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ON A NEW CLASS OF s-TYPE OPERATORS

EMRAH EVREN KARA AND MERVE İLKHAN

Abstract. In this paper, we introduce a new class of operators by using s-numbers and the sequence space $Z(u,v;\ell_p)$ for $1<p<\infty$. We prove that this class is a quasi-Banach operator ideal. Also, we give some properties of the quasi-Banach operator ideal. Lastly, we establish some inclusion relations among the operator ideals formed by different s-number sequences.

1. Introduction

By ω, we denote the space of all real-valued sequences. Any vector subspace of ω is called a sequence space. We write ℓ_p for the sequence space of p-absolutely convergent series.

Maddox [6] defined the linear space $\ell(p)$ as follows:

$$\ell(p)=\left\{x\in\omega:\sum_{n=1}^{\infty}|x_n|^{p_n}<\infty\right\},$$

where (p_n) is a bounded sequence of strictly positive real numbers.

Altay and Baar [1] introduced the sequence space $\ell(u,v;p)$ which is the set of all sequences whose generalized weighted mean transforms are in the space $\ell(p)$, that is,

$$\ell(u,v;p)=\left\{x\in\omega:\sum_{n=1}^{\infty}\left|u_n\sum_{k=1}^{n}v_kx_k\right|^{p_n}<\infty\right\},$$

where $u_n,v_k\neq 0$ for all $n,k\in\mathbb{N}$.

If $(p_n)=(p)$, $\ell(u,v;p)=Z(u,v;\ell_p)$ which is defined by Malkowsky and Sava [8] as follows:

$$Z(u,v;\ell_p)=\left\{x\in\omega:\sum_{n=1}^{\infty}\left|u_n\sum_{k=1}^{n}v_kx_k\right|^{p}<\infty\right\},$$

where $1<p<\infty$.

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Cesàro sequence space was defined by Shiue [13] as

$$ces_p = \left\{ x \in \omega : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p < \infty \right\}$$

for $1 < p < \infty$.

In [7], the weighted Cesàro sequence space $ces(p,q)$ is defined as

$$ces(p,q) = \left\{ x \in \omega : \sum_{n=1}^{\infty} \left(\frac{1}{Q_n} \sum_{k=1}^{n} |q_k x_k| \right)^p < \infty \right\},$$

where $q = (q_k)$ is a bounded sequence of positive real numbers, $Q_n = \sum_{k=1}^{n} q_k$ and $1 < p < \infty$.

In the literature, various operator ideals were defined by using sequences of different s-numbers of bounded linear operators. For example, Pietsch [9] defined the class of ℓ_p type operators for $0 < p < \infty$. A bounded linear operator T is in this class if $\sum_{n=1}^{\infty} (a_n(T))^p < \infty$. By using the Cesàro sequence space, Constantin [3] introduced the class of $ces - p$ type operators which satisfy the following condition:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} a_k(T) \right)^p < \infty$$

for $1 < p < \infty$. s-type $ces(p,q)$ operators studied by Maji and Srivastava [7] as a general case of $ces - p$ type operators. A bounded linear operator T is of s-type $ces(p,q)$ operator if

$$\sum_{n=1}^{\infty} \left(\frac{1}{Q_n} \sum_{k=1}^{n} q_k s_k(T) \right)^p < \infty$$

for $1 < p < \infty$.

The main purpose of this paper is to introduce a more general class of s-type operators by using the sequence space $Z(u,v;\ell_p)$. We show that the class of s-type $Z(u,v;\ell_p)$ operators is an operator ideal and a quasi-norm is defined on this class. Moreover, we give some properties and inclusion relations related to the operator ideals formed by different s-number sequences.

2. Preliminaries and Background

Firstly, we give basic notations used throughout this paper. By $\mathcal{B}$, we denote the class of all bounded linear operators between any two Banach spaces. $\mathcal{B}(X,Y)$ is the space of all bounded linear operators from X to Y, where X and Y Banach spaces. X' is composed of continuous linear functionals on X, that is, X' is the dual of X. The map $x' \otimes y : X \to Y$ is defined by $(x' \otimes y)(x) = x'(x)y$, where $x' \in X'$ and $y \in Y$. By $\mathbb{N}$ and $\mathbb{R}^+$, we denote the set of all natural numbers and all nonnegative real nubers, respectively.

Now, we give some definitions and results about s-number sequences and operator ideals.

Definition 2.1. [7] A finite rank operator is a bounded linear operator whose dimension of the range space is finite.

