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A GENERALIZED VERSION OF FOSTER AND STUART'S d-STATISTIC

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Abstract. Assume that only the lists of upper k-records and lower k-records of a finite sequence are available and the existence of a monotonic trend in location is interested in. In this study, a distribution-free test based on the difference between the numbers of upper and lower k-records is proposed for this situation. The exact and asymptotic distributions of the proposed test statistic are obtained for a random continuous sequence which is independent and identically distributed (i.i.d.). Also, a comparison between the proposed test and some well-known distribution-free tests is made in terms of empirical powers.

1. Introduction

Statistical detection of a monotonic upward or downward trend in location over time is a crucial subject in many applications. For example, in meteorology, relating to global warming, it is inevitable and important to ask whether the mean annual temperatures have been rising over a long-time period. Similar questions related to the monotonic changes can also be found in many other fields. Considering the problems of a monotonic trend detection; in general, two groups of methods, i.e., parametric and nonparametric, are discussed in the literature. Parametric methods are more powerful in detecting trends compare to nonparametric methods if observations come from a normal distribution. However, when the distributional assumption of normality fails to be the case, it is statistically appropriate to use a nonparametric method. For this reason, various distribution-free tests related to this issue were proposed by many authors such as Wallis and Moore [24], Moore and Wallis [16], Wald and Wolfowitz [23], Mann [15], Daniels [6], Foster and Stuart [9], Cox and Stuart [5], Aiyer et. al. [2], Diersen and Trenkler [7], and Hofmann and Balakrishnan [10]. If each observation is recorded one by one in time, one of the

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rank-correlation tests like Mann-Kendall [15], Spearman's rho [6], and Hofmann-Balakrishnan [10] can be used for the trend detection due to their high power under variety of distributions (see [10], [21], and [22]). On the other hand, sometimes, only extreme values, i.e., record values (see [4]), are sequentially recorded and taken into consideration in many areas like hydrology, meteorology, epidemiology, and sports (see for details; [1], [3], [12], [17], [18], and [20]). In the case of such data, the record based distribution-free tests are uniquely appropriate in order to detect a monotonic trend in location. First examples of this kind of tests were proposed by Foster and Stuart [9]. In the more recent literature, Diersen and Trenkler [7] proposed more powerful versions of these tests.

In some applications like insurance claims in non-life insurance, kth-records (kth largest or smallest values) rather than records among a sequence may be considered (see [11] and [19]). The distributional theory of kth-records was first introduced by Dziubdziela and Kopocinski [8]. There are many papers on the kth-records in the literature but it appears that none of them has been related to the nonparametric trend detection. For this reason, in this study, a nonparametric test based on the difference between the numbers of upper and lower kth-records in a finite continuous sequence will be proposed as a generalized version of the d-test in Foster and Stuart [9]. This generalized version can be thought to be suitable in some situations. For example, in sports, generally, only the k best and the k worst scores among all the performances are instantly reported at the time when one of the k lists is updated. Note that such k lists are called "bottom-k-list" and "top-k-list" (see [13] and [14]). Sometimes, the changing numbers of both lists and the total number of the performances from the beginning may be accessible even if all of the performances are not available. Such situations also exist in some other fields like meteorology and hydrology. In this context, the proposed distribution-free test will be uniquely appropriate for detecting a monotonic increasing or decreasing trend in location.

The paper is organized as follows: The proposed test statistic is defined and its exact distribution is derived for i.i.d. case in Section 2. In the following section, the asymptotic distribution of the test statistic is also obtained. An illustrative example is given in Section 4. In the last section, the comparative results of the proposed test against Mann-Kendall's and Foster and Stuart's tests are presented via Monte-Carlo simulations.

