [3].

Wavelet theory provides a unified framework for a number of techniques which had been developed independently for various signal processing applications [4]. Most studies including wavelet theory have been recently recognized as different views of a signal theory. In [5], Chang and Qu represented wavelet estimation of partially linear models using penalized least squares approach which avoids the restrictive smoothness requirements for the nonparametric function of the traditional smoothing approaches such as smoothing spline, kernel and piecewise polynomial methods. [6] introduced adaptive wavelet multivariate ($x^* \in [0,1]^d, d > 1$) nonparametric regression with errors in variables. They devise an adaptive estimator based on projection kernels on wavelets and a deconvolution operator.

Curse of dimensionality in nonparametric models causes that converge of any estimator to the true value is very slow. Because of this complexity, we introduce wavelet semiparametric regression. We extend the wavelet partially linear regression procedure to semiparametric errors in variables model. We consider the partially linear model from the data (X, X^*, Y) with mean function $X^T\beta + g(X^*)$ when X^* has measurement error.

Consider the semiparametric partially linear model:

$$Y_u = X_u^T\beta + g(X_u^*) + \Delta y_u, \quad u = 1, 2, ..., n \tag{1}$$

where Y_u are observations, X_u are $n \times p$ dimensional known design points, X_u^* is a random variable defined on $[0, 1]$, β in an unknown $p-$dimensional parameter vector, $g(.)$ is unknown and the random model errors Δy_u are iid with $N(0, \sigma^2)$. Here

$$\chi_u = X_u^* + \Delta\chi_u, \tag{2}$$

where $\Delta\chi_u$ are iid measurement errors. It is assumed that $\Delta\chi$ has a known distribution which is proposed by [7] for nonparametric model and [8] for semiparametric model. This model is also studied by [9] for semiparametric model on the assumption that $\Delta\chi$ has an unknown distribution.

This study is structured as mentioned. In section 2 we briefly describe the semiparametric errors in variables method for one dimensional wavelets ($d = 1$). We will redefine an estimator of linear parts' parameter in Section 3. Section 4 provides some Monte Carlo simulation studies to investigate finite sample properties of estimators. Finally conclusion and some remarks are given in Section 5.

2. Approximation Kernels and Family of Estimators for Nonparametric Function

If parametric part is embedded into response variable in a semiparametric regression model, one can get nonparametric regression model:

$$\underbrace{Y - X^T\widehat{\beta}_n}_{Y^*} \quad = \quad g(x^*) + \Delta y$$

for $E[\Delta y|x^*] = 0$. If β is known nonparametric function can be estimated using nonparametric methods. Hence, using the theoretical arguments of the nonparametric wavelet estimation we can define a wavelet approach to estimate partially linear errors in variables model.

Let $g(x^*) = E(Y^*|X^* = x^*) = \frac{\int y^* g_{X^*,Y^*}(x^*,y^*)dy^*}{f(x^*)} = \frac{(gf)(x^*)}{f(x^*)}$ where $f(x^*)$ is defined as a classical deconvolution problem. We firstly estimate $(gf)(x^*)$. Denote $p(x^*) := g(x^*) \times f(x^*)$ and consider a father wavelet φ on the real line satisfying

- φ is compactly supported on $[-A, A]$; A is a positive integer.
- Denote $\varphi_{0k}(x^*) = \varphi(x^* - k)$. There exists a positive integer N, such that for any x^* and k
$$\int \sum_{k\in\mathbb{Z}} \underbrace{\varphi(x^* - k)}_{\varphi_{0k}(x^*)} \underbrace{\varphi(y^* - k)}_{\varphi_{0k}(y^*)} (y^* - x^*)^l dy^* = \delta_{0l}, \quad l = 0, ..., N.$$
where δ_{0l} is the Kronecker delta which is defined as [11]
$$\delta_{0l} = \begin{cases} 1 & , \quad 0l = 0, \\ 0 & , \quad otherwise. \end{cases}$$
- φ is of class the space of functions having all continuous derivatives $\mathcal{C}^r$, where $r \geq 2$.