Definition 2.2. [2] A map

$$s : T \to (s_n(T)),$$

which assigns a non-negative scalar sequence to each operator, is called an s-number sequence if for all Banach spaces X, Y, Z and W the following conditions are satisfied:

(i) $\|T\| = s_1(T) \geq s_2(T) \geq ... \geq 0$ for all $T \in \mathcal{B}(X,Y)$.

(ii) $s_{n+m-1}(T+S) \leq s_n(T) + s_m(S)$ for $T, S \in \mathcal{B}(X,Y)$.

(iii)$s_n(RST) \leq \|R\| s_n(S) \|T\|$ for all $R \in \mathcal{B}(Z,W)$, $S \in \mathcal{B}(Y,Z)$, $T \in \mathcal{B}(X,Y)$.

(iv) If rank$(T) < n$, then $s_n(T) = 0$ for all $T \in \mathcal{B}(X,Y)$.

(v) $s_n(I_n) = 1$, where I_n is the identity map of n-dimensional Hilbert space ℓ_2^n to itself.

$s_n(T)$ is called the nth s-number of T.

Let $T \in \mathcal{B}(X,Y)$ and $n \in \mathbb{N}$. $(a_n(T))$, $(c_n(T))$, $(d_n(T))$, $(x_n(T))$, $(y_n(T))$ and $(h_n(T))$ are the sequences of nth approximation number, nth Gel'fand number, nth Kolmogorov number, Weyl number, Chang number and Hilbert number, respectively. These sequences are some examples of s-number sequences of a bounded linear operator. For the definition of these sequences, see [7, 2].

Definition 2.3. [4, p. 440] A subcollection $\mathcal{M}$ of $\mathcal{B}$ is said to be an operator ideal if the following conditions are satisfied:
(OI-1) $x' \otimes y : X \to Y \in \mathcal{M}(X,Y)$ for $x' \in X'$ and $y \in Y$.
(OI-2) $T + S \in \mathcal{M}(X,Y)$ for $T, S \in \mathcal{M}(X,Y)$.
(OI-3) $RST \in \mathcal{M}(X_0,Y_0)$ for $S \in \mathcal{M}(X,Y)$, $T \in \mathcal{M}(X_0,X)$ and $R \in \mathcal{M}(Y,Y_0)$.

Definition 2.4. [10] A function $\alpha : \mathcal{M} \to \mathbb{R}^+$ is said to be a quasi-norm on the operator ideal $\mathcal{M}$ if the following conditions hold:
(QN-1) If $x' \in X'$ and $y \in Y$, then $\alpha(x' \otimes y) = \|x'\| \|y\|$.
(QN-2) If $S, T \in \mathcal{M}(X,Y)$, then there exists a constant $C \geq 1$ such that $\alpha(S+T) \leq C[\alpha(S) + \alpha(T)]$.
(QN-3) If $S \in \mathcal{M}(X,Y)$, $T \in \mathcal{M}(X_0,X)$ and $R \in \mathcal{M}(Y,Y_0)$, then $\alpha(RST) \leq \|R\| \alpha(S) \|T\|$.

In particular if $C = 1$ then α becomes a norm on the operator ideal $\mathcal{M}$.

Let $\mathcal{M}$ be an ideal and α be a quasi-norm on the ideal $\mathcal{M}$. $[\mathcal{M}, \alpha]$ is said to be a quasi-Banach operator ideal if each $\mathcal{M}(X,Y)$ is complete under the quasi-norm α.

Lemma 2.1. [5] *If Y is a Banach space, then $\mathcal{B}(X,Y)$ is a Banach space.*

Lemma 2.2. [11] *Let $T, S \in \mathcal{B}(X,Y)$. Then $|s_n(T) - s_n(S)| \leq \|T - S\|$ for $n = 1, 2, ...$*

Definition 2.5. [12, p. 90] An s-number sequence $s = (s_n)$ is called injective if, given any metric injection $I \in \mathcal{B}(Y,Y_0)$, $s_n(T) = s_n(IT)$ for all $T \in \mathcal{B}(X,Y)$.

A quasi-normed operator ideal $[\mathcal{M}, \alpha]$ is called injective if $T \in \mathcal{M}(X,Y)$ and $\alpha(IT) = \alpha(T)$ as $IT \in \mathcal{M}(X,Y_0)$, where $T \in \mathcal{B}(X,Y)$ and $I \in \mathcal{B}(Y,Y_0)$ is a metric injection.