2. The Proposed Test Statistic and Its Exact Distribution

Let $X_1, X_2, ..., X_n$ be independent continuous random variables with distribution functions $F_1, F_2, ..., F_n$, respectively. The proposed test statistic is defined as

$$T_{k,m} = \sum_{r=1}^{k} \xi_{k,r} + \sum_{r=k+1}^{m} \eta_{k,r}$$

where $\sum_{\varnothing} = 0$, $1 \leq k < n$, $m = n - k$,

$$\xi_{k,r} = \begin{cases} -1 & \text{if the } (k+r)\text{th observation changes only current lower } k\text{th-record,} \\ & \text{i.e., } X_{k+r} < X_{r:k+r-1} \\ 1 & \text{if the } (k+r)\text{th observation changes only current upper } k\text{th-record,} \\ & \text{i.e., } X_{k+r} > X_{k:k+r-1} \\ 0 & \text{if the } (k+r)\text{th observation changes both of current upper } k\text{th-record} \\ & \text{and current lower } k\text{th-record, i.e., } X_{r:k+r-1} < X_{k+r} < X_{k:k+r-1}, \end{cases}$$

and

$$\eta_{k,r} = \begin{cases} -1 & \text{if the } (k+r)\text{th observation changes only current lower } k\text{th-record,} \\ & \text{i.e., } X_{k+r} < X_{k:k+r-1} \\ 1 & \text{if the } (k+r)\text{th observation changes only current upper } k\text{th-record,} \\ & \text{i.e., } X_{k+r} > X_{r:k+r-1} \\ 0 & \text{if the } (k+r)\text{th observation changes neither current upper } k\text{th-record} \\ & \text{nor current lower } k\text{th-record, i.e., } X_{k:k+r-1} < X_{k+r} < X_{r:k+r-1}. \end{cases}$$

It is clear from the definition that the statistic of $T_{k,m}$ indicates the difference between the numbers of upper kth-records and lower kth-records in the sequence $X_1, X_2, ..., X_n$. Note that, for $k = 1$, this definition reduces to the definition of the d-statistic in [9].

In i.i.d. case, since every arrangement of independent observations has an equal probability, the probability generating functions of $\xi_{k,r}$ and $\eta_{k,r}$ are obtained as follows:

$$\begin{aligned} E\left(s^{\xi_{k,r}}\right) &= \sum_{i=-1}^{1} P\left(\xi_{k,r} = i\right) s^i \\ &= \frac{r}{k+r} s^{-1} + \frac{(k-r)}{k+r} + \frac{r}{k+r} s \end{aligned} \tag{2.1}$$

for $1 \leq r \leq k$ and

$$\begin{aligned} E\left(s^{\eta_{k,r}}\right) &= \sum_{i=-1}^{1} P\left(\eta_{k,r} = i\right) s^i \\ &= \frac{k}{k+r} s^{-1} + \frac{r-k}{k+r} + \frac{k}{k+r} s \end{aligned} \tag{2.2}$$

for $k < r \leq m$. Under the null hypothesis $H_0 : F_1 = F_2 = ... = F_n$, since the event that the rth observation of the sequence is an upper or lower kth-record is independent of the order among themselves of the preceding observations, one can write the probability mass function of $T_{k,m}$ as

$$P\left(T_{k,m}=t\right) = \begin{cases} P\left(T_{k,m-1}=t-1\right)P\left(\xi_{k,m}=1\right) \\ +P\left(T_{k,m-1}=t\right)P\left(\xi_{k,m}=0\right) \\ +P\left(T_{k,m-1}=t+1\right)P\left(\xi_{k,m}=-1\right) & ,1\leq m\leq k \\ P\left(T_{k,m-1}=t-1\right)P\left(\eta_{k,m}=1\right) \\ +P\left(T_{k,m-1}=t\right)P\left(\eta_{k,m}=0\right) \\ +P\left(T_{k,m-1}=t+1\right)P\left(\eta_{k,m}=-1\right) & ,1\leq k<m \end{cases}$$

