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A fixed point approach to the stability of a nonlinear volterra integrodifferential equation with delay

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Abstract

By using a fixed point method, we prove the Hyers–Ulam–Rassias stability and the Hyers–Ulam stability of a nonlinear Volterra integrodifferential equation with delay. Two examples are presented to support the usability of our results.

Keywords: Hyers-Ulam-Rassias stability, Hyers-Ulam stability, Volterra integrodifferential equation with delay, fixed point approach.

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1. Introduction

In 1940, Ulam posed the following problem related to the stability of functional equations: "Under what conditions does there exist an additive mapping near an approximately additive mapping?" see for more detail [15]. One year later, Hyers [8] gave an answer to the problem of Ulam for the case of functional equation for homomorphism between the Banach spaces. In 1978, Rassias [13] proved the existence of unique linear mapping near approximate additive mapping that provides generalization of the Hyers result. Jung [9] applied the fixed point method to the investigation of Volterra integral equation by using the idea of Cadariu and Radu in [2]. S. M. Jung proved that if a continuous function $v\colon I \to \mathbb{C}$ is such that

$$\left| v(t) - \int_c^t G(\xi, v(\xi))d\xi \right| \leq \phi(t)$$

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for all $t \in I$, then there exists a unique continuous function $v_0\colon I \to \mathbb{C}$ and a constant $K > 0$ such that

$$v_0(t) = \int_c^t G(\xi, v_0(\xi))d\xi \quad \text{and} \quad d\left(v(t), v_0(t)\right) \le L\phi(t),$$

for all $t \in I$, it is important to obtain a precise L because it is clear that L will lead us to the error between the actual solution $v_0(t)$ and the approximate solution $v(t)$. In 2013, Jung *et al.* proved that if $g\colon I \to \mathbb{R}$, $h\colon I \to \mathbb{R}$, $G\colon I \to \mathbb{R}$ and $\phi\colon I \to \mathbb{R}$ are sufficiently smooth functions and if a continuously differentiable function $v\colon I \to \mathbb{R}$ satisfies the perturbed Volterra integrodifferential equation

$$\left| v'(t) + g(t)v(t) + h(t) + \int_c^t K(t,\eta)v(\eta)d\eta \right| \le \phi(t),$$

for some $t \in I$, then there exists a unique solution $v_0\colon I \to \mathbb{R}$ of the Volterra integrodifferential equation

$$v'(t) + g(t)v(t) + h(t) + \int_c^t K(t,\eta)v(\eta)d\eta = 0,$$

such that

$$d(v(t), v_0(t)) \le \exp\Big\{ - \int_c^t g(\eta)d\eta \Big\} \int_t^b \phi(\varsigma) \exp\Big\{ \int_c^\varsigma g(\eta)d\eta \Big\} d\varsigma,$$

for all $t \in I$. If the reader wishes more details, we recommend [1, 3, 4, 6, 7, 12, 14, 16].

The main purpose of the paper is to investigate the Hyers-Ulam-Rassias stability and the Hyers-Ulam stability of following nonlinear Volterra integrodifferential equation with delay:

$$v'(t) = g(t, v(t), v(\alpha(t))) + \int_0^t k(t, s, v(s), v(\alpha(s)))ds, \tag{1.1}$$

for all $t \in I = [0, T]$, where the function $g(t, v(t), v(\alpha(t)))$ is continuous function with respect to variables t and v on $I \times \mathbb{R} \times \mathbb{R}$, $k(t, s, v(t), v(\alpha(t)))$ is continuous with respect to variables t, s and v on $I \times I \times \mathbb{R} \times \mathbb{R}$, β is any constant and $\alpha : [0, T] \to [0, T]$ is a continuous delay function with $\alpha(t) \le t$.

1.1. Definition. If for each continuously differentiable function $v(t)$ satisfying

$$\left| v'(t) - g(t, v(t), v(\alpha(t))) - \int_0^t k(t, s, v(s), v(\alpha(s)))ds \right| \le \phi(t),$$

for some $\phi\colon [0, T] \to (0, \infty)$, there exists a solution $v_0(t)$ of the Volterra integrodifferential equations with delay (1.1) and a constant $K > 0$ (independent of $v(t)$ and $v_0(t)$) with

$$|v(t) - v_0(t)| \le K\phi(t),$$

for all $t \in I$, then we can say that the equation (1.1) is Hyers-Ulam-Rassias stable on I. If $\phi(t)$ is constant function then we say that the equation (1.1) has Hyers-Ulam stability on I.

