Fundamental Journal of Mathematics and Applications, 1 (1) (2018) 49-56

FUJMA

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma

Characterizations of slant and spherical helices due to pseudo-Sabban frame

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Article Info

Keywords: *Geodesic curvature, Pseudosphere, Hyperbolic plane, Helix, Sabban frame, Minkowski space*
2010 AMS: *51B20, 53A04*
Received: *3 April 2018*
Accepted: *19 April 2018*
Available online: *30 June 2018*

Abstract

In this paper, we investigate that under which conditions of the geodesic curvature of unit speed curve γ that lies on S_1^2 or H^2, the curve α which is obtained by using γ, is a spherical helix or slant helix in Minkowski 3-space.

1. Introduction

There are several studies in literature examining methodology to use spherical curves to construct some specialized curves. For example, Izuyama and Takeuchi [7], defined a way to construct Bertrand curves from the spherical curve whose spherical evolute coincides with the spherical Darboux image of the Bertrand curve. In addition to this paper, Encheva and Georgiev [4] showed a way to construct all *Frenet curves* $(\kappa > 0)$ by the following formula

$$\alpha(s) = b\int e^{\int k(s)ds}\gamma(s)ds + a$$

where b is a constant number, a is a constant vector, γ is a unit speed curve on S^2 with the *Sabban frame* and $k : I \to R$ is a function of class C^1. Moreover, they showed that the spherical curve γ is a circle if and only if the corresponding *Frenet curves* are cylindrical helices. Previously, we have found some characterizations to construct spherical helices and slant helices in Euclidean space by using these methods [2].
This paper is organized in the following way. In section 2 basic concepts of Minkowski 3-space R_1^3 are given. In section 3, spherical helices in R_1^3 are discussed by indicating some examples. Similarly, in section 4, slant helices in R_1^3 are examined.

2. Basic Concepts

Let us consider the Minkowski 3-space R_1^3 with the Lorentzian inner product

$$\langle x,y\rangle = x_1y_1 + x_2y_2 - x_3y_3$$

where $x = (x_1,x_2,x_3)$ and $y = (y_1,y_2,y_3) \in {R_1}^3$. The pseudo-norm of a vector x is given by $\|x\| = \sqrt{|\langle x,x\rangle|}$.
In the space R_1^3, the Lorentzian cross-product is defined as follows

$$x\wedge y = \begin{vmatrix} e_1 & e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2y_3 - x_3y_2, \quad x_3y_1 - x_1y_3, \quad x_2y_1 - x_1y_2).$$

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It's clearly seen that the cross-product has the following properties [3],

$$\begin{aligned}
&(i) \quad x\wedge y = -(y\wedge x)\\
&(ii) \quad \langle x\wedge y, z\rangle = det(x,y,z)\\
&(iii) \quad x\wedge (y\wedge z) = \langle x,y\rangle z - \langle x,z\rangle y\\
&(iv) \quad \langle x\wedge y, x\wedge y\rangle = (\langle x,y\rangle)^2 - \langle x,x\rangle \langle y,y\rangle\\
&(v) \quad \langle x\wedge y, x\rangle = 0 \quad , \quad \langle x\wedge y, y\rangle = 0
\end{aligned}$$

where $x,y,z \in {R_1}^3$.
A vector $x\in {R_1}^3$ is called spacelike if $\langle x,x\rangle > 0$ or $x=0$, timelike if $\langle x,x\rangle < 0$, lightlike if $\langle x,x\rangle = 0$ and $x\neq 0$ [8].
In [8], the hyperbolic plane (resp. pseudosphere) center $q\in R_1^3$ and of radius $r>0$ are defined by,

$$H^2(r;q) = \left\{x=(x_1,x_2,x_3)\in R_1^3 : \langle x-q, x-q\rangle = -r^2, x_3-q_3>0\right\},$$

$$S_1^2(r;q) = \left\{(x_1,x_2,x_3)\in R_1^3 : \langle x-q, x-q\rangle = r^2\right\}.$$

