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FLAT STRONG δ-COVERS OF MODULES

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ABSTRACT. We say that a ring R is right generalized δ-semiperfect if every simple right R-module is an epimorphic image of a flat right R-module with δ-small kernel. This definition gives a generalization of both right δ-semiperfect rings and right generalized semiperfect rings. We provide examples involving such rings along with some of their properties. We introduce flat strong δ-cover of a module as a flat cover which is also a flat δ-cover and use flat strong δ-covers in characterizing right A-perfect rings, right B-perfect rings and right perfect rings.

1. Introduction

Flat cover of a module M is introduced by E. Enochs (see [10]). It is a homomorphism $\alpha : F \longrightarrow M$ with the following properties.

(i) *F is a flat module.*
(ii) *for any homomorphism $\beta : F' \longrightarrow M$ with F' a flat module, there is a homomorphism $\gamma : F' \longrightarrow F$ such that $\alpha \circ \gamma = \beta$.*
(iii) *if θ is an endomorphism of F satisfying $\alpha \circ \theta = \alpha$, then θ is an automorphism.*

In [1] the term flat cover is used for another concept. A flat cover of a module M is defined as an epimorphism $f : F \longrightarrow M$ from a flat module F with a small kernel. In [9], such covers of modules are called flat B-covers to distinguish between these two definitions, since this definition is derived from the definition of a projective cover in the sense of H. Bass (see [6]). We stick to the notation used in [9] concerning flat covers.

As a generalization of right perfect rings, right generalized perfect rings are introduced in [1] as rings whose modules have flat B-covers. In [9], right generalized semiperfect (shortly G-semiperfect) rings are defined with the same condition restricted to the class of all simple modules. Some properties and examples of such

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rings can be found in [1] and [9]. In [9, §3], a flat cover of a module which is also a flat B-cover is called a flat strong cover.

Right A-perfect rings and right B-perfect rings are defined using the projectivity of flat covers of certain modules (see [2] and [7]). One of the equivalent conditions for a ring R to be right A-perfect (right B-perfect resp.) is that flat covers of cyclic (simple resp.) modules are projective. It is shown in [9] that certain modules having flat strong covers are related to the ring being right A-perfect or right B-perfect.

Y. Zhou introduced δ-small submodules and defined δ-covers as epimorphisms with δ-small kernel (see [15]). Rings whose (simple resp.) modules have projective δ-covers are defined as right δ-perfect (right δ-semiperfect resp.) rings in the same work. In [5], flat δ-covers are introduced as a generalization of both projective δ-covers and flat B-covers. Rings over which every module has a flat δ-cover are called right generalized δ-perfect and properties and examples illustrating relation between such rings, perfect rings and δ-perfect rings are given in [5].

In the first part of this work, we follow the ideas used in [5] and define right generalized δ-semiperfect rings as a generalization of both δ-semiperfect rings and generalized δ-perfect rings by restricting the property of "having flat δ-covers"to simple modules. For this reason, most of the results given in section 2 depend on and/or uses the ones given in [5] for generalized δ-perfect rings. In this section, we give some properties of right generalized δ-semiperfect rings and provide some examples. Such rings are closed under quotients and finite direct products. We show that a commutative domain is right generalized δ-semiperfect if and only if it is local which generalizes [9, Proposition 2.10]. We also give a direct proof to the fact that a ring is right perfect if and only if it is semilocal and every semisimple module has a flat δ-cover so that semilocal right generalized δ-perfect rings are right perfect.

Generalizing the notion of flat strong covers, we also define flat strong δ-covers of modules as flat covers which are also flat δ-covers. We show that flat cover of a module M is projective if and only if M has a projective cover and a flat strong δ-cover. Using this result we characterize right A-perfect rings, right B-perfect rings and right perfect rings as semilocal rings over which every cyclic, simple and semisimple module has a flat strong δ-cover, respectively.

For a ring R, J denotes the Jacobson radical of the ring R and by saying a regular ring we mean a von Neumann regular ring. Let M be a module and N be a submodule of M. N is said to be δ-small in M if $N + K \neq M$ for every proper submodule K of M with M/K singular (see [15]). It generalizes the notion of small submodules in which the condition M/K being singular in the definition is omitted. The submodule $\delta(M)$ is the sum of all δ-small submodules of M. If M is projective, then by [15, Lemma 1.9], $\delta(M)$ is the intersection of all essential maximal submodules of M. We use δ_r instead of $\delta(R_R)$. $\mathrm{Rad}(M)$ denotes the Jacobson radical of M and the notations $\leq$, $\ll$ and $\ll_\delta$ are used to indicate submodule, small submodule and δ-small submodule, respectively. The following results are used in the sequel.