For a nonempty set Y, the generalized metric on Y is defined as follow:

1.2. Definition. A function $d : Y \times Y \to [0, \infty]$ is called a generalized metric on Y if and only if for all $u,\ v,\ w \in Y$ d satisfies the following conditions:

(1) $d(u, v) = 0$ if and only if $u = v$.
(2) $d(u, v) = d(v, u)$.
(3) $d(u, w) \le d(u, v) + d(v, w)$.

Now, we are going to introduce one of the most crucial result of fixed point theory that will play an important role in proving our main results.

1.3. Theorem. ([5]) Let (Y,d) be a generalized complete metric space. Assume that $\Theta : Y \to Y$ is a strictly contractive operator with $L < 1$ as Lipschitz constant. If there exists a nonnegative integer k such that $d(\Theta^{k+1}u, \Theta^k u) < \infty$ for some $u \in Y$, then the following conditions are true:

- The sequence $\Theta^n u$ converges to a fixed point u^* of Θ;
- u^* is the unique fixed point of Θ in

  $$Y^* = \Big\{v \in Y \mid d(\Theta^k u, v) < \infty\Big\};$$

- If $v \in Y^*$, then

  $$d(v,u^*) \le \frac{1}{1-L}\, d(\Theta v, v).$$

2. Hyers-Ulam-Rassias stability

In this section, we prove the Hyers-Ulam-Rassias stability of the nonlinear Volterra integrodifferential equation with delay (1.1).

2.1. Theorem. Let $I = [0,T]$ be a closed and bounded interval for a given $T > 0$ and let N, L_g and L_k be nonnegative constants with $0 < NL_g + N^2 L_k < 1$. Assume that $g\colon I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies a Lipschitz condition

$$|g(t,v_1,v_1(\alpha(t))) - g(t,v_2,v_2(\alpha(t)))| \le L_g|v_1 - v_2|, \tag{2.1}$$

for all $t \in I$ and $v_1, v_2 \in \mathbb{R}$. Let $k\colon I \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies a Lipchitz condition

$$|k(t,s,v_1,v_1(\alpha(s))) - k(t,s,v_2,v_2(\alpha(s))| \le L_k|\, v_1 - v_2|, \tag{2.2}$$

for all $t,s \in I$, $v_1, v_2 \in \mathbb{R}$. If $v\colon I \to \mathbb{R}$ a continuously differentiable function satisfies

$$\left| v'(t) - g(t,v(t),v(\alpha(t))) - \int_0^t k(t,s,v(s),v(\alpha(s)))ds \right| \le \phi(t), \tag{2.3}$$

for all $t \in I$, where $\phi\colon I \to (0,\infty)$ is a continuous function with

$$\int_0^t \phi(\zeta)d\zeta \le N\phi(t) \tag{2.4}$$

for all $t \in I$, then there exists a unique continuous function $v_0\colon I \to \mathbb{R}$ such that

$$v_0(t) = \int_0^t g(\zeta, v_0(\zeta), v_0(\alpha(\zeta)))d\zeta + \int_0^t\int_0^s k(t,\zeta,v_0(\zeta),v_0(\alpha(\zeta))d\zeta ds \tag{2.5}$$

and

$$|v(t) - v_0(t)| \le \frac{N}{1-(NL_g + N^2L_k)}\phi(t) \tag{2.6}$$

for all $t \in I$.