Definition 2.6. [12, p. 95] An s-number sequence $s = (s_n)$ is called surjective if, given any metric surjection $S \in \mathcal{B}(X_0,X)$, $s_n(T) = s_n(TS)$ for all $T \in \mathcal{B}(X,Y)$.

A quasi-normed operator ideal $[\mathcal{M}, \alpha]$ is called surjective if $T \in \mathcal{M}(X,Y)$ and $\alpha(TS) = \alpha(T)$ as $TS \in \mathcal{M}(X_0,Y)$, where $T \in \mathcal{B}(X,Y)$ and $S \in \mathcal{B}(X_0,X)$ is a metric surjection.

Definition 2.7. [10, p. 152] Let T' be the dual of T. An s-number sequence is called symmetric if $s(T) \geq s_n(T')$ for all $T \in \mathcal{B}$. If $s(T) = s_n(T')$ then the s-number sequence is said to be completely symmetric.

Definition 2.8. [10] For every operator ideal $\mathcal{M}$, the dual operator ideal denoted by $\mathcal{M}'$ is defined as

$$\mathcal{M}'(X,Y) = T \in \mathcal{B}(X,Y) : T' \in \mathcal{M}'(Y', X'),$$

where T' is the dual of T, X' and Y' are duals of X and Y, respectively.

Definition 2.9. [10, p. 68] An operator ideal $\mathcal{M}$ is called symmetric if $\mathcal{M} \subset \mathcal{M}'$. If $\mathcal{M} = \mathcal{M}'$, the operator ideal $\mathcal{M}$ is called completely symmetric.

3. s-type $Z(u, v; \ell_p)$ operators

Let $u = (u_n)$ and $v = (v_n)$ be sequences of positive real numbers. An operator $T \in \mathcal{B}(X,Y)$ is in the class of s-type $Z(u, v; \ell_p)$ if

$$\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(T) \right)^p < \infty, \qquad 1 < p < \infty.$$

The class of all s-type $Z(u, v; \ell_p)$ operators is denoted by $\mathcal{G}_p^{(s)}$.

If $u_n = \frac{1}{Q_n}$ and $v_k = q_k$ are taken for all $n, k \in \mathbb{N}$, then the class of s-type $Z(u, v; \ell_p)$ operators reduces to the class of s-type $ces(p, q)$ operators.

Theorem 3.1. *Let $v = (v_k)$ be a sequence of positive numbers such that*

$$v_{2k-1} + v_{2k} \leq M v_k \quad \text{for all } k = 1, 2, ..., \tag{3.1}$$

where $M > 0$. If $\sum_{n=1}^{\infty} (u_n)^p < \infty$, then the class $\mathcal{G}_p^{(s)}$ is an operator ideal for $1 < p < \infty$.

Proof. Let X and Y be any two Banach spaces and $1 < p < \infty$. For $x' \in X'$ and $y \in Y$, the rank of the operator $x' \otimes y$ is one which means $s_n(x' \otimes y) = 0$ for all $n \geq 2$. We obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(x' \otimes y) \right)^p &= \sum_{n=1}^{\infty} \left(u_n v_1 s_1(x' \otimes y) \right)^p \\ &= \left(v_1 s_1(x' \otimes y) \right)^p \sum_{n=1}^{\infty} (u_n)^p < \infty. \end{aligned}$$

Hence $x' \otimes y \in \mathcal{G}_p^{(s)}(X,Y)$.

Let $T, S \in \mathcal{G}_p^{(s)}(X,Y)$. Then

$$\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(T) \right)^p < \infty, \quad \sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(S) \right)^p < \infty.$$

By using the inequality (3.1) with monotonicity and additivity of s-number sequence,

$$\begin{aligned}
\sum_{k=1}^{n} v_k s_k(T+S) &= \sum_{k=1}^{n} v_{2k-1} s_{2k-1}(T+S) + \sum_{k=1}^{n} v_{2k} s_{2k}(T+S) \\
&\leq \sum_{k=1}^{n} (v_{2k-1} + v_{2k}) s_{2k-1}(T+S) \\
&\leq M \sum_{k=1}^{n} v_k s_{2k-1}(T+S) \\
&\leq M \left(\sum_{k=1}^{n} v_k s_k(T) + \sum_{k=1}^{n} v_k s_k(S) \right).
\end{aligned}$$

From Minkowsky inequality, we have

$$\begin{aligned}
\left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(T+S) \right)^p \right)^{1/p} &\leq M \left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(T) + u_n \sum_{k=1}^{n} v_k s_k(S) \right)^p \right)^{1/p} \\
&\leq M \left[\left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(T) \right)^p \right)^{1/p} + \left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(S) \right)^p \right)^{1/p} \right].
\end{aligned}$$

Thus $T+S \in \mathcal{G}_p^{(s)}(X,Y)$.