Lemma 1. [15, Lemma 1.2] *Let N be a submodule of a module M. Then the following are equivalent.*

(i) $N \ll_\delta M$.
(ii) *If $X + N = M$, then $M = X \oplus Y$ for a projective semisimple submodule Y with $Y \subseteq N$.*

Lemma 2. *Let M be a module. If* $\mathrm{Rad}(M) \ll_\delta M$*, then* $\mathrm{Rad}(M) \ll M$.

Proof. Assume that $\mathrm{Rad}(M) + N = M$ for some $N \subsetneqq M$. By Lemma 1, $M = N \oplus (\bigoplus_{i\in I} S_i)$ for some index set I, where S_i is simple for every $i \in I$. Since $N \neq M$, I is nonempty. For $i_0 \in I$, $K = N \oplus (\bigoplus_{\substack{i\in I\\ i\neq i_0}} S_i)$ is a maximal submodule of M and we have $M = \mathrm{Rad}(M) + N \leq K$ which is a contradiction. □

2. Generalized δ-Semiperfect Rings

Flat δ-covers of modules are introduced in [5]. Rings over which every module has a flat δ-cover are defined as right generalized δ-perfect (briefly right G-δ-perfect) rings in the same work. Most of the results given in this section depend on and/or uses the ones given in [5] for generalized δ-perfect rings. Related results from this work are cited wherever they are used. We restrict the property of "having a flat δ-cover" to simple modules and give the following definition.

Definition 1. *We call a ring R right generalized δ-semiperfect (right G-δ-semiperfect, for short) if every simple right R-module has a flat δ-cover. Left G-δ-semiperfect rings are defined similarly. If R is both right and left G-δ-semiperfect, we call R a G-δ-semiperfect ring.*

We now give some examples of right G-δ-semiperfect rings to see their relation to those already studied. Let us recall that a ring R is called a right V-ring if every simple module is flat.

Example 1.

(a) *Every right perfect ring is right G-δ-perfect and every semiperfect ring is G-δ-semiperfect.*
(b) *Every flat module is a flat δ-cover of itself, therefore every right V-ring is right G-δ-semiperfect.*
(c) *Every right G-semiperfect ring is right G-δ-semiperfect.*
(d) *Following the proof for* [5, Example 3.4]*, we can shown that $\mathbb{Z}$ is not a G-δ-semiperfect ring. Let p be a prime number and $f : F \longrightarrow \mathbb{Z}/p\mathbb{Z}$ be a flat δ-cover of $\mathbb{Z}/p\mathbb{Z}$. By the use of* [5, Lemma 2.5]*, $F \cong \mathbb{Z}/K$ for some submodule K of $\mathbb{Z}$, since projective semisimple abelian groups are zero. Then $\mathbb{Z}/K$*

is a cyclic flat abelian group. Since $\mathbb{Z}$ *is noetherian, it is projective so that* $K = 0$ *and* $F \cong \mathbb{Z}$. *Then for the isomorphism* $g : F \longrightarrow \mathbb{Z}$, *we have* $g(\operatorname{Ker} f) \ll_\delta \mathbb{Z}$ *by* [15, Lemma 1.3(2)], *since* $\operatorname{Ker} f \ll_\delta \mathbb{Z}$. g *is an isomorphism and* $\delta(\mathbb{Z}) = 0$ *imply that* $\operatorname{Ker} f = 0$ *and so* $\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z}$ *which is a contradiction. Therefore* $\mathbb{Z}/p\mathbb{Z}$ *does not have a flat* δ*-cover.*

Example 2. *For a regular ring* K, *Let* $R = \prod_{i=1}^{\infty} K_i$ *with* $K_i = K$ *for* $i = 1, 2, 3, \ldots$. *It is shown in* [11, §10.4] *that* R *is a regular ring which is not semisimple. Then* R *is a regular ring which is not semiperfect. Hence* R *is a right* G-δ*-semiperfect ring which is not semiperfect.*