$$= \begin{cases} \frac{P(T_{k,m-1}=t-1)}{\frac{k+m}{m}}+\frac{P(T_{k,m-1}=t)}{\frac{k+m}{k-m}}+\frac{P(T_{k,m-1}=t+1)}{\frac{k+m}{m}} & ,1\leq m\leq k \\ \frac{P(T_{k,m-1}=t-1)}{\frac{k+m}{k}}+\frac{P(T_{k,m-1}=t)}{\frac{k+m}{m-k}}+\frac{P(T_{k,m-1}=t+1)}{\frac{k+m}{k}} & ,1\leq k<m \end{cases}$$

where $t\in\{-m,-m+1,...,m\}$ and $P\left(T_{k,0}=0\right)=1$. In addition, the probability generating function of $T_{k,m}$ can be obtained using (2.1) and (2.2) as follows:

$$E\left(s^{T_{k,m}}\right)=\begin{cases} \frac{k!\prod\limits_{r=1}^{m}\left(rs^{-1}+k-r+rs\right)}{(k+m)!} & ,\text{for } 1\leq m\leq k \\ \frac{k!\prod\limits_{r=1}^{k}\left(rs^{-1}+k-r+rs\right)\prod\limits_{r=k+1}^{m}\left(ks^{-1}+r-k+ks\right)}{(k+m)!} & ,\text{for } 1\leq k<m. \end{cases} \tag{2.3}$$

Note that the probability of $\{T_{k,m}=t\}$ is the coefficient of s^{t+m} in $s^mE\left(s^{T_{k,m}}\right)$. Furthermore, substituting $s=e^{iu}$ in (2.3), the following characteristic function of $T_{k,m}$ is obtained as

$$\varphi_{T_{k,m}}(u)=\begin{cases} \frac{k!\prod\limits_{r=1}^{m}(k-r+2r\cos u)}{(k+m)!} & ,\text{for } 1\leq m\leq k \\ \frac{k!\prod\limits_{r=1}^{k}(k-r+2r\cos u)\prod\limits_{r=k+1}^{m}(r-k+2k\cos u)}{(k+m)!} & ,\text{for } 1\leq k<m. \end{cases} \tag{2.4}$$

Thanks to the characteristic function in (2.4), the following first three cumulants are derived:

$$\mu=E\left(T_{k,m}\right)=\left[\frac{\partial\log\varphi_{T_{k,m}}(u)}{\partial(iu)}\right]_{u=0}=0,$$

$$\sigma^2_{k,m}=E\left(T^2_{k,m}\right)=\left[\frac{\partial^2\log\varphi_{T_{k,m}}(u)}{\partial(iu)^2}\right]_{u=0}$$

$$=\begin{cases} 2k\left(\frac{m}{k}-\sum\limits_{r=k+1}^{k+m}\frac{1}{r}\right) & ,\text{for } 1\leq m\leq k \\ 2k\left(\frac{2-2k}{3k+1.5}+\sum\limits_{r=2}^{k+m}\frac{1}{r}-\sum\limits_{r=2}^{k}\frac{1}{r+0.5}\right) & ,\text{for } 1\leq k<m \end{cases}$$

where $\sum_{\varnothing}=0$, and

$$E\left(T_{k,m}^{3}\right)=\left[\frac{\partial^{3}\log\varphi_{T_{k,m}}(u)}{\partial(iu)^{3}}\right]_{u=0}=0.$$

Finally, if there is an increasing trend in $X_1, X_2, ..., X_n$, it is expected that the upper kth-records will be observed more than the lower kth-records. In other words, in such trends, it is expected that the proposed test statistic $T_{k,m}$ will be large enough. Thus, for testing against the existence of a monotonic increasing trend in location, the critical value at α level of significance can be defined as

$$T_{k,m}^{\alpha}=\min\left\{j\in\{-m,-m+1,...,m\}:P\left(T_{k,m}\geq j\right)\leq\alpha\right\}.$$

If there is no such j that $P\left(T_{k,m}\geq j\right)\leq\alpha$, it can not be tested at α level of significance. For $m\leq 5$, $k\leq 20$, and $\alpha=0.05$, the critical values which are derived using (2.3) can be given as in Table 1. Note that this table can also be used for left-tailed and two-tailed trend tests since $T_{k,m}$ is symmetrically distributed.