Let $S \in \mathcal{G}_p^{(s)}(X,Y)$, $T \in \mathcal{G}_p^{(s)}(X_0,X)$ and $R \in \mathcal{G}_p^{(s)}(Y,Y_0)$. Since s-number sequence has ideal property, we obtain that

$$\left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(RST) \right)^p \right)^{1/p} \leq \|R\|.\|T\|. \left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(S) \right)^p \right)^{1/p} < \infty.$$

Hence $RST \in \mathcal{G}_p^{(s)}(X_0,Y_0)$.

We have proved that the class $\mathcal{G}_p^{(s)}$ satisfied the conditions (OI-1) to (OI-3) and so $\mathcal{G}_p^{(s)}$ is an operator ideal. □

Proposition 3.1. *The inclusion $\mathcal{G}_p^{(s)} \subseteq \mathcal{G}_q^{(s)}$ holds for $1 < p \leq q < \infty$.*

Proof. Since $\ell_p \subseteq \ell_q$ for $1 < p \leq q < \infty$, we have $\mathcal{G}_p^{(s)} \subseteq \mathcal{G}_q^{(s)}$. □

Now, let $\mathcal{G}_p^{(s)}$ be an operator ideal. Define the maps $\Gamma_p^{(s)} : \mathcal{G}_p^{(s)} \to \mathbb{R}^+$ and $\widehat{\Gamma}_p^{(s)} : \mathcal{G}_p^{(s)} \to \mathbb{R}^+$ for $1 < p < \infty$ by

$$\Gamma_p^{(s)}(T) = \left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(T) \right)^p \right)^{1/p} \text{ and } \widehat{\Gamma}_p^{(s)}(T) = \frac{\Gamma_p^{(s)}(T)}{\left(\sum_{n=1}^{\infty} (v_1 u_n)^p\right)^{1/p}}.$$

Theorem 3.2. *Let $v = (v_k)$ be a sequence of positive numbers satisfying inequality (3.1). If $\sum_{n=1}^{\infty} (u_n)^p < \infty$, then the function $\widehat{\Gamma}_p^{(s)}$ is a quasi-norm on $\mathcal{G}_p^{(s)}$.*

Proof. Let X and Y be two Banach spaces. Then $x'\otimes y : X \to Y$ is a rank one operator, that is, $s_n(x'\otimes y)=0$ for all $n\geq 2$. Hence, we have

$$\begin{aligned}\Gamma_p^{(s)}(x'\otimes y) &= \left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n} v_k s_k(x'\otimes y)\right)^p\right)^{1/p}\\ &= \left(\sum_{n=1}^{\infty}(u_n v_1 s_1(x'\otimes y))^p\right)^{1/p}\\ &= \|x'\otimes y\|\left(\sum_{n=1}^{\infty}(v_1u_n)^p\right)^{1/p}.\end{aligned}$$

Since $\sup_{\|x\|=1}\|x'\otimes y\| = \sup_{\|x\|=1}\|x'(x)y\| = \|y\|\sup_{\|x\|=1}|x'(x)| = \|x'\|\|y\|$, we have

$$\widehat{\Gamma}_p^{(s)}(x'\otimes y)=\|x'\|\|y\|.$$

Since the following inequality holds

$$\left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n} v_k s_k(T+S)\right)^p\right)^{1/p} \leq M\left[\left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n} v_k s_k(T)\right)^p\right)^{1/p}+\left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n} v_k s_k(S)\right)^p\right)^{1/p}\right],$$

that is, $\Gamma_p^{(s)}(T+S)\leq M\left[\Gamma_p^{(s)}(T)+\Gamma_p^{(s)}(S)\right]$ for $T,S\in\mathcal{G}_p^{(s)}(X,Y)$, we have

$$\begin{aligned}\widehat{\Gamma}_p^{(s)}(T+S) &= \frac{\Gamma_p^{(s)}(T+S)}{\left(\sum_{n=1}^{\infty}(v_1u_n)^p\right)^{1/p}}\\ &\leq M\frac{\left[\Gamma_p^{(s)}(T)+\Gamma_p^{(s)}(S)\right]}{\left(\sum_{n=1}^{\infty}(v_1u_n)^p\right)^{1/p}}\\ &= M\left[\widehat{\Gamma}_p^{(s)}(T)+\widehat{\Gamma}_p^{(s)}(S)\right].\end{aligned}$$