Proposition 1. *Let* R *and* S *be right* G-δ*-semiperfect rings. Then the following hold.*

(i) *A ring Morita equivalent to* R *is right* G-δ*-semiperfect.*
(ii) *Every factor ring of* R *is right* G-δ*-semiperfect.*
(iii) $R \times S$ *is right* G-δ*-semiperfect.*

Proof. The proof for (i) is almost the same as the one for right G-δ-perfect rings in [5, Proposition 3.7]. Its proof is given in details, so we omit it to avoid repetition. A proof similar to that for [9, Proposition 2.8] implies (ii) and (iii). □

Remark 1. *Over a right noetherian ring, finitely generated modules are finitely presented and so a flat* δ*-cover of a finitely generated module* M *is also a projective* δ*-cover of* M *by* [5, Lemma 2.6]. *It follows from* [8, Remark 4.4] *that a right noetherian ring is semiperfect if and only if it is right* G-δ*-semiperfect.*

The following result is a consequence of [5, Theorem 4.3] and [8, Corollary 4.3]. We include it for future references.

Theorem 1. *The following are equivalent for a ring* R.

(i) R *is semiperfect.*
(ii) R *is semilocal and every simple module has a flat* B*-cover.*
(iii) R *is semilocal and every simple module has a flat* δ*-cover.*

Example 3. *(Remark in* [13]*) Let* $R = S^{-1}\mathbb{Z}$ *with* $S = \mathbb{Z} \backslash (p\mathbb{Z} \cup q\mathbb{Z})$ *for prime numbers* $p \neq q$. *Then* R *is semilocal but not* G-δ*-semiperfect.*

Proposition 2. *Let* R *be a right* G-δ*-semiperfect ring and* J *be nil. Then* R *is right noetherian if and only if* R *is right artinian.*

Proof. It is a consequence of [9, Proposition 2.15] and Remark 1. □

Corollary 1. $R[x]$ *is not a* G-δ*-semiperfect ring for every commutative noetherian ring* R.

Proposition 3. *Let* R *be a commutative domain. Then the following statements are equivalent.*

(i) *R is local.*
(ii) *R is semiperfect.*
(iii) *R is G-semiperfect.*
(iv) *R is G-δ-semiperfect.*

Proof. Only $(iv)\Rightarrow(i)$ needs to be proved. We follow the proof for [5, Lemma 2.6] to show that every simple module has a flat δ-cover which is cyclic. Let $f : F \longrightarrow S$ be a flat δ-cover of a simple module S and $g : R \longrightarrow S$ be the canonical epimorphism. Since R is projective, there is a homomorphism $h : R \longrightarrow F$ satisfying $fh = g$. Since $\operatorname{Ker} f + \operatorname{Im} h = F$ and $\operatorname{Ker} f \ll_\delta F$, we have that $\operatorname{Im} h$ is a direct summand of F by Lemma 1 and $\operatorname{Ker} f \cap \operatorname{Im} h \ll_\delta$ by [5, Lemma 2.4]. Then $f|_{\operatorname{Im} h} : \operatorname{Im} h \longrightarrow S$ is a flat δ-cover of S. Moreover, $\operatorname{Im} h$ is cyclic as a factor module of R and so $\operatorname{Im} h \cong R$, since $\operatorname{Im} h$ is torsion-free. By using [15, Lemma 1.3(2)] there is a maximal ideal M of R with $M \ll_\delta R$. Hence R is local. □

Note that Proposition 3 gives another way to show that $\mathbb{Z}$ is not a G-δ-semiperfect ring.

Proposition 4. *Let R be a commutative ring and S be a multiplicatively closed subset of R such that every maximal ideal of the ring $S^{-1}R$ is of the form $S^{-1}M$ for some maximal ideal M of R. If R is G-δ-semiperfect then so is $S^{-1}R$.*

Proof. Let U be a maximal ideal of $S^{-1}R$ with $U = S^{-1}M$ for some maximal ideal M of R. Since R is G-δ-semiperfect, by [5, Lemma 2.6] there is a cyclic flat δ-cover R/I of R/M. Since R/I is a flat R-module, $S^{-1}R/S^{-1}I \cong S^{-1}(R/I)$ is a flat $S^{-1}R$-module. Let $S^{-1}N/S^{-1}I$ be a maximal ideal of $S^{-1}R/S^{-1}I$ other than $S^{-1}M/S^{-1}I$. Then $S^{-1}M/S^{-1}I + S^{-1}N/S^{-1}I = S^{-1}R/S^{-1}I$ and so $M + N = R$. Since $M/I \ll_\delta R/I$, $(N + I)/I$ is a direct summand in R so that $S^{-1}N/S^{-1}I$ is a direct summand in $S^{-1}R/S^{-1}I$.