The associated projection kernel on the space

$$V_j := span\{\varphi_{jk}, k \in \mathbb{Z}\}, \quad j \in \mathbb{N},$$

is given for any x^* and y^* by

$$K_j(x^*, y^*) = \sum_k \varphi_{jk}(x^*)\varphi_{jk}(y^*),$$

where

$$\varphi_{jk}(x^*) = 2^{\frac{j}{2}}\varphi(2^j x^* - k), \quad j \in \mathbb{N}, k \in \mathbb{Z}.$$

Then the projection of $p(x^*)$ on V_j can be written as,

$$p_j(x^*) = K_j(p)(x^*) := \int K_j(x^*, y^*)p(y^*)dy^* = \sum_k p_{jk}\varphi_{jk}(x^*)$$

where

$$p_{jk} = \int p(y^*)\varphi_{jk}(y^*)dy^*.$$

In [6] the authors adapt the kernel approach proposed by [7] in their wavelet context and they introduced

$$\hat{p}_{jk} := \frac{1}{n}\sum_{u=1}^{n} Y_u^* \times (\mathcal{D}_j\varphi)_{j,k}(\chi_u) = 2^{\frac{j}{2}}\frac{1}{n}\sum_{u=1}^{n} Y_u^* \int \exp(-i2^j\chi_u - k)\frac{\phi_\varphi(s)}{\phi_{\Delta y}(2^j s)}ds, \quad (3)$$

$$\hat{p}_j(x^*) = \frac{1}{n}\sum_k \sum_{u=1}^{n} Y_u^* \times (\mathcal{D}_j\varphi)_{j,k}(\chi_u)\varphi_{jk}(x^*), \quad (4)$$

where $\phi_\varphi(s)$ is the Fourier transform of wavelet φ_{jk} and $\mathcal{D}_j$ is the deconvolution operator which is demonstrated by K_n in [7] and defined as follows

$$(\mathcal{D}_j\varphi)(\chi) = \int \exp(-is\chi)\frac{\phi_\varphi(s)}{\phi_{\Delta y}(2^j s)}ds. \quad (5)$$

The authors also proposed a resolution level j selecting rule depending Goldenshluger and Lepski methodology [10]. After this step which helps to introduce the unknown function of the nonparametric part, we can estimate the parameter of the parametric part.

3. Construction of Estimators

For the model (1.1) let us denote the densities of χ and x^* by $f_\chi(.)$ and $f_{x^*}(.)$ respectively. Then the estimator of $f_{x^*}(.)$ is described as,

$$\widehat{f}_n(x^*) = \frac{1}{n}\sum_k \sum_{u=1}^{n} (\mathcal{D}_j\varphi)_{j,k}(\chi_u)\varphi_{jk}(x^*), \quad (6)$$

where $(\mathcal{D}_j\varphi)_{j,k}$ is demonstrated in equation (2.3).

Let

$$\omega_{nu}(.) = \frac{(\mathcal{D}_j\varphi)_{j,k}(.)}{\sum_u (\mathcal{D}_j\varphi)_{j,k}(.)} = \frac{1}{n}\frac{(\mathcal{D}_j\varphi)_{j,k}(.)}{\widehat{f}_n(.)}. \quad (7)$$

As we mentioned above if β is known then the estimation of $g(.)$ can be found using nonparametric errors in variables method which uses classical Nadaraya-Watson Kernel estimator as follows [12, 13]

$$g_n(x^*) = \sum_k \sum_{u=1}^{n} \omega_{nu}(x^*)(Y_u^* - X_u^T\beta). \quad (8)$$

Hence before to find the estimation of nonparametric function we need to estimate linear parts' parameter. Here the generalized least squares estimates of β can be used to estimate $\widehat{\beta}_n$ as shown below,

$$\widehat{\beta}_n = (\widetilde{X}^T\widetilde{X})^{-1}(\widetilde{X}^T\widetilde{Y}) \quad (9)$$

where $\widetilde{Y}_u = Y_u - \sum_k \sum_{u=1}^{n} \omega_{nu}(\chi_u)Y_u$ for $\widetilde{Y} = (\widetilde{Y}_1, ..., \widetilde{Y}_n)$ and $\widetilde{X}_u = X_u - \sum_k \sum_{u=1}^{n} \omega_{nu}(\chi_u)X_u$ for $\widetilde{X} = (\widetilde{X}_1, ..., \widetilde{X}_n)$.

4. Simulation Study

In this section, finite sample properties of the estimators were investigated by Monte Carlo simulation approach. To carry out calculations and give some simulation results we used MATLAB.