3. Asymptotic Distribution

One can see that it is difficult to obtain the critical values for large m by using the probability generating function of $T_{k,m}$ in (2.3). For that reason, the asymptotic distribution of $T_{k,m}$ is derived in this section. Let $T'_{k,m}=T_{k,m}/\sigma_{k,m}$. Considering (2.4), the characteristic function of $T'_{k,m}$ for $1\leq k<m$ can be obtained as

$$\begin{aligned}\varphi_{T'_{k,m}}(u) &= \varphi_{T_{k,m}}\left(\frac{u}{\sigma_{k,m}}\right)=\prod_{r=1}^{k}\frac{k-r+2r\cos\frac{u}{\sigma_{k,m}}}{k+r}\\ &\quad\times\prod_{r=k+1}^{m}\frac{r-k+2k\cos\frac{u}{\sigma_{k,m}}}{k+r}.\end{aligned}\tag{3.1}$$

Furthermore, using (3.1), the cumulative function can be derived as

$$\Psi_{T'_{k,m}}(u)=\log\varphi_{T'_{k,m}}(u)=\sum_{r=1}^{k}\log\frac{k-r+2r\cos\frac{u}{\sigma_{k,m}}}{k+r}+\sum_{r=k+1}^{m}\log\frac{r-k+2k\cos\frac{u}{\sigma_{k,m}}}{k+r}.$$

It is clear that since $\sigma_{k,m}^{2}\to\infty$ while $m\to\infty$, $\sum_{r=1}^{k}\log\frac{k-r+2r\cos\frac{u}{\sigma_{k,m}}}{k+r}$ convergences zero. Therefore, one can write

$$\lim_{m\to\infty}\Psi_{T'_{k,m}}(u)=\lim_{m\to\infty}\sum_{r=k+1}^{m}\log\frac{r-k+2k\cos\frac{u}{\sigma_{k,m}}}{k+r}.\tag{3.2}$$

Also, (3.2) can be rewritten as

$$\lim_{m\to\infty}\Psi_{T'_{k,m}}(u)=\lim_{m\to\infty}\sum_{r=k+1}^{m}\log\left(1+\frac{2k}{k+r}v_{k,m}(u)\right)$$

where $v_{k,m}(u)=\cos\frac{u}{\sigma_{k,m}}-1=\frac{u^2i^2}{\sigma_{k,m}^2 2!}+\frac{u^4i^4}{\sigma_{k,m}^4 4!}+\frac{u^6i^6}{\sigma_{k,m}^6 6!}+....$ In addition, using the Taylor expansion, one can write

$$\log\left(1+\frac{2k}{k+r}v_{k,m}(u)\right)=\frac{2k}{k+r}v_{k,m}(u)+\sum_{j=2}^{\infty}\frac{\left(\frac{2k}{k+r}v_{k,m}(u)\right)^j}{j}. \tag{3.3}$$

Considering (3.3), one has

$$\lim_{m\to\infty}\Psi_{T'_{k,m}}(u)=\lim_{m\to\infty}\sum_{r=k+1}^{m}\frac{2k}{k+r}v_{k,m}(u)+\sum_{r=k+1}^{m}\sum_{j=2}^{\infty}\frac{\left(\frac{2k}{k+r}v_{k,m}(u)\right)^j}{j}. \tag{3.4}$$