When $r=1$ and p is the origin, the *hyperbolic plane* is denoted by H^2 and the *pseudosphere* is denoted by S_1^2. ,
In this paper, when a helix lies on $H^2(r;q)$ or $S_1^2(r;q)$, we call it spherical curve.
Given a regular curve $\alpha(t) : I\subset R \to R_1^3$. We say that α is spacelike (resp. timelike, lightlike) at t if $\alpha'(t)$ is a spacelike (resp. timelike, lightlike) vector. The curve α is called spacelike (resp. timelike, lightlike) if it is for any $t\in I$ [8].
A non-lightlike curve $\alpha : I\subset R \longrightarrow {E_1}^3$ is said to be parametrized by the pseudo arclength parameter s, if $|\langle \alpha'(s), \alpha'(s)\rangle| = 1$. In this case, we call α is a unit speed curve.
For a unit speed non-lightlike curve α with a spacelike or timelike normal vector $N(s)$, the Frenet formulae are given in [8]. It's easy to calculate the formulae for arbitrary speed non-lightlike curves as follows.
If α is a timelike curve,

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa\nu & 0 \\ \kappa\nu & 0 & \tau\nu \\ 0 & -\tau\nu & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.1}$$

If α is a spacelike curve with a spacelike normal vector $N(t)$,

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa\nu & 0 \\ -\kappa\nu & 0 & \tau\nu \\ 0 & \tau\nu & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.2}$$

If α is a spacelike curve with a timelike normal vector $N(t)$,

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa\nu & 0 \\ \kappa\nu & 0 & \tau\nu \\ 0 & \tau\nu & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.3}$$

where

$$\kappa = \frac{\|\alpha'\wedge\alpha''\|}{\|\alpha'\|^3}, \tau = \frac{det(\alpha', \alpha'', \alpha''')}{\|\alpha'\wedge\alpha''\|^2}, \nu = \sqrt{|\langle \alpha', \alpha'\rangle|}. \tag{2.4}$$

In the formulae above, we denote unit tangent vector with $T(t)$, unit binormal vector with $B(t)$, unit normal vector with $N(t)$.
A regular timelike or spacelike curve α is a helix, if τ/κ is a constant function.
For a unit speed curve α in R_1^3, slant helix characterization is given in [1]. Also, some characterizations of Lorentzian unit speed curves which lies on H^2 or S_1^2 were investigated in [9, 10, 11, 12]. With the help of these papers, we easily have the Lemmas for arbitrary speed curves below.

Lemma 2.1. *Let α be a timelike curve in R_1^3. Then, α is a slant helix if and only if either one of the next two functions*

$$\frac{\kappa^2}{\nu\left(\tau^2-\kappa^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)' \quad or \quad \frac{\kappa^2}{\nu\left(\kappa^2-\tau^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)' \tag{2.5}$$

is constant everywhere $\tau^2-\kappa^2$ does not vanish.

Lemma 2.2. *Let α be a spacelike curve in R_1^3 with a spacelike normal vector. Then, α is a slant helix if and only if either one of the next two functions*

$$\frac{\kappa^2}{\nu\left(\tau^2-\kappa^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)' \quad or \quad \frac{\kappa^2}{\nu\left(\kappa^2-\tau^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)' \tag{2.6}$$

is constant everywhere $\tau^2-\kappa^2$ does not vanish.

Lemma 2.3. *Let α be a spacelike curve in R_1^3 with a timelike normal vector. Then, α is a slant helix if and only if the function*

$$\frac{\kappa^2}{\nu\left(\tau^2+\kappa^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)' \tag{2.7}$$

is constant.

Lemma 2.4. *Let α be a spacelike curve in R_1^3 with a spacelike normal vector. Image of α lies on the pseudosphere (resp. hyperbolic plane) of radius r and center q if and only if*

$$\frac{1}{\kappa^2}-\left(\frac{1}{\nu\tau}\left(\frac{1}{\kappa}\right)'\right)^2=\pm r^2(resp.) \tag{2.8}$$

where $r>0\in R, \kappa\neq 0, \tau\neq 0$.

Lemma 2.5. *Let α be a timelike curve in R_1^3. Image of α lies on the pseudosphere of radius r and center q if and only if*

$$\frac{1}{\kappa^2}+\left(\frac{1}{\nu\tau}\left(\frac{1}{\kappa}\right)'\right)^2=r^2 \tag{2.9}$$

where $r>0\in R, \kappa\neq 0, \tau\neq 0$.

Lemma 2.6. *Let α be a spacelike curve in R_1^3 with a timelike normal vector. Image of α lies on the hyperbolic plane of radius r and center q if and only if*

$$\frac{-1}{\kappa^2}+\left(\frac{1}{\nu\tau}\left(\frac{1}{\kappa}\right)'\right)^2=-r^2 \tag{2.10}$$

where $r>0\in R, \kappa\neq 0, \tau\neq 0$.