Proof. Let Y be the set of all real valued continuous functions on closed and bounded interval I. For all $u, w \in Y$, we set

$$d(u,w) = inf\{K \in [0,\infty] : |u(t) - w(t)| \le K\phi(t), \text{ for all } t \in I\}. \tag{2.7}$$

The metric space (Y,d) is a complete generalized metric space, see [10]. Consider the operator $\Theta : Y \to Y$ defined by

$$(\Theta u)(t) = v(0) + \int_0^t g(\zeta, u(\zeta), u(\alpha(\zeta)))d\zeta + \int_0^t\int_0^s k(t,\zeta,u(\zeta),u(\alpha(\zeta)))d\zeta ds \tag{2.8}$$

for all $t \in I$. We show that the operator Θ is strictly contractive. Let $K_{uw} \in [0, \infty]$ be a constant with $d(u, w) \leq K_{uw}$ for $u, w \in Y$. By (2.7), we can write

$$|u(t) - w(t)| \leq K_{uw}\phi(t) \text{ for all } t \in I. \tag{2.9}$$

From inequalities (2.1)), (2.2), (2.4), (2.8) and (2.9) it follows that for all $t \in I$ we have

$$\begin{aligned} |(\Theta u)(t) - (\Theta w)(t)| = \Big| & \int_0^t \Big\{ g(\zeta, u(\zeta), u(\alpha(\zeta))) - g(\zeta, w(\zeta), w(\alpha(\zeta))) \Big\} \, d\zeta \\ & + \int_0^t \int_0^s \Big\{ k(t, \zeta, u(\zeta), u(\alpha(\zeta))) - k(t, \zeta, w(\zeta), w(\alpha(\zeta))) \Big\} \, d\zeta \, ds \Big| \\ & \leq \int_0^t \Big| g(\zeta, u(\zeta), u(\alpha(\zeta))) - g(\zeta, w(\zeta), w(\alpha(\zeta))) \Big| \, d\zeta \\ & + \int_0^t \int_0^s \Big| k(t, \zeta, u(\zeta), u(\alpha(\zeta))) - k(t, \zeta, w(\zeta), w(\alpha(\zeta))) \Big| \, d\zeta \, ds \\ & \leq L_g \int_0^t \big| u(\zeta) - w(\zeta) \big| d\zeta + L_h \int_0^t \int_0^s \big| u(\zeta) - w(\zeta) \big| \, d\zeta \, ds \\ & \leq L_g K_{uw} \int_0^t \phi(\zeta) d\zeta + L_k K_{uw} \int_0^t \int_0^s \phi(\zeta) \, d\zeta \, ds \\ & \leq K_{uw} \phi(t) \left(N L_g + N^2 L_k \right), \end{aligned}$$

i.e. $d(\Theta u, \Theta w) \leq K_{uw}\phi(t)(NL_g + N^2 L_k)$. Hence, we may conclude that $d(\Theta u, \Theta w) \leq (NL_g + N^2 L_k) d(u, w)$ for any $u, w \in Y$, where $0 < NL_g + N^2 L_k < 1$.

It follows from (2.8) that for any arbitrary $w_0 \in Y$, there exists a constant $K \in [0, \infty]$ with

$$\begin{aligned} |\Theta w_0(t) - w_0(t)| = \Big| & v(0) + \int_0^t g(\zeta, w_0(\zeta), w_0(\alpha(\zeta))) \, d\zeta \\ & + \int_0^t \int_0^s k(t, \zeta, u_0(\zeta), u_0(\alpha(\zeta))) \, d\zeta \, ds - w_0 \Big| \\ & \leq K\phi(t), \text{ for all } t \in I. \end{aligned}$$

Since $g(\zeta, w_0(\zeta), w_0(\alpha(\zeta))), k(t, \zeta, u_0(\zeta), u_0(\alpha(\zeta)))$ and w_0 are bounded and $\min_{t \in I} \phi(t) > 0$. Thus, (2.7) implies that

$$d(\Theta w_0, w_0) < \infty.$$

So, according to Theorem 1.3, there exists a continuous function $v_0 \colon I \to \mathbb{R}$ in a way that $\Theta^n w_0 \to v_0$ in (Y, d) and $\Theta v_0 = v_0$, i.e., v_0 satisfies (2.5) for all $t \in I$. Since we know that w and w_0 are bounded on closed interval I for any $w \in Y$ and $\min_{t \in I} \phi(t) > 0$, then there exists a constant $K_w \in [0, \infty]$ such that