Since $\sum_{r=k+1}^{m}\sum_{j=2}^{\infty}\frac{\left(\frac{2k}{k+r}v_{k,m}(u)\right)^j}{j}\to 0$ while $m\to\infty$, (3.4) can be rewritten as

$$\begin{aligned}\lim_{m\to\infty}\Psi_{T'_{k,m}}(u)&=\lim_{m\to\infty}\rho_{k,m}v_{k,m}(u)\\&=\lim_{m\to\infty}\rho_{k,m}\left(\frac{u^2i^2}{\sigma_{k,m}^2 2!}+\frac{u^4i^4}{\sigma_{k,m}^4 4!}+\frac{u^6i^6}{\sigma_{k,m}^6 6!}+...\right)\end{aligned}$$

where $\rho_{k,m}=\sum_{r=k+1}^{m}\frac{2k}{k+r}$. While $m\to\infty$, $\rho_{k,m}/\sigma_{k,m}^2\to 1$ and $\rho_{k,m}/\sigma_{k,m}^j\to 0$ for $j>2$. Thus, the following result is obtained as

$$\lim_{m\to\infty}\Psi_{T'_{k,m}}(u)=\frac{-u^2}{2}$$

which indicates that $T'_{k,m}=T_{k,m}/\sigma_{k,m}$ asymptotically follows the standard normal distribution under the null hypothesis. In a similar way, while $m\to\infty$, one can deduce that the asymptotic distribution of $T'_{k,m}$ for $1\le m\le k$ is also standard normal. In practice, the following standardized test statistic with a continuity correction can be used for large m:

$$Z_{k,m}=\begin{cases}\frac{T_{k,m}-0.5}{\sigma_{k,m}} & ,\ \text{for } T_{k,m}>0\\ 0 & ,\ \text{for } T_{k,m}=0\\ \frac{T_{k,m}+0.5}{\sigma_{k,m}} & ,\ \text{for } T_{k,m}<0.\end{cases} \tag{3.5}$$

In order to see the practical usage of this result, the exact and the asymptotic cumulative distribution functions of $T_{k,m}$ are given in Table 2 for $m=6$, $k=1,5,10,15,20$, and nonnegative values of $T_{k,m}$. In this table, some numerical values of the exact and the asymptotic cumulative distributions are derived by using (2.3) and (3.5), respectively. It will be seen that the approximation to the distribution of $T_{k,m}$ is remarkably good at $m = 6$. Therefore, the asymptotic distribution can be used instead of the exact distribution for $m \geq 6$.

4. Illustrative Example

Let the data which represent the amount of annual rainfall in inches at the Los Angeles Civic Center during the 100-year period from 1890 until 1989 (see [3], p.180) be considered. For $k = 1, 2$, and 3, the upper kth-records and lower kth-records extracted from these data can be tabulated as in Table 3. Also, the summary statistics of the monotonic increasing trend tests based on 1th-, 2nd-, and 3rd-records have been presented as in Table 4. From this table, for each one of the tests, it can be statistically said that there is no monotonic increasing trend in location at $\alpha = 0.05$ level of significance.

5. Empirical Power

This section has been motivated by two different ways: (i) to compare empirical powers of the $T_{k,m}$ statistic and some well-known statistics (Mann-Kendall's Q statistic [15] and Foster and Stuart's d and D statistics [9]) if all of the observations are available, and (ii) to give empirical powers of the $T_{k,m}$ statistic for some fixed values of k in the case that only the lists of upper kth-records and lower kth-records of a finite sequence are available.

Let $X_1, X_2, ..., X_n$ be independent continuous random variables with distribution functions $F_1, F_2, ..., F_n$, respectively. Recall that the Mann-Kendall's test statistic is

$$Q = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} I_{i,j}$$

where

$$I_{i,j} = \begin{cases} -1 & , \text{ for } X_i > X_j \\ 1 & , \text{ for } X_i < X_j \\ 0 & , \text{ otherwise.} \end{cases}$$

and the statistics of d and D can also be defined as

$$d = \sum_{i=2}^{n} I_i$$

and

$$D = d - \sum_{j=2}^{n} J_j$$

where

$$I_i = \begin{cases} -1 & , \text{ for } X_i < X_{1:i-1} \\ 1 & , \text{ for } X_i > X_{i-1:i-1} \\ 0 & , \text{ otherwise} \end{cases}$$

$$J_j = \begin{cases} -1 & , \text{ for } Y_i < Y_{1:i-1} \\ 1 & , \text{ for } Y_i > Y_{i-1:i-1} \\ 0 & , \text{ otherwise,} \end{cases}$$

and $Y_i = X_{n+1-i}$ for $i = 1, 2, ..., n$.