Let γ be a non-lightlike unit speed spherical curve with the arc-length parameter s and denote $\gamma'=t$ where $\gamma'=d\gamma/ds$. If we set a vector $p=\gamma\wedge t$, by definition we have an orthonormal frame $\{\gamma,t,p\}$. This frame is called the pseudo-Sabban frame of γ [5, 6]. Thus, we have the following Lemma .

Lemma 2.7. *Let $\gamma(s)$ be a unit speed spherical curve in R_1^3, then*
(i) If γ is a timelike curve on S_1^2 then,

$$\begin{aligned}\gamma' &= t\\ t' &= k_g p+\gamma\\ p' &= k_g t\end{aligned} \tag{2.11}$$

(ii) If γ is a spacelike curve on S_1^2, then

$$\begin{aligned}\gamma' &= t\\ t' &= -k_g p-\gamma\\ p' &= -k_g t\end{aligned} \tag{2.12}$$

(iii) If γ is a spacelike curve on H^2, then

$$\begin{aligned}\gamma' &= t\\ t' &= k_g p+\gamma\\ p' &= -k_g t\end{aligned} \tag{2.13}$$

where $k_g=det\left(\gamma,t,t'\right)$ the geodesic curvature of curve γ.

3. Spherical helices on $S_1^2(r;p)$ and $H^2(r;p)$

Let us take the curve

$$\alpha(s)=b\int e^{\int k(s)ds}\gamma(s)ds+a \tag{3.1}$$

at [4]. If we make the neccessary calculations, we have

$$\begin{aligned}\alpha'(s) &= be^{\int k(s)ds}\gamma(s),\\ \alpha''(s) &= be^{\int k(s)ds}\left(k(s)\gamma(s)+\gamma'(s)\right),\\ \alpha'''(s) &= be^{\int k(s)ds}\left(\left(k^2(s)+k'(s)\right)\gamma(s)+2k(s)\gamma'(s)+\gamma''(s)\right).\end{aligned} \tag{3.2}$$

If we calculate κ, τ, and ν of the curve α by using the equations at (2.4) and (3.2), we find

$$\begin{aligned}\kappa(s) &= \tfrac{1}{be^{\int k(s)ds}},\\ \tau(s) &= \tfrac{k_g(s)}{be^{\int k(s)ds}},\\ \nu(s) &= be^{\int k(s)ds}.\end{aligned} \tag{3.3}$$

It's easy to see

$$\begin{aligned} \langle\alpha'(s),\alpha'(s)\rangle &= b^2 e^{2\int k(s)ds}\langle\gamma(s),\gamma(s)\rangle, \\ T(s) &= \gamma(s), \\ T'(s) &= t(s). \end{aligned} \tag{3.4}$$

So, we can say if γ is a unit speed spacelike curve which lies on S_1^2, then α is a spacelike curve with a spacelike normal vector N.
If γ is a unit speed spacelike curve which lies on H^2, then α is a timelike curve with a spacelike normal vector N.
If γ is a unit speed timelike curve which lies on S_1^2 then α is a spacelike curve with a timelike normal vector N.
Now, we want to show, under which circumstances the curve α at equation (3.1) is a spherical helix on $S_1^2(r;p)$.

Theorem 3.1. *If the curve γ is a unit speed spacelike curve with a constant geodesic curvature, which lies on S_1^2, the curve α defined by (3.1) is a spherical helix which lies on the pseudosphere of the radius $|bd|$ and of the center origin if and only if the function $k(s)=k_g tanh\left[(k_g)(s-c)\right]$ where $b,c,d\in R$.*

Proof. From (3.2), (3.3), and (3.4), we know the curve

$$\alpha(s)=b\int e^{\int k(s)ds}\gamma(s)ds+a$$

is a spacelike curve with a spacelike normal vector $N(s)$. So we need to use (2.8). Let's take the derivate of (2.8) with respect to s. Then, we have

$$\left(\frac{1}{\nu}\left[\frac{1}{\nu\tau}\left(\frac{1}{\kappa}\right)'\right]'-\frac{\tau}{\kappa}\right)(s)=0$$