$$d(w_0(t), w(t)) \leq K_w \phi(t)$$

for any $t \in I$. We have $|w_0(t) - w(t)| < \infty$ for any $w \in Y$. Therefore, we get that $\{w \in Y | d(w_0, w) < \infty\}$ is equal to Y. From Theorem 1.3, we conclude that v_0, given by (2.5), is the unique continuous function. Again from (2.3), we get

$$-\phi(t) \leq v'(t) - g(t, v(t), v(\alpha(t))) - \int_0^t k(t, s, v(s), v(\alpha(s))) ds \leq \phi(t) \tag{2.10}$$

for all $t \in I$. By integrating each term of inequality (2.10) from 0 to t, we get

$$\Big| v(t) - v(0) - \int_0^t g(\zeta, v(\zeta), v(\alpha(\zeta))) \, d\zeta - \int_0^t \int_0^s k(t, \zeta, v(\zeta), v(\alpha(\zeta))) \, d\zeta \, ds \Big|$$

$$\leq \int_0^t \phi(\zeta)\, d\zeta,$$

for all $t \in I$. From (2.4) and (2.8), we get

$$|v(t) - (\Theta v)(t)| \leq \int_0^t \phi(\zeta)\, d\zeta \leq N\phi(t) \tag{2.11}$$

for all $t \in I$, which implies that $d(v, \Theta v) \leq N$. Next by making the use of Theorem 1.3 and inequality (2.11), we conclude that

$$d(v, v_0) \leq \frac{1}{1-(NL_g + N^2 L_k)}\, d(\Theta v, v) \leq \frac{N}{1-(NL_g + N^2 L_k)}.$$

Consequently, this yields the inequality (2.6) for all $t \in I$. □

In the above Theorem 2.1 we examined the Hyers-Ulam-Rassias stability of (1.1) on a closed and bounded interval. Now, we are going to show that Theorem 2.1 is also valid for the case of unbounded interval.

2.2. Theorem. Suppose that I denote either $\mathbb{R}$ or $[0, \infty)$ or $(-\infty, T]$ for a given nonnegative real number T. Let L_g, L_k and N be positive constants with $0 < NL_g + N^2 L_k < 1$ and $\alpha\colon I \to \mathbb{R}$ be a continuous delay function such that $\alpha(t) \leq t$ for all $t \in I$. Assume that $g\colon I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying condition (2.1) for all $t \in I$, $v_1, v_2 \in \mathbb{R}$. If a continuously differentiable function $v\colon I \to \mathbb{R}$ satisfies inequality (2.3) for all $t \in I$, where $\phi\colon I \to (0, \infty)$ is a continuous function satisfying condition (2.4) for all $t \in I$, then there exists a unique continuous function $v_0\colon I \to \mathbb{R}$ which satisfies (2.5) and (2.6) for each $t \in I$.

Proof. First we assume that $I = \mathbb{R}$ and we are going to show that v_0 is a continuous function. For any $n \in \mathbb{N}$, we define the interval $I_n = [-n, n]$. In accordance with Theorem 2.1, there exists a unique continuous function $v_n : I_n \to \mathbb{R}$ in such a way that

$$v_n(t) = v(0) + \int_0^t g(\zeta, v_n(\zeta), v_n(\alpha(\zeta)))d\zeta + \int_0^t\!\int_0^s k(t, \zeta, v_n(\zeta), v_n(\alpha(\zeta))\, d\zeta\, ds \tag{2.12}$$

and

$$|v(t) - v_n(t)| \leq \frac{N}{1-(NL_g + N^2 L_k)}\phi(t) \text{ for all } t \in I. \tag{2.13}$$

The uniqueness of the function v_n implies that if $t \in I_n$, then

$$v_n(t) = v_{n+1}(t) = v_{n+2}(t) = \cdots . \tag{2.14}$$

For $t \in \mathbb{R}$, define $n(t) \in \mathbb{N}$ as

$$n(t) = \min\{n \in \mathbb{N} |\, t \in I_n\}.$$

Next, we define a function $v_0\colon \mathbb{R} \to \mathbb{R}$ by

$$v_0(t) = v_{n(t)}(t), \tag{2.15}$$

and claim that v_0 is continuous. For any $t_1 \in \mathbb{R}$ we take the integer $n_1 = n(t_1)$. Then, t_1 belongs to the interior of the interval I_{n_1+1} and there exists positive $\varepsilon > 0$ such that $v_0(t) = v_{n_1+1}(t)$ for all t with $t_1 - \varepsilon < t < t_1 + \varepsilon$. Since v_{n_1+1} is continuous at t_1, v_0 is continuous at t_1 for $t_1 \in \mathbb{R}$.