We consider a model which has 2 dimensional linear variable and report on the simulation results with the sample sizes $n = 32, 64, 128$ and 256. We consider different nonparametric functions for the nonparametric component as follows

$$g_1(x^*) = 4.26(\exp(-3.25x^*) - 4\exp(-6.5x^*) + 3\exp(-9.75x^*)),$$

$$g_2(x^*) = \begin{cases} 4x^{*2}(3 - 4x^*) & , \quad 0 \leq x^* \leq 0.5, \\ \frac{4}{3}x^*(4x^{*2} - 10x^* + 7) - 1.5 & , \quad 0.5 < x^* \leq 0.75, \\ \frac{16}{3}x^*(1 - x^*)^2 & , \quad 0.75 < x^* \leq 1. \end{cases}$$

Here g_1 is a smoothing function which is given by [14] and g_2 is a piecewise polynomial with discontinuity which is given by [15]. In all examples the density of both the true regressor x^* and the measurement error $\Delta\chi$ are chosen as the most common combinations of ordinarily smooth distributions which are summarized in Table 1.

1	**2**	**3**
$x^* \rightarrow Beta\,(2,2)$	$x^* \rightarrow Beta\,(0.5,2)$	$x^* \rightarrow Uniform\,[0,1]$
$\Delta\chi \rightarrow L\,(0,0.25)$	$\Delta\chi \rightarrow L\,(0,0.25)$	$\Delta\chi \rightarrow L\,(0,0.25)$
$\Delta y \rightarrow N\,(0,0.25)$	$\Delta y \rightarrow N\,(0,0.25)$	$\Delta y \rightarrow N\,(0,0.25)$
$\beta = (1,2)$	$\beta = (1,2)$	$\beta = (1,2)$
$\sigma^2_{\Delta y} = 0.25$	$\sigma^2_{\Delta y} = 0.25$	$\sigma^2_{\Delta y} = 0.25$
$X \rightarrow N_2\,(0, I_2)$	$X \rightarrow N_2\,(0, I_2)$	$X \rightarrow N_2\,(0, I_2)$

Table 1. Functions of variables and values of parameters in each example driven in simulation

We consider the normal distribution as an example of a supersmooth distribution, and the Laplace (or double exponential) distribution, uniform distribution and beta distribution for the ordinarily smooth case. Because $Beta\,(2,2)$ and $Beta\,(0.5,2)$ distributions reflect two different behaviors on $[0,1]$ we use them. Finally, following the asymptotic considerations given in [6], we choose the primary resolution level j that we have used throughout our simulations as $j(n) = \log_2(\log(n)) + 1$.

To compare the results of the cases considering and ignoring measurement errors average values of $N = 100$ replicates of mean squared error (MSE) and estimates of the estimators considered in four different sample sizes are given in Table 2 for g_1 and in Table 3 for g_2 (NoME: No Measurement Error). It can be easily seen that the results are encouraging.

5. Conclusion

This paper presents the case wavelet estimation of partially linear model when nonparametric part has measurement error. If the measurement error is known it

	n = 32		n = 64		n = 128		n = 256	
Example 1								
	MSE	$\hat{\beta}$	**MSE**	$\hat{\beta}$	**MSE**	$\hat{\beta}$	**MSE**	$\hat{\beta}$
Our	0.0079	$\begin{pmatrix}0.9943\\1.9543\end{pmatrix}$	0.0027	$\begin{pmatrix}1.0309\\2.0062\end{pmatrix}$	0.0013	$\begin{pmatrix}0.9815\\2.0071\end{pmatrix}$	0.0008	$\begin{pmatrix}0.9846\\2.0141\end{pmatrix}$
NoME	0.0188	$\begin{pmatrix}1.0091\\1.8848\end{pmatrix}$	0.0059	$\begin{pmatrix}0.9747\\2.0271\end{pmatrix}$	0.0022	$\begin{pmatrix}0.9356\\1.9803\end{pmatrix}$	0.0015	$\begin{pmatrix}0.9846\\2.0155\end{pmatrix}$
Example 2								
	MSE	$\hat{\beta}$	**MSE**	$\hat{\beta}$	**MSE**	$\hat{\beta}$	**MSE**	$\hat{\beta}$
Our	0.0130	$\begin{pmatrix}1.0919\\2.0000\end{pmatrix}$	0.0074	$\begin{pmatrix}0.9794\\1.8561\end{pmatrix}$	0.0028	$\begin{pmatrix}0.9628\\1.8337\end{pmatrix}$	0.0015	$\begin{pmatrix}0.9954\\1.9974\end{pmatrix}$
NoME	0.0237	$\begin{pmatrix}1.1162\\2.0073\end{pmatrix}$	0.0058	$\begin{pmatrix}0.8911\\1.9115\end{pmatrix}$	0.0021	$\begin{pmatrix}0.9447\\1.9188\end{pmatrix}$	0.0014	$\begin{pmatrix}1.0384\\1.9981\end{pmatrix}$
Example 3								
	MSE	$\hat{\beta}$	**MSE**	$\hat{\beta}$	**MSE**	$\hat{\beta}$	**MSE**	$\hat{\beta}$
Our	0.0081	$\begin{pmatrix}0.9485\\2.0048\end{pmatrix}$	0.0030	$\begin{pmatrix}1.0618\\1.8745\end{pmatrix}$	0.0018	$\begin{pmatrix}1.0312\\2.0733\end{pmatrix}$	0.0010	$\begin{pmatrix}1.0743\\1.9438\end{pmatrix}$
NoME	0.0221	$\begin{pmatrix}0.9631\\1.8586\end{pmatrix}$	0.0060	$\begin{pmatrix}1.1039\\1.7965\end{pmatrix}$	0.0029	$\begin{pmatrix}1.0369\\2.1189\end{pmatrix}$	0.0019	$\begin{pmatrix}1.0941\\1.9345\end{pmatrix}$