Let $S\in\mathcal{G}_p^{(s)}(X,Y)$, $T\in\mathcal{G}_p^{(s)}(X_0,X)$ and $R\in\mathcal{G}_p^{(s)}(Y,Y_0)$. Then, we have

$$\begin{aligned}\Gamma_p^{(s)}(RST) &= \left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n} v_k s_k(RST)\right)^p\right)^{1/p}\\ &\leq \|R\|\|T\|\left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n} v_k s_k(S)\right)^p\right)^{1/p}\\ &= \|R\|\|T\|\Gamma_p^{(s)}(S).\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\widehat{\Gamma}_p^{(s)}(RST) &= \frac{\Gamma_p^{(s)}(RST)}{\left(\sum_{n=1}^{\infty}(v_1u_n)^p\right)^{1/p}}\\ &\leq \frac{\|R\|\|T\|\Gamma_p^{(s)}(S)}{\left(\sum_{n=1}^{\infty}(v_1u_n)^p\right)^{1/p}} = \|R\|\|T\|\widehat{\Gamma}_p^{(s)}(S).\end{aligned}$$

Consequently, $\widehat{\Gamma}_p^{(s)}$ is a quasi-norm on $\mathcal{G}_p^{(s)}$. $\square$

Theorem 3.3. *Let $1 < p < \infty$. $\left[\mathcal{G}_p^{(s)}, \widehat{\Gamma}_p^{(s)}\right]$ is a quasi-Banach operator ideal.*

Proof. Let X and Y be any two Banach spaces and $1 < p < \infty$. The following inequality holds

$$\begin{aligned}\Gamma_p^{(s)}(T) &= \left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_k s_k(T)\right)^p\right)^{1/p} \\ &\geq \left(\sum_{n=1}^{\infty}\left(u_n v_1 s_1(T)\right)^p\right)^{1/p} = \|T\|\left(\sum_{n=1}^{\infty}\left(v_1 u_n\right)^p\right)^{1/p}\end{aligned}$$

for $T \in \mathcal{G}_p^{(s)}(X,Y)$. Hence, we have

$$\|T\| \leq \widehat{\Gamma}_p^{(s)}(T). \tag{3.2}$$

Let (T_m) be a Cauchy sequence in $\mathcal{G}_p^{(s)}(X,Y)$. Then for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\widehat{\Gamma}_p^{(s)}(T_m - T_l) < \varepsilon \tag{3.3}$$

for $\forall m, l \geq n_0$. It follows that

$$\|T_m - T_l\| \leq \widehat{\Gamma}_p^{(s)}(T_m - T_l) < \varepsilon$$

from the inequality (3.2). Then (T_m) is a Cauchy sequence in $\mathcal{B}(X,Y)$. According to Lemma 2.1, $\mathcal{B}(X,Y)$ is a Banach space since Y is a Banach space. Therefore $\| T_m - T \| \to 0$ as $m \to \infty$ for $T \in \mathcal{B}(X,Y)$. Now, we show that $\widehat{\Gamma}_p^{(s)}(T_m - T) \to 0$ as $m \to \infty$ for $T \in \mathcal{G}_p^{(s)}(X,Y)$.

The operators $T_l - T_m$, $T - T_m$ are in the class $\mathcal{B}(X,Y)$ for $T_m, T_l, T \in \mathcal{B}(X,Y)$. From Lemma 2.2, we have

$$\begin{aligned}|s_n(T_l - T_m) - s_n(T - T_m)| &\leq \|T_l - T_m - (T - T_m)\| \\ &= \|T_l - T\|.\end{aligned}$$