Now we have that either $S^{-1}M/S^{-1}I$ is a direct summand in $S^{-1}R/S^{-1}I$ or $S^{-1}M/S^{-1}I$ is essential in $S^{-1}R/S^{-1}I$. Then either $S^{-1}R/S^{-1}I$ is semisimple or $S^{-1}M/S^{-1}I \ll_\delta S^{-1}R/S^{-1}I$ by [15, Lemma 1.9]. Hence $S^{-1}R/S^{-1}I$ is a flat δ-cover of $S^{-1}R/S^{-1}M$. □

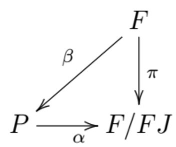
The following result is a consequence of Theorem 1 and Proposition 4.

Corollary 2. *Let R be a commutative G-δ-semiperfect ring. Then for every finite number of maximal ideals $M_1, M_2, \cdots, M_n$ and $S = R \setminus \bigcup_{i=1}^{n} M_i$, $S^{-1}R$ is semiperfect.*

The following result can be given as a consequence of [5, Remark 3.21] and [5, Theorem 4.8]. Here we give a direct proof of this fact.

Theorem 2. *Let R be a semilocal ring. Then R is right perfect if and only if every semisimple R-module has a flat δ-cover.*

Proof. Necessity part is clear, since flat modules are projective over a right perfect ring and a flat cover is also a flat B-cover, hence a flat δ-cover by [14, Theorem 1.2.12] in this case. For sufficiency let F be a free right R-module. Since R/J is semisimple, F/FJ is a semisimple right R-module. By assumption F/FJ has a flat δ-cover $\alpha : P \longrightarrow F/FJ$ for some flat right R-module P. Since F is projective, we have the commutative diagram



where $\pi : F \longrightarrow F/FJ$ is the canonical epimorphism. Since π is an epimorphism, we have $\operatorname{Ker}\alpha + \operatorname{Im}\beta = P$. Since $\operatorname{Ker}\alpha \ll_\delta P$, $\operatorname{Im}\beta$ is a direct summand of P by Lemma 1 and so $\operatorname{Im}\beta$ is flat. Then $\overline{\alpha} : \operatorname{Im}\beta \longrightarrow F/FJ$ induced by α is a flat δ-cover of F/FJ, since $\operatorname{Ker}\alpha \cap \operatorname{Im}\beta \ll_\delta \operatorname{Im}\beta$ by [5, Lemma 2.4]. Since F is projective, $\operatorname{Im}\beta$ is flat and $\operatorname{Ker}\beta \leq \operatorname{Ker}\pi = FJ = \operatorname{Rad}F$, we have $\operatorname{Ker}\beta = 0$ by [12, Exercise 4.20], so $\tilde{\beta} : F \longrightarrow \operatorname{Im}\beta$ induced by β is an isomorphism. $\operatorname{Rad}F = FJ = \tilde{\beta}^{-1}(\operatorname{Ker}\alpha \cap \operatorname{Im}\beta) \ll_\delta F$ by [15, Lemma 1.3]. Lemma 2 implies that $\operatorname{Rad}F \ll F$. By [3, Lemma 28.3], J is right T-nilpotent. Hence R is right perfect. □

The following result is a consequence of Theorem 2 and [15, Theorem 3.8].

Corollary 3. *Let R be a semilocal ring. Then the following are equivalent.*

(i) *R is right perfect.*
(ii) *R is right δ-perfect.*
(iii) *R is right G-δ-perfect.*

Example 4. *Let R be a semiperfect ring which is not right perfect. Then by Corollary 3, R is a right G-δ-semiperfect ring which is not right G-δ-perfect.*

3. Flat Strong δ-covers

Flat strong covers of modules are introduced in [9] as flat covers which are also flat B-covers. They are used in uniqueness (up to isomorphism) of flat B-covers under some conditions in the same work. Here we define flat strong δ-covers of modules as a generalization.