Now, we prove that the continuous function v_0 satisfies (2.5) and (2.7) for all $t \in \mathbb{R}$. Assume that $n(t)$ is an integer for any $t \in \mathbb{R}$. Then, by the use of (2.12) and (2.15), we have $t \in I_{n(t)}$ and

$$\begin{aligned} v_0(t) = v_{n(t)}(t) &= v(0) + \int_0^t g(\zeta, v_n(\zeta), v_n(\alpha(\zeta)))\, d\zeta \\ &+ \int_0^t \int_0^s k(t, \zeta, v_n(\zeta), v_n(\alpha(\zeta))\, d\zeta\, ds \\ &= v(0) + \int_0^t g(\zeta, v_0(\zeta), v_0(\alpha(\zeta)))\, d\zeta \\ &+ \int_0^t \int_0^s k(t, \zeta, v_0(\zeta), v_0(\alpha(\zeta)))\, d\zeta\, ds. \end{aligned}$$

where the last equality holds true because $n(\zeta) \le n(t)$ for all $\zeta \in I_{n(t)}$ and from equations (2.14) and (2.15) we get that

$$v_{n(t)}(\zeta) = v_{n(\zeta)}(\zeta) = v_0(\zeta).$$

Since $t \in I_{n(t)}$ for all $t \in \mathbb{R}$, so from (2.13) and (2.15), we have

$$|v(t) - v_0(t)| \le |v(t) - v_{n(t)}(t)| \le \frac{N}{1 - (NL_g + N^2 L_k)} \phi(t),$$

for any $t \in \mathbb{R}$.

Finally, we are going to prove that v_0 is unique. To do this we consider another continuous function $u_0 \colon \mathbb{R} \to \mathbb{R}$ which satisfies (2.5) and (2.7), with u_0 instead of v_0, for all $t \in \mathbb{R}$. Let $t \in \mathbb{R}$ be an arbitrary number. Since the restrictions $v_0|_{I_{n(t)}}$ and $u_0|_{I_{n(t)}}$ satisfies (2.5) and (2.7) for each $t \in I_{n(t)}$, the uniqueness of $v_n(t) = v_0|_{I_{n(t)}}$ suggest that

$$v_0(t) = v_0|_{I_{n(t)}}(t) = u_0|_{I_{n(t)}}(t) = u_0(t).$$

We can prove similarly for the cases $I = (-\infty, T]$ and $I = [0, \infty)$. □

3. Hyers-Ulam stability

In this section, we prove the Hyers-Ulam stability of the nonlinear Volterra integrodifferential equation with delay (1.1).

3.1. Theorem. Let $I = [0, T]$ be a non-degenerated interval, L_g and L_k be nonnegative constants such that $0 < TL_g + \frac{T^2}{2} L_k < 1$. Assume that $g \colon I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function which satisfies the Lipschitz condition (2.1) and $k \colon I \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies the Lipchitz condition (2.2). If for some $\sigma \ge 0$ and a continuously differentiable function $v \colon I \to \mathbb{R}$ we have

$$\left| v'(t) - g(t, v(t), v(\alpha(t))) - \int_0^t k(t, s, v(s), v(\alpha(s)))\, ds \right| \le \sigma \text{ for all } t \in I, \tag{3.1}$$

then there exists a unique continuous function $v_0 : I \to \mathbb{R}$ satisfying the equation (2.5) and

$$|v(t) - v_0(t)| \le \frac{T}{1 - (TL_g + \frac{T^2}{2} L_k)} \sigma, \text{ for all } t \in I. \tag{3.2}$$

Proof. Let Y be the set of all real valued continuous functions on closed and bounded interval I. For all $u, w \in Y$, we define a metric on Y by

$$d(u, w) = \inf \{K \in [0, \infty] \colon |u(t) - w(t)| \le K \text{ for all } t \in I\}. \tag{3.3}$$