TABLE 2. Monte Carlo simulation results for g_1

	n = 32		n = 64		n = 128		n = 256	
Example 1								
	MSE	$\hat{\beta}$	**MSE**	$\hat{\beta}$	**MSE**	$\hat{\beta}$	**MSE**	$\hat{\beta}$
Our	0.0064	$\begin{pmatrix}1.0708\\1.9043\end{pmatrix}$	0.0035	$\begin{pmatrix}1.0299\\1.9672\end{pmatrix}$	0.0021	$\begin{pmatrix}1.0160\\2.0243\end{pmatrix}$	0.0009	$\begin{pmatrix}0.9825\\2.0090\end{pmatrix}$
NoME	0.0093	$\begin{pmatrix}1.0135\\1.8829\end{pmatrix}$	0.0036	$\begin{pmatrix}0.9565\\2.0048\end{pmatrix}$	0.0021	$\begin{pmatrix}0.9393\\2.0243\end{pmatrix}$	0.0010	$\begin{pmatrix}0.9958\\2.0427\end{pmatrix}$
Example 2								
	MSE	$\hat{\beta}$	**MSE**	$\hat{\beta}$	**MSE**	$\hat{\beta}$	**MSE**	$\hat{\beta}$
Our	0.0185	$\begin{pmatrix}1.1512\\1.8998\end{pmatrix}$	0.0119	$\begin{pmatrix}1.0085\\1.8975\end{pmatrix}$	0.0065	$\begin{pmatrix}1.0384\\1.7115\end{pmatrix}$	0.0033	$\begin{pmatrix}0.9320\\1.9629\end{pmatrix}$
NoME	0.0385	$\begin{pmatrix}1.2572\\1.9170\end{pmatrix}$	0.0192	$\begin{pmatrix}0.8755\\1.9425\end{pmatrix}$	0.0081	$\begin{pmatrix}1.0450\\1.7743\end{pmatrix}$	0.0042	$\begin{pmatrix}1.0148\\1.9589\end{pmatrix}$
Example 3								
	MSE	$\hat{\beta}$	**MSE**	$\hat{\beta}$	**MSE**	$\hat{\beta}$	**MSE**	$\hat{\beta}$
Our	0.0078	$\begin{pmatrix}1.0143\\2.0204\end{pmatrix}$	0.0038	$\begin{pmatrix}1.0571\\1.8656\end{pmatrix}$	0.0027	$\begin{pmatrix}1.0263\\2.0203\end{pmatrix}$	0.0013	$\begin{pmatrix}1.0453\\1.9602\end{pmatrix}$
NoME	0.0238	$\begin{pmatrix}0.9632\\1.8918\end{pmatrix}$	0.0054	$\begin{pmatrix}1.0999\\1.8119\end{pmatrix}$	0.0088	$\begin{pmatrix}0.9970\\2.0913\end{pmatrix}$	0.0018	$\begin{pmatrix}1.0517\\1.9696\end{pmatrix}$

TABLE 3. Monte Carlo simulation results for g_2

is possible to estimate linear parts' parameter using well known Nadaraya-Watson estimator. In this study we introduced generalized least square estimator of β based on projection kernels on wavelets and borrowing the ideas of deconvolution technique. It is discussed in the simulation that the resulting rates are comparable to no measurement error case. Asymptotic normality of proposed estimator is still open one and should be investigated.

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