By putting (3.3) in this equation, we have

$$\left(\frac{1}{be^{\int kds}}\left[\frac{1}{k_g}\left(be^{\int kds}\right)'\right]'-k_g\right)(s)=0$$

$$k'(s)+k^2(s)={k_g}^2.$$

If we solve this differential equation, we have

$$k(s)=k_g tanh\left[(k_g)(s-c)\right]$$

Conversely, if we take $k(s)=k_g tanh\left[(k_g)(s-c)\right]$ in (14), then

$$\int k(s)ds=\int k_g tanh\left[(k_g)(s-c)\right]ds.$$

Let $u=k_g(s-c)=k_g s-k_g c$ then $k_g ds=du$, by using these equations

$$\begin{aligned}\int k(s)ds &= \int \tanh u du \\ &= \ln\cosh u+\ln d \\ &= \ln\left[d\cosh\left(k_g(s-c)\right)\right]\end{aligned}$$

we have

$$\begin{aligned}\alpha(s) &= b\int e^{\int k(s)ds}\gamma(s)ds+a \\ &= b\int e^{\ln\left[d\cosh\left(k_g(s-c)\right)\right]}\gamma(s)ds+a \\ &= b\int d\cosh\left(k_g(s-c)\right)\gamma(s)ds+a\end{aligned}$$

where $c,d\in R$.
Now, we must show that curve α is spherical. If we use (2.8) to do it, we have

$$\begin{aligned} r^2 &= \left(\left(\frac{1}{\kappa^2}-\left(\frac{1}{\nu\tau}\left(\frac{1}{\kappa}\right)'\right)\right)^2\right)(s) \\ &= \left(b^2e^{2\int kds}\left(1-\frac{k^2}{k_g^2}\right)\right)(s) \\ &= b^2d^2\cosh^2\left(k_g(s-c)\right)\left(\frac{1}{\cosh^2\left(k_g(s-c)\right)}\right) \\ &= b^2d^2.\end{aligned}$$

Therefore, it can be said that the curve α lies on S_1^2 which has a radius $|bd|$. $\square$

Now, we can give another theorem.

Theorem 3.2. *If the curve γ is a unit speed spacelike curve with a constant geodesic curvature, which lies on H^2, the curve α defined by (3.1) is a spherical helix which lies on the pseudosphere of the radius $|bd|$ and of the center origin if and only if the function $k(s) = k_g tan\left[\left(k_g\right)(s-c)\right]$ where $b,c,d \in R$.*

Proof. By using (2.9) instead of (2.8) in Theorem 3.1, the proof is similar. □

Theorem 3.3. *If the curve γ is a unit speed timelike curve with a constant geodesic curvature, which lies on S_1^2, the curve α defined by (3.1) is a spherical helix which lies on the hyperbolic plane of the radius $|bd|$ and of the center origin if and only if the function $k(s) = k_g tanh\left[\left(k_g\right)(s-c)\right]$ where $b,c,d \in R$.*

Proof. By using (2.10) instead of (2.8) in Theorem 3.1, the proof is similar. □

Example 3.4. *Let's take $\gamma(s) = \left\{\sqrt{2}\cos\left(s/\sqrt{2}\right), \sqrt{2}\sin\left(s/\sqrt{2}\right), 1\right\}$, we know that γ is a spacelike curve on S_1^2 with the geodesic curvature $\sqrt{2}$. Then due to Theorem 3.1,*

$$k(s) = k_g tanh\left[\left(k_g\right)(s-c)\right]$$

and

$$\alpha(s) = b\int d\cosh\left(k_g(s-c)\right)\gamma(s)ds + a$$

where $b,c,d \in R$. If we take $b=2, c=0, d=1$; then, we have

$$\begin{aligned}
\alpha_1(s) &= 2\cosh\left(s/\sqrt{2}\right)\sin\left(s/\sqrt{2}\right) + 2\cos\left(s/\sqrt{2}\right)\sinh\left(s/\sqrt{2}\right)\\
\alpha_2(s) &= -2\cos\left(s/\sqrt{2}\right)\cosh\left(s/\sqrt{2}\right) - 2\sin\left(s/\sqrt{2}\right)\sinh\left(s/\sqrt{2}\right)\\
\alpha_3(s) &= 2\sqrt{2}\sinh\left(s/\sqrt{2}\right)
\end{aligned}$$