The metric space (Y, d) is a complete generalized metric space, see [10]. Consider the operator $\Theta \colon Y \to Y$ defined by

$$(\Theta u)(t) = v(0) + \int_0^t g(\zeta, u(\zeta), u(\alpha(\zeta))) d\zeta + \int_0^t \int_0^s k(t, \zeta, u(\zeta), u(\alpha(\zeta)))\, d\zeta\, ds \tag{3.4}$$

for all $t \in I$ and for all $u \in Y$. Next, we need to check that the operator Θ is strictly contractive on the set Y. Suppose that $K_{uw} \in [0, \infty]$ be a constant with $d(u,w) \leq K_{uw}$ for any $u, w \in Y$. We have,

$$|u(t) - w(t)| \leq K_{uw}, \text{ for all } t \in I. \tag{3.5}$$

By making the use of (2.1), (2.2), (3.4) and (3.5), we get

$$\begin{aligned}
|(\Theta u)(t) - (\Theta w)(t)| &= \Big| \int_0^t \Big\{ g(\zeta, u(\zeta), u(\alpha(\zeta))) - g(\zeta, w(\zeta), w(\alpha(\zeta))) \Big\} d\zeta \\
&\quad + \int_0^t \int_0^s \Big\{ k(t, \zeta, u(\zeta), u(\alpha(\zeta))) - k(t, \zeta, w(\zeta), w(\alpha(\zeta))) \Big\} d\zeta\, ds \Big| \\
&\leq \int_0^t \Big| g(\zeta, u(\zeta), u(\alpha(\zeta))) - g(\zeta, w(\zeta), w(\alpha(\zeta))) \Big| d\zeta \\
&\quad + \int_0^t \int_0^s \Big| k(t, \zeta, u(\zeta), u(\alpha(\zeta))) - k(t, \zeta, w(\zeta), w(\alpha(\zeta))) \Big| d\zeta\, ds \\
&\leq L_g \int_0^t \big| u(\zeta) - w(\zeta) \big| d\zeta + L_k \int_0^t \int_0^s \big| u(\zeta) - w(\zeta) \big| d\zeta\, ds \\
&\leq L_g K_{uw} T + L_k K_{uw} \frac{T^2}{2} \\
&\leq K_{uw}(T L_g + \frac{T^2}{2} L_k), \text{ for all } t \in I,
\end{aligned}$$

i.e., $d(\Theta u, \Theta w) \leq K_{uw}(T L_g + \frac{T^2}{2} L_k)$. Hence, we may conclude that $d(\Theta u, \Theta w) \leq (T L_g + \frac{T^2}{2} L_k) d(u, w)$ for any $u, w \in Y$, where $0 < T L_g + \frac{T^2}{2} L_k < 1$. Suppose $w_0 \in Y$ be arbitrary, there exists a constant $K \in [0, \infty]$ with

$$\begin{aligned}
|\Theta w_0(t) - w_0(t)| &= \Big| v(0) + \int_0^t g(\zeta, w_0(\zeta), w_0(\alpha(\zeta))) d\zeta \\
&\quad + \int_0^t \int_0^s k(t, \zeta, u_0(\zeta), u_0(\alpha(\zeta))) d\zeta ds - w_0 \Big| \\
&\leq K, \text{ for all } t \in I.
\end{aligned}$$

Since $g(\zeta, w_0(\zeta), w_0(\alpha(\zeta))), k(t, \zeta, u_0(\zeta), u_0(\alpha(\zeta)))$ and w_0 are bounded. Thus, equation (3.3) implies that

$$d(\Theta w_0, w_0) < \infty.$$

So, according to Theorem 1.3, there exists a continuous function $v_0 \colon I \to \mathbb{R}$ in a way that $\Theta^n w_0 \to v_0$ in (Y, d) and $\Theta v_0 = v_0$, i.e., v_0 satisfies (2.5) for all $t \in I$. As in the proof of Theorem 2.1, it can be verify easily that $\{w \in Y | d(w_0, w) < \infty\}$ is equal to Y. From Theorem 1.3, we conclude that v_0, given by equation (2.5), is the unique continuous function. Again from equation (2.3), we get