Definition 2. *A right R-module M is said to have a flat strong δ-cover if a flat cover $f : F \longrightarrow M$ of M is also a flat δ-cover. In this case, we also say that F is a flat strong δ-cover of M.*

Flat δ-cover of a module need not be unique, in general, as [1, Example 3.1] shows. As a consequence of the example mentioned, one can deduce the following result.

Proposition 5. *Let R be a regular ring and flat δ-covers of modules be unique (up to isomorphism), then R is a right V-ring.*

The property having flat strong δ-covers is not inherited by submodules, in general. The following result demonstrates a special case. Note that a homomorphism $\alpha : F \longrightarrow M$ satisfying the first two conditions in the definition of a flat cover is called a flat precover of M.

Proposition 6. *Let R be a ring such that $\delta(M) = M\delta_r \ll_\delta M$ for every module M. Let $K \leq L$ and L/K be flat. If L has a flat strong δ-cover, then so does K.*

Proof. Let $f : F \longrightarrow L$ be a flat strong δ-cover of L. Following the proof for [14, Lemma 3.1.3] with $P = f^{-1}(K)$, $f' : P \longrightarrow K$ induced by f is a flat precover of K. By [14, Theorem 1.2.7], $P = X \oplus Y$ for submodules X and Y such that ${f'}_{|X} : X \longrightarrow K$ is a flat cover of K and $Y \leq \operatorname{Ker} f' = \operatorname{Ker} f$.

Since $\operatorname{Ker} f \ll_\delta F$, $F/P \cong L/K$ is flat and δ_r is two sided, by [3, Lemma 19.18] we have

$$\operatorname{Ker} f \leq F\delta_r \cap P = P\delta_r \ll_\delta P.$$

Let $W + \operatorname{Ker} {f'}_{|X} = W + (\operatorname{Ker} f \cap X) = X$ for some submodule W of X. Then $\operatorname{Ker} f = \operatorname{Ker} f \cap P = \operatorname{Ker} f \cap (X+Y) = (\operatorname{Ker} f \cap X) + Y$ and $P = X + Y = W + (\operatorname{Ker} f \cap X) + Y = W + \operatorname{Ker} f$. Since $\operatorname{Ker} f \ll_\delta P$, $P = W \oplus U$ for some projective semisimple submodule U of $\operatorname{Ker} f$ by Lemma 1. Then $X = X \cap P = X \cap (W \oplus U) = W \oplus (X \cap U)$ with $X \cap U$ is projective semisimple and contained in $\operatorname{Ker} f'_{|X}$. The use of Lemma 1 once again implies that $\operatorname{Ker} {f'}_{|X} \ll_\delta X$. Hence ${f'}_{|X} : X \longrightarrow K$ is a flat strong δ-cover of K. □

Rings over which flat covers of cyclic modules are projective are introduced in [2] as right A-perfect rings. Right B-perfect rings are defined with the same condition restricted to simple modules in [7].

Proposition 7. *If flat cover of a module M is projective, then flat δ-covers of M are projective.*

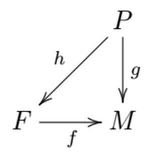
Proof. Let $f : F \longrightarrow M$ be a flat cover of M and $g : P \longrightarrow M$ be a flat δ-cover of M. Since F is projective, there is a homomorphism $h : F \longrightarrow P$ such that $gh = f$. Since $\operatorname{Ker} g + \operatorname{Im} h = P$ and $\operatorname{Ker} g \ll_\delta P$, we have by Lemma 1 that $P = \operatorname{Im} h \oplus Y$ for some projective semisimple module Y. Then $F/\operatorname{Ker} h \cong \operatorname{Im} h$ is flat and $\operatorname{Ker} h \leq \operatorname{Ker} f \ll F$ which implies that $\operatorname{Ker} h = 0$ and $\operatorname{Im} h \cong F$ is projective by [12, Exercise 4.20]. Therefore, $P = \operatorname{Im} h \oplus Y$ is projective. □

Corollary 4. *Over a right A-perfect (right B-perfect resp.) ring, flat δ-covers of cyclic (simple resp.) modules are projective.*

Flat strong covers are used in characterizing right A-perfect rings, right B-perfect rings and right perfect rings in [9]. It turns out that flat strong δ-covers are also related to such rings. We need the following result, which is a generalization of [9, Lemma 3.6], before proceeding.