Since $T_l \to T$ as $l \to \infty$, that is $\|T_l - T\| < \varepsilon$, we obtain

$$s_n(T_l - T_m) \to s_n(T - T_m) \text{ as } l \to \infty. \tag{3.4}$$

It follows from (3.3) that the statement

$$\begin{aligned}\widehat{\Gamma}_p^{(s)}(T_m - T_l) &= \frac{\Gamma_p^{(s)}(T_m - T_l)}{\left(\sum_{n=1}^{\infty}(v_1 u_n)^p\right)^{1/p}} \\ &= \frac{\left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_k s_k(T_m - T_l)\right)^p\right)^{1/p}}{\left(\sum_{n=1}^{\infty}(v_1 u_n)^p\right)^{1/p}} < \varepsilon\end{aligned}$$

holds for $\forall m, l \geq n_0$. Then,

$$\left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_k s_k(T_m - T_l)\right)^p\right)^{1/p} < \varepsilon\left(\sum_{n=1}^{\infty}(v_1 u_n)^p\right)^{1/p}$$

for $\forall m, l \geq n_0$. We obtain from (3.4) that

$$\left(\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_k s_k(T_m - T)\right)^p\right)^{1/p} < \varepsilon\left(\sum_{n=1}^{\infty}(v_1 u_n)^p\right)^{1/p}$$

as $l \to \infty$ for $\forall m \geq n_0$. Hence, we have $\widehat{\Gamma}_p^{(s)}(T_m - T) < \varepsilon$ for $\forall m \geq n_0$.

Finally, we show that $T \in \mathcal{G}_p^{(s)}(X,Y)$. From the inequality (3.4) and conditions (i), (ii) of Definition 2.2, we obtain

$$\begin{aligned}
\sum_{k=1}^{n} v_k s_k(T) &= \sum_{k=1}^{n} v_{2k-1} s_{2k-1}(T) + \sum_{k=1}^{n} v_{2k} s_{2k}(T) \\
&\leq \sum_{k=1}^{n} (v_{2k-1} + v_{2k}) s_{2k-1}(T) \\
&\leq M \sum_{k=1}^{n} v_k s_{2k-1}(T) \\
&= M \sum_{k=1}^{n} v_k s_{k+k-1}(T - T_m + T_m) \\
&\leq M \left[\sum_{k=1}^{n} v_k s_k(T - T_m) + \sum_{k=1}^{n} v_k s_k(T_m) \right].
\end{aligned}$$

By using Minkowsky inequality, since $T_m \in \mathcal{G}_p^{(s)}(X,Y)$ for all m and $\widehat{\Gamma}_p^{(s)}(T_m - T) \to \infty$ as $m \to \infty$, we have

$$\begin{aligned}
\left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(T) \right)^p \right)^{1/p} &\leq M \left[\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(T - T_m) + u_n \sum_{k=1}^{n} v_k s_k(T_m) \right)^p \right]^{1/p} \\
&\leq M \left[\left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(T - T_m) \right)^p \right)^{1/p} + \left(\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(T_m) \right)^p \right)^{1/p} \right] \\
&< \infty
\end{aligned}$$

which means $T \in \mathcal{G}_p^{(s)}(X,Y)$. □

Let $\left[\mathcal{G}_p^{(a)}, \widehat{\Gamma}_p^{(a)}\right]$, $\left[\mathcal{G}_p^{(c)}, \widehat{\Gamma}_p^{(c)}\right]$, $\left[\mathcal{G}_p^{(d)}, \widehat{\Gamma}_p^{(d)}\right]$, $\left[\mathcal{G}_p^{(x)}, \widehat{\Gamma}_p^{(x)}\right]$, $\left[\mathcal{G}_p^{(y)}, \widehat{\Gamma}_p^{(y)}\right]$ and $\left[\mathcal{G}_p^{(h)}, \widehat{\Gamma}_p^{(h)}\right]$ be the quasi-Banach operator ideals corresponding to the approximation numbers $a = (a_n)$, Gel'fand numbers $c = (c_n)$, Kolmogorov numbers $d = (d_n)$, Weyl numbers $x = (x_n)$, Chang numbers $y = (y_n)$ and Hilbert numbers $h = (h_n)$, respectively.

Theorem 3.4. *If s-number sequence is injective, then the quasi-Banach operator ideal* $\left[\mathcal{G}_p^{(s)}, \widehat{\Gamma}_p^{(s)}\right]$ *is injective for* $1 < p < \infty$.