In this empirical study, we have restricted the alternative hypothesis to H_1 : $F_1 > F_2 > ... > F_n$. Also, we have selected the increasing trend model as

$$X_i = U_i + \theta t(i), \qquad i = 1, 2, 3, ..., n$$

where $\theta > 0$, $U_1, U_2, ..., U_n$ are i.i.d. random variables, and $t(i)$ is a strictly increasing trend function. In addition, the trend function has been determined as $t(i) = i$ for linear trend, $t(i) = \sqrt{2ni}$ for concave trend, and $t(i) = \frac{i^2}{2n}$ for convex trend. Furthermore, standard normal, standard logistic, and standard exponential distributions have been used as underlying models for U_i's. Here, $n = 10, 50$, $\alpha = 0.05$ level of significance, and some selected k values among $\{2, 3, ..., n-1\}$ have been considered. Moreover, $100,000$ simulations for $n = 50$ and $300,000$ simulations for $n = 10$ have been carried out in Matlab to obtain empirical powers of the selected tests.

The simulation results are summarized in Table 5 and 6. In the tables, the values shown as bold represent the largest two empirical powers in each row. It can be said that simulated α's are sufficiently closest to the true α's and the Q-test seems to be the most powerful. In general, the proposed test is clearly better than the d-test. For $n = 50$, the $T_{k,m}$ test is observed to be more powerful than the D-test for almost every selected k. On the other hand, for $n = 10$, it can be generally said that it is less powerful than the D-test. The reason for this result may be the fact that the true α's of the $T_{k,m}$ tests are considerably smaller than 0.05.

6. Appendix

TABLE 1. Positive critical values for $T_{k,m}$ derived from (2) for $m \leq 5$ and $k \leq 20$

k	α	$T^{\alpha}_{k,1}$	$T^{\alpha}_{k,2}$	$T^{\alpha}_{k,3}$	$T^{\alpha}_{k,4}$	$T^{\alpha}_{k,5}$	k	α	$T^{\alpha}_{k,1}$	$T^{\alpha}_{k,2}$	$T^{\alpha}_{k,3}$	$T^{\alpha}_{k,4}$	$T^{\alpha}_{k,5}$
1	.100	–	–	3	3	3	6	.100	–	2	3	3	3
	.050	–	–	3	4	4		.050	–	2	3	3	4
	.025	–	–	–	4	4		.025	–	–	3	4	4
	.010	–	–	–	4	5		.010	–	–	–	4	5
	.005	–	–	–	–	5		.005	–	–	–	4	5
2	.100	–	–	3	3	3	7	.100	–	2	2	3	3
	.050	–	–	–	4	4		.050	–	2	3	3	4
	.025	–	–	–	4	5		.025	–	–	3	4	4
	.010	–	–	–	–	5		.010	–	–	3	4	5
	.005	–	–	–	–	–		.005	–	–	–	4	5
3	.100	–	–	–	3	4	8	.100	–	2	2	3	3
	.050	–	–	–	4	4		.050	–	2	3	3	4
	.025	–	–	–	4	5		.025	–	2	3	4	4
	.010	–	–	–	–	5		.010	–	–	3	4	5
	.005	–	–	–	–	–		.005	–	–	–	4	5
4	.100	–	2	3	3	4	9	.100	1	2	2	3	3
	.050	–	–	3	4	4		.050	–	2	3	3	4
	.025	–	–	–	4	5		.025	–	2	3	3	4
	.010	–	–	–	–	–		.010	–	–	3	4	4
	.005	–	–	–	–	–		.005	–	–	3	4	5
5	.100	–	2	3	3	3	10	.100	1	2	2	3	3
	.050	–	2	3	4	4		.050	–	2	3	3	3
	.025	–	–	3	4	5		.025	–	2	3	3	4
	.010	–	–	–	4	5		.010	–	–	3	4	4
	.005	–	–	–	–	5		.005	–	–	3	4	5
11	.100	1	2	2	3	3	16	.100	1	2	2	2	3
	.050	–	2	2	3	3		.050	–	2	2	3	3
	.025	–	2	3	3	4		.025	–	2	3	3	3