$$-\sigma \leq v'(t) - g(t, v(t), v(\alpha(t))) - \int_0^t k(t, s, v(s), v(\alpha(s)))\, ds \leq \sigma, \tag{3.6}$$

for all $t \in I$. By integrating each term of inequality (3.6) from 0 to t, then we get

$$\begin{aligned}
&\Big| v(t) - v(0) - \int_0^t g(\zeta, v_0(\zeta), v_0(\alpha(\zeta))) d\zeta - \int_0^t \int_0^s k(t, \zeta, v_0(\zeta), v_0(\alpha(\zeta)))\, d\zeta\, ds \Big| \\
&\leq \sigma T
\end{aligned}$$

for all $t \in I$. So, it satisfies that $|v - \Theta v| \leq \sigma T$. Finally, Theorem 1.3 together with (2.11) implies that

$$d(v, v_0) \leq \frac{T}{1 - (TL_g + \frac{T^2}{2}L_k)} d(\Theta v, v) \leq \frac{T}{1 - (TL_g + \frac{T^2}{2}L_k)}\sigma.$$

□

Now we present two examples which indicate how our theorems can be applied to concrete problems.

3.2. Example. Let $a > 1$, $q \in (0, \infty)$ and p are arbitrary but fixed constants. Consider the class of equations

$$u'(t) = g(t) + \frac{1}{2}\int_0^t \lambda(t-s)^p (u(s) + u\big(\alpha(s) + f(s)\big)\, ds, t \in [0, T] \tag{3.7}$$

for $\lambda < \frac{(q \ln a)^2}{T^p}$, where $f(t)$ and $g(t)$ are any continuous functions. Here

$$k(t, s, u(s), u(\alpha(s))) = \frac{1}{2}\lambda(t-s)^p (u(s) + u(\alpha(s)) + f(s)).$$

Clearly

$$\begin{aligned}
&|k(t, s, u_1(s), u_1(\alpha(s))) - k(t, s, u_2(s), u_2(\alpha(s)))| \\
&\leq \frac{1}{2}\lambda \left|(t-s)^p\right| \left(|u_1(s) - u_2(s)| + |u_1(\alpha(s) - u_2(\alpha(s))|\right) \\
&\leq \lambda T^p \|u_1 - u_2\|.
\end{aligned}$$

Let $v : I \to R$ be such that

$$\left| v'(t) - g(t) - \frac{1}{2}\int_0^t (\lambda(t-s)^p (v(s) + v(\alpha(s)) + f(s)))ds \right| \leq \sigma(t) = a^{qt}, t \in [0, T].$$

Clearly,

$$\left| \int_0^t \sigma(t)dt \right| = \left| \int_0^t a^{qt}dt \right| = \frac{1}{q \ln a} a^{qt} = \frac{1}{q \ln a}\sigma(t)$$

for all $t \in [0, T]$. Theorem 2.1 ensures the existence of a unique continuous function $v : I \to R$ that solves (3.7) and

$$\big|v(t) - v_0(t)\big| \leq \frac{q \ln a}{(q \ln a)^2 - T^p \lambda} a^{qt},\ t \in [0, T].$$

3.3. Example. Consider the above class of problems for $\lambda < \frac{2}{T^{p+2}}$. Let for some $\sigma > 0$ and $v : I \to R$ we have

$$\Big|v'(t) - g(t) + \frac{1}{2}\int_0^t \lambda(t-s)^p (v(s) + v(\alpha(s)) + f(s))ds\Big| \leq \sigma,\ t \in [0, T].$$

In the light of Theorem 3.1, there exists a unique continuous function $v : I \to R$ that solves (3.7) for $\lambda < \frac{2}{T^{p+2}}$ and

$$\big|v(t) - v_0(t)\big| \leq \frac{2T}{2 - T^{p+2}\lambda}\sigma,\ t \in [0, T].$$

Conclusion. In this manuscript, we investigated the Hyers–Ulam–Rassias stability and Hyers–Ulam stability of a nonlinear Volterra integrodifferential equation with delay by using a fixed point method.

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