where $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$ and $a = (0,0,0)$

Example 3.5. *Let's take $\gamma(s) = \left\{cos(s), sin(s), \sqrt{2}\right\}$, we know that γ is a spacelike curve on H^2 with the geodesic curvature $\sqrt{2}$. Then, due to Theorem 3.2,*

$$k(s) = k_g tan\left[\left(k_g\right)(s-c)\right]$$

and

$$\alpha(s) = b\int d\cos\left(k_g(s-c)\right)\gamma(s)ds + a$$

where $b,c,d \in R$. If we take $b=2, c=0, d=1$; then, we have

$$\begin{aligned}
\alpha_1(s) &= -2\cos\left(\sqrt{2}s\right)\sin(s) + 2\sqrt{2}\cos(s)\sin\left(\sqrt{2}s\right)\\
\alpha_2(s) &= 2\cos(s)\cos\left(\sqrt{2}s\right) + 2\sqrt{2}\sin(s)\sin\left(\sqrt{2}s\right)\\
\alpha_3(s) &= 2\sin\left(\sqrt{2}s\right)
\end{aligned}$$

where $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$ and $a = (0,0,0)$

Example 3.6. *Let's take $\gamma(s) = \left\{\frac{1}{\sqrt{3}}\cosh\left(\sqrt{3}s\right), \frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\sinh\left(\sqrt{3}s\right)\right\}$, we know that γ is a timelike curve on S_1^2 with the geodesic curvature $\sqrt{2}$. Then, due to Theorem 3.3,*

$$k(s) = k_g tanh\left[\left(k_g\right)(s-c)\right]$$

and

$$\alpha(s) = b\int d\cosh\left(k_g(s-c)\right)\gamma(s)ds + a$$

where $b,c,d \in R$. If we take $b=2, c=0, d=1$; then, we have

$$\begin{aligned}
\alpha_1(s) &= -2\sqrt{\tfrac{2}{3}}\cosh\left(\sqrt{3}s\right)\sinh\left(\sqrt{2}s\right) + 2\cosh\left(\sqrt{2}s\right)\sinh\left(\sqrt{3}s\right)\\
\alpha_2(s) &= \frac{2\sinh\left(\sqrt{2}s\right)}{\sqrt{3}}\\
\alpha_3(s) &= 2\cosh\left(\sqrt{2}s\right)\cosh\left(\sqrt{3}s\right) - 2\sqrt{\tfrac{2}{3}}\sinh\left(\sqrt{2}s\right)\sinh\left(\sqrt{3}s\right)
\end{aligned}$$

where $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$ and $a = (0,0,0)$

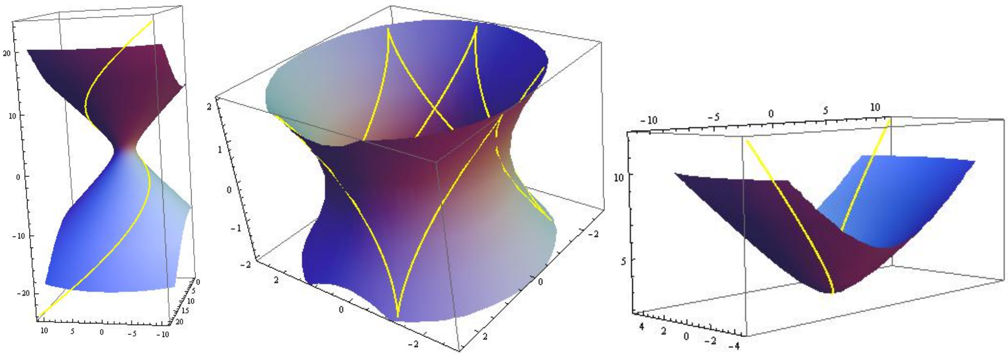

Figure 3.1: Spherical Helices (Resp. Example 1,2, and 3)

4. Constructing slant helices from unit speed spherical curves

In this section, we want to give some characterizations about slant helices.

Theorem 4.1. *Let $\gamma(s)$ be a unit speed spacelike curve on S_1^2; b,m,n be constant numbers; and a be a constant vector. The geodesic curvature of $\gamma(s)$ satisfies*

$$k_g{}^2(s) = \frac{(ms+n)^2}{1+(ms+n)^2}$$

if and only if

$$\alpha(s) = b\int e^{\int k(s)ds}\gamma(s)ds + a$$

is a spacelike slant helix with a spacelike normal vector.