TABLE 1. (Continued) Positive critical values for $T_{k,m}$ derived from (2) for $m \leq 5$ and $k \leq 20$

k	α	$T^{\alpha}_{k,1}$	$T^{\alpha}_{k,2}$	$T^{\alpha}_{k,3}$	$T^{\alpha}_{k,4}$	$T^{\alpha}_{k,5}$	k	α	$T^{\alpha}_{k,1}$	$T^{\alpha}_{k,2}$	$T^{\alpha}_{k,3}$	$T^{\alpha}_{k,4}$	$T^{\alpha}_{k,5}$
	.010	–	–	3	4	4		.010	–	2	3	3	4
	.005	–	–	3	4	4		.005	–	–	3	4	4
12	.100	1	2	2	2	3	17	.100	1	2	2	2	3
	.050	–	2	2	3	3		.050	–	2	2	3	3
	.025	–	2	3	3	4		.025	–	2	3	3	3
	.010	–	–	3	4	4		.010	–	2	3	3	4
	.005	–	–	3	4	4		.005	–	–	3	4	4
13	.100	1	2	2	2	3	18	.100	1	2	2	2	2
	.050	–	2	2	3	3		.050	–	2	2	3	3
	.025	–	2	3	3	4		.025	–	2	2	3	3
	.010	–	2	3	4	4		.010	–	2	3	3	4
	.005	–	–	3	4	4		.005	–	–	3	3	4
14	.100	1	2	2	2	3	19	.100	–	2	2	2	2
	.050	–	2	2	3	3		.050	–	2	2	3	3
	.025	–	2	3	3	3		.025	–	2	2	3	3
	.010	–	2	3	3	4		.010	–	2	3	3	4
	.005	–	–	3	4	4		.005	–	2	3	3	4
15	.100	1	2	2	2	3	20	.100	1	2	2	2	2
	.050	–	2	2	3	3		.050	1	2	2	2	3
	.025	–	2	3	3	3		.025	–	2	2	3	3
	.010	–	2	3	3	4		.010	–	2	3	3	4
	.005	–	–	3	4	4		.005	–	2	3	3	4

TABLE 2. Exact and asymptotic probabilities of the event $\{T_{k,6} \leq t\}$ for $k = 1, 5, 10, 15, 20$ and nonnegative integer t.

t	$k : \sigma_{k,6}$ 1 : 1.7849	5 : 2.1102	10 : 1.7219	15 : 1.4785	20 : 1.3164
0	.6069	.5898	.6121	.6325	.6517
0.5**	.6103*	.5937*	.6142*	.6324*	.6480*
1	.7944	.7563	.8054	.8466	.8784
1.5**	.7997*	.7614*	.8082*	.8448*	.8728*
2	.9192	.8776	.9276	.9577	.9741
2.5**	.9193*	.8819*	.9267*	.9546*	.9712*
3	.9788	.9529	.9821	.9931	.9969
3.5**	.9751*	.9514*	.9790*	.9910*	.9961*
4	.9968	.9863	.9976	.9994	.9998
4.5**	.9942*	.9835*	.9955*	.9998*	.9997*
5	.9998	.9982	.9998	.9999	.9999
5.5**	.9990*	.9954*	.9993*	.9999*	.9999*
6	1.0000	1.0000	1.0000	1.0000	1.0000
6.5**	.9999*	.9990*	.9999*	.9999*	.9999*

* Normal approximation for selected k. ** Corrected t for continuity.