Proof. Let $T \in \mathcal{B}(X,Y)$ and $I \in \mathcal{B}(Y,Y_0)$ be any metric injections. If $IT \in \mathcal{G}_p^{(s)}(X,Y_0)$, then

$$\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(IT) \right)^p < \infty.$$

Since $s = (s_n)$ is injective, we have $s_n(T) = s_n(IT)$ for all $T \in \mathcal{B}(X,Y)$. Thus, we obtain

$$\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(T) \right)^p = \sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} v_k s_k(IT) \right)^p < \infty,$$

that is, $T \in \mathcal{G}_p^{(s)}(X,Y)$. Clearly, we have $\widehat{\Gamma}_p^{(s)}(IT) = \widehat{\Gamma}_p^{(s)}(T)$. □

Conclusion 3.1. *The quasi-Banach operator ideals* $\left[\mathcal{G}_p^{(c)}, \widehat{\Gamma}_p^{(c)}\right]$ *and* $\left[\mathcal{G}_p^{(x)}, \widehat{\Gamma}_p^{(x)}\right]$ *are injective since the Gel'fand numbers and the Weyl numbers are injective (See* [12, p. 90-94]*).*

Theorem 3.5. *If s-number sequence is surjective, then the quasi-Banach operator ideal* $\left[\mathcal{G}_p^{(s)}, \widehat{\Gamma}_p^{(s)}\right]$ *is surjective for* $1 < p < \infty$*.*

Proof. Let $T \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(X_0, X)$ be any metric surjections. If $TS \in \mathcal{G}_p^{(s)}(X_0, Y)$, then

$$\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n} v_k s_k(TS)\right)^p < \infty.$$

Since $s = (s_n)$ is surjective, we have $s_n(T) = s_n(TS)$ for all $T \in \mathcal{B}(X, Y)$. Thus, we obtain

$$\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n} v_k s_k(T)\right)^p = \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n} v_k s_k(TS)\right)^p < \infty,$$

that is, $T \in \mathcal{G}_p^{(s)}(X, Y)$. Clearly, we have $\widehat{\Gamma}_p^{(s)}(TS) = \widehat{\Gamma}_p^{(s)}(T)$. □

Conclusion 3.2. *The quasi-Banach operator ideals* $\left[\mathcal{G}_p^{(d)}, \widehat{\Gamma}_p^{(d)}\right]$ *and* $\left[\mathcal{G}_p^{(y)}, \widehat{\Gamma}_p^{(y)}\right]$ *are surjective since the Kolmogorov numbers and the Chang numbers are surjective (See* [12, p. 95]*).*

Now, we give some inclusion relations among the operator ideals $\mathcal{G}_p^{(a)}$, $\mathcal{G}_p^{(c)}$, $\mathcal{G}_p^{(d)}$, $\mathcal{G}_p^{(x)}$, $\mathcal{G}_p^{(y)}$ and $\mathcal{G}_p^{(h)}$.

Theorem 3.6. *The following inclusion relations*

(i) $\mathcal{G}_p^{(a)} \subseteq \mathcal{G}_p^{(c)} \subseteq \mathcal{G}_p^{(x)} \subseteq \mathcal{G}_p^{(h)}$,
(ii) $\mathcal{G}_p^{(a)} \subseteq \mathcal{G}_p^{(d)} \subseteq \mathcal{G}_p^{(y)} \subseteq \mathcal{G}_p^{(h)}$

hold for $1 < p < \infty$*.*

Proof. Let $T \in \mathcal{G}_p^{(a)}$. Then

$$\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n} v_k a_k(T)\right)^p < \infty,$$

where $1 < p < \infty$. It follows from [12, p. 115] that $h_n(T) \leq x_n(T) \leq c_n(T) \leq a_n(T)$ and $h_n(T) \leq y_n(T) \leq d_n(T) \leq a_n(T)$. Hence, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n} v_k h_k(T)\right)^p &\leq \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n} v_k x_k(T)\right)^p \\ &\leq \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n} v_k c_k(T)\right)^p \\ &\leq \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n} v_k a_k(T)\right)^p < \infty\end{aligned}$$

and

$$\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_kh_k(T)\right)^p \le \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_ky_k(T)\right)^p$$
$$\le \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_kv_k(T)\right)^p$$
$$\le \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_ka_k(T)\right)^p < \infty.$$

Thus, the inclusions are clear. □

Theorem 3.7. *Let $1 < p < \infty$. The operator ideal $\mathcal{G}_p^{(a)}$ is symmetric and the operator ideal $\mathcal{G}_p^{(h)}$ is completely symmetric.*

Proof. Let $1 < p < \infty$.