Lemma 3. *Let M be an R-module. Then flat cover of M is projective if and only if M has a projective cover and a flat strong δ-cover.*

Proof. Necessity part is clear by [14, Theorem 1.2.12]. For sufficiency let $f : F \longrightarrow M$ be a flat strong δ-cover of a right R-module M and $g : P \longrightarrow M$ be a projective cover of M. Since P is projective, we have the commutative diagram



with $\operatorname{Ker} f \ll_\delta F$. Since $\operatorname{Im} h + \operatorname{Ker} f = F$, it follows from Lemma 1 that $F = \operatorname{Im} h \oplus K$ for some projective semisimple submodule K of F. Since P is projective, $\operatorname{Ker} h \leq \operatorname{Ker} g \ll P$ and $P/\operatorname{Ker} h \cong \operatorname{Im} h$ is flat as a direct summand of F, we have $\operatorname{Ker} h = 0$ by [12, Exercise 4.20] and so $\operatorname{Im} h$ is projective. Hence $F = \operatorname{Im} h \oplus K$ is projective. □

Over a right noetherian ring, a flat δ-cover of a cyclic module is also a projective δ-cover by Remark 1. If we assume that R is right noetherian and M is cyclic in the proof of Lemma 3, then projectivity of $\operatorname{Im} h$ follows from [5, Proposition 2.15] in this case. Using these facts, we obtain the following result.

Corollary 5. *Let R be a right noetherian ring and M be a cyclic module. Then flat cover of M is projective if and only if M has a flat strong δ-cover.*

Now we can give characterizations for right A-perfect rings, right B-perfect rings and right perfect rings using flat strong δ-covers, respectively.

Theorem 3. *The following statements are equivalent for a ring R.*

(i) *Flat covers of cyclic modules are projective.*
(ii) *R is semilocal and every cyclic module has a flat strong δ-cover.*

Proof. $(i)\Rightarrow(ii)$: R is semilocal by [2, Theorem 3.7]. If C is a cyclic module and $f : F \longrightarrow C$ is flat cover of C with F projective, then $\operatorname{Ker} f \ll F$ by [14, Theorem 1.2.12]. Then $f : F \longrightarrow C$ is a flat strong cover and hence a flat strong δ-cover of C.

$(ii)\Rightarrow(i)$: Let C be a cyclic module. R is semiperfect by Theorem 1. Therefore C has a projective cover. Then C has a projective cover and a flat strong δ-cover. By Lemma 3, flat cover of C is projective. □

Theorem 4. *The following statements are equivalent for a ring R.*

(i) *Flat covers of simple modules are projective.*
(ii) *R is semilocal and every simple module has a flat strong δ-cover.*

Proof. $(i)\Rightarrow(ii)$: R is semilocal by [7, Theorem 2.4]. If S is a simple module and $f : F \longrightarrow S$ is flat cover of S with F projective, then $\operatorname{Ker} f \ll F$ by [14, Theorem 1.2.12]. Then $f : F \longrightarrow S$ is a flat strong cover and hence a flat strong δ-cover of S.

$(ii)\Rightarrow(i)$: Just let C be simple in the proof for Theorem 3 $ii)\Rightarrow i)$. □

Theorem 5. *The following statements are equivalent for a ring R.*

(i) *R is right perfect.*
(ii) *Flat covers of semisimple modules are projective.*
(iii) *R is semilocal and every semisimple module has a flat strong δ-cover.*
(iv) *Every semisimple module has a flat δ-cover and flat covers of simple modules are projective.*

Proof. Proofs for $(i)\Rightarrow(ii)$ and $(ii)\Rightarrow(iii)$ are given in [9, Theorem 3.9].

$(iii)\Rightarrow(iv)$ is a consequence of Theorem 4.

$(iv)\Rightarrow(i)$: R is semilocal by Theorem 4. Theorem 2 completes the proof. □

Note that when R is right noetherian, then using Corollary 5, the condition for R being semilocal can be dropped in Theorem 3, Theorem 4 and Theorem 5 so that such rings can be characterized as rings whose certain modules have flat strong δ-covers.

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