Proof. Let, for γ

$$k_g{}^2(s) = \frac{(ms+n)^2}{1+(ms+n)^2}. \tag{4.1}$$

From (3.2), (3.3), and (3.4), we know α is a spacelike curve with a spacelike normal vector N. So; from (2.6), the geodesic curvature of the spherical image of the principal normal indicatrix of α is as follows

$$\begin{aligned}\sigma(s) &= \left(\frac{\kappa^2}{\nu\left(\kappa^2-\tau^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)'\right)(s)\\ &= \left(\frac{\frac{1}{\nu^2}}{\nu\left(\frac{1}{\nu^2}-\frac{k_g{}^2}{\nu^2}\right)^{3/2}}k_g{}'\right)(s).\end{aligned}$$

So, we have

$$\sigma(s) = \frac{k_g{}'(s)}{\left(1-k_g{}^2(s)\right)^{3/2}} \tag{4.2}$$

Now, let's take $u(s) = ms+n$, then we have (4.1)

$$k_g{}^2(s) = \frac{u^2(s)}{1+u^2(s)}. \tag{4.3}$$

If we take the derivates of the both sides of (4.3) with respect to s, we have

$$\begin{aligned}2k_g(s)\,k_g{}'(s) &= \left(\frac{2uu'\left(1+u^2\right)-\left(2uu'\right)u^2}{\left(1+u^2\right)^2}\right)(s)\\ k_g(s)\,k_g{}'(s) &= \left(\frac{uu'}{\left(1+u^2\right)^2}\right)(s)\end{aligned}$$

$$k_g{}'(s) = \left(\left(\frac{uu'}{\left(1+u^2\right)^2}\right)\left(\varepsilon\sqrt{\frac{1+u^2}{u^2}}\right)\right)(s) \tag{4.4}$$

where $\varepsilon = \pm 1$. Putting (4.3) and (4.4) in (4.3), we have

$$\begin{aligned}\sigma(s) &= \frac{{k_g}'(s)}{\left(1-{k_g}^2(s)\right)^{3/2}} \\ &= \left(\varepsilon \frac{\sqrt{1+u^2}uu'}{|u|\left(1+u^2\right)^2}\left(1+u^2\right)^{3/2}\right)(s) \\ &= \varepsilon \frac{ms+n}{|ms+n|}m \\ &= \varepsilon m\end{aligned}$$

which is constant.

Conversely, let $\alpha(s)$ be a spacelike slant helix, then the geodesic curvature of the spherical image of the principal normal indicatrix of α is a constant function. So, we can take

$$\sigma(s) = \left(\frac{\kappa^2}{\nu\left(\kappa^2-\tau^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)'\right)(s) = m$$

where $m \in R$. Therefore, from (4.2)

$$\begin{aligned}m &= \left(\frac{\kappa^2}{\nu\left(\kappa^2-\tau^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)'\right)(s) \\ &= \frac{{k_g}'(s)}{\left(1-{k_g}^2(s)\right)^{3/2}}\end{aligned}$$

If we solve this differential equation, we have

$$\frac{k_g(s)}{\sqrt{1-{k_g}^2(s)}} = ms+n$$

where $n \in R$. Then,

$${k_g}^2(s) = \frac{(ms+n)^2}{1+(ms+n)^2}.$$

□

Theorem 4.2. *Let $\gamma(s)$ be a unit speed spacelike curve on H^2; b,m,n be constant numbers; and a be a constant vector. The geodesic curvature of $\gamma(s)$ satisfies*

$${k_g}^2(s) = \frac{(ms+n)^2}{1+(ms+n)^2}$$

if and only if

$$\alpha(s) = b\int e^{\int k(s)ds}\gamma(s)ds + a$$

is a timelike slant helix with a spacelike normal vector.

Proof. By using (2.5) instead of (2.6) in Theorem 4.1, the proof is similar. □

Theorem 4.3. *Let $\gamma(s)$ be a unit speed timelike curve on S_1^2; b,m,n be constant numbers; and a be a constant vector. The geodesic curvature of $\gamma(s)$ satisfies*

$${k_g}^2(s) = \frac{(ms+n)^2}{1-(ms+n)^2}$$

if and only if

$$\alpha(s) = b\int e^{\int k(s)ds}\gamma(s)ds + a$$

is a spacelike slant helix with a timelike normal vector.

Proof. By using (2.7) instead of (2.6) in Theorem 4.1, the proof is similar. □

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