Firstly, we prove that the inclusion $\mathcal{G}_p^{(a)} \subseteq \left(\mathcal{G}_p^{(a)}\right)'$ holds. Let $T \in \mathcal{G}_p^{(a)}$. Then

$$\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_ka_k(T)\right)^p < \infty.$$

It follows from [10, p. 152] that $a_n(T') \le a_n(T)$ for $T \in \mathcal{B}$. Hence, we have

$$\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_ka_k(T')\right)^p \le \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_ka_k(T)\right)^p < \infty,$$

that is, $T \in \left(\mathcal{G}_p^{(a)}\right)'$. Thus, $\mathcal{G}_p^{(a)}$ is symmetric.

Now, we prove that the equation $\mathcal{G}_p^{(h)} = \left(\mathcal{G}_p^{(h)}\right)'$ holds. It follows from [12, p. 97] that $h_n(T') = h_n(T)$ for $T \in \mathcal{B}$. Hence, we have

$$\sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}v_kh_k(T')\right)^p = \sum_{n=1}^{\infty}\left(u_n\sum_{k=1}^{n}h_ka_k(T)\right)^p.$$

Thus, $\mathcal{G}_p^{(h)}$ is completely symmetric. □

Theorem 3.8. *Let $1 < p < \infty$. The equation $\mathcal{G}_p^{(c)} = \left(\mathcal{G}_p^{(d)}\right)'$ and the inclusion $\mathcal{G}_p^{(d)} \subseteq \left(\mathcal{G}_p^{(c)}\right)'$ hold. Also, the equation $\mathcal{G}_p^{(d)} = \left(\mathcal{G}_p^{(c)}\right)'$ holds for any compact operators.*

Proof. Let $1 < p < \infty$. We have from [12, p. 95] that $c_n(T) = d_n(T')$ and $c_n(T') \le d_n(T)$ for $T \in \mathcal{B}$. Also, the equality $c_n(T') = d_n(T)$ holds, where T is a compact operator. Thus the proof is clear. □

Theorem 3.9. *Let $1 < p < \infty$. The equations $\mathcal{G}_p^{(x)} = \left(\mathcal{G}_p^{(y)}\right)'$ and $\mathcal{G}_p^{(y)} = \left(\mathcal{G}_p^{(x)}\right)'$ hold.*

Proof. Let $1 < p < \infty$. We have from [12, p. 96] that $x_n(T) = y_n(T')$ and $y_n(T) = x_n(T')$ for $T \in \mathcal{B}$. Thus the proof is clear. □

References

[1] B. Altay and F. Başar, Generalization of the sequence space $\ell(p)$ derived by weighted mean, J. Math. Anal. Appl. Vol:330 (2007), 174–185.

[2] J. Burgoyne, Denseness of the generalized eigenvectors of a discrete operator in a Banach space, J. Operator Theory Vol:33 (1995), 279–297.

[3] Gh. Constantin, Operators of $ces-p$ type, Rend. Acc. Naz. Lincei. Vol:52 No.8 (1972), 875–878.

[4] W. B. Johnson and J. Lindenstrauss, *Handbook of the geometry of banach spaces*, Elsevier Science B. V., North-Holand, 2001.

[5] E. Kreyszig, *Introductory functional analysis with applications*, John Wiley Sons. Inc., The United States of America, 1978.

[6] I. J. Maddox, Spaces of strongly summable sequences, Quart. J. Math. Oxforf Vol:18 No.2 (1967), 345–355.

[7] A. Maji and P. D. Srivastava, On operator ideals using weighted Cesàro sequence space, Egyp. Math. Soc., Vol:22 (2014), 446–452.

[8] E. Malkowsky and E. Sava, Matrix transformations between sequence spaces of generalized weighted means, Appl. Math. Comp. Vol:147 (2004), 333–345.

[9] A. Pietsch, Einige neue klassen von kompakten linearen Abbildungen, Rev. Math. Pures Appl. Vol:8 (1963), 427–447.

[10] A. Pietsch, *Operator Ideals*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[11] A. Pietsch, s-numbers of operators in Banach spaces, Studia Math. Vol:51 (1974), 201–223.

[12] A. Pietsch, *Eigenvalues and s-numbers*, Cambridge University Press, New York, 1986.

[13] J. S. Shiue, On the Cesàro sequence spaces, Tamkang J. Math. Vol:1 No.1 (1970), 19–25.

Duzce University, Science and Art Faculty, Department of Mathematics, Duzce-TURKEY

E-mail address: `eevrenkara@hotmail.com`

Duzce University, Science and Art Faculty, Department of Mathematics, Duzce-TURKEY

E-mail address: `merveilkhan@gmail.com`