TABLE 3. Upper and lower k-records extracted from the data set in Arnold et al. (1998, p.180)

	$k = 1$		$k = 2$		$k = 3$	
Number	Upper	Lower	Upper	Lower	Upper	Lower
1	12.69	12.69	12.69	12.84	12.69	18.72
2	12.84	7.51	12.84	12.69	12.84	12.84
3	18.72	4.83	18.72	12.55	14.28	12.69
4	21.96	4.13	19.19	11.80	14.77	12.55
5	23.92	4.08	21.46	7.51	18.72	11.80
6	27.16	–	21.96	4.89	19.19	8.69
7	31.28	–	23.21	4.83	21.46	7.51
8	34.04	–	23.29	4.13	21.96	6.25
9	–	–	23.92	–	23.21	4.89
10	–	–	27.16	–	23.29	4.83
11	–	–	30.57	–	23.92	4.56
12	–	–	31.28	–	24.95	–
13	–	–	–	–	26.81	–
14	–	–	–	–	27.16	–
15	–	–	–	–	30.57	–

Table 4. Summary statistics

k, m	$T_{k,m}$	$\sigma_{k,m}$	$Z_{k,m}$	asymptotic p-value
1, 99	3	2.894	.8638	.1938
2, 98	4	3.753	.9326	.1755
3, 97	4	4.327	.8089	.2093

Table 5. Empirical power comparison for $n = 50$.

Trend	Distribution*	θ	Test Statistics: Q .050**	D .033**	d .043**	$T_{5,45}$ .043**	$T_{10,40}$ .043**	$T_{15,35}$ .044**	$T_{20,30}$ .033**	$T_{30,20}$ .038**
None	Any Dist.	0	.049	.032	.043	.043	.042	.044	.032	.039
Linear	Normal	.010	.248	.081	.086	.148	.177	.188	.153	.157
		.019	.578	.155	.143	.318	.411	.439	.383	.373
		.025	.781	.226	.195	.469	.593	.628	.568	.542
	Logistic	.018	.267	.071	.079	.148	.192	.208	.172	.168
		.031	.548	.117	.114	.284	.396	.434	.378	.350
		.041	.754	.164	.149	.417	.576	.626	.565	.516
	Exponential	.006	.247	.189	.110	.184	.178	.169	.136	.154
		.012	.570	.368	.144	.343	.416	.417	.360	.387
		.016	.759	.484	.169	.446	.577	.605	.555	.569
Concave	Normal	.009	.247	.084	.100	.169	.189	.190	.146	.141
		.017	.569	.169	.182	.380	.442	.438	.359	.319
		.022	.760	.240	.245	.538	.621	.616	.525	.457
	Logistic	.016	.261	.076	.091	.169	.203	.206	.161	.148
		.028	.547	.129	.145	.344	.431	.439	.360	.305
		.037	.752	.183	.196	.502	.622	.631	.537	.450
	Exponential	.005	.225	.194	.129	.196	.171	.157	.122	.126
		.011	.575	.412	.192	.430	.460	.421	.346	.333
		.015	.768	.542	.230	.556	.642	.621	.533	.502
Convex	Normal	.020	.254	.083	.075	.125	.161	.183	.159	.186
		.035	.532	.146	.110	.231	.322	.377	.352	.402
		.047	.746	.212	.145	.341	.481	.558	.542	.603
	Logistic	.034	.253	.068	.068	.117	.161	.188	.168	.187
		.062	.560	.120	.094	.225	.345	.418	.397	.432
		.081	.757	.166	.117	.320	.498	.593	.580	.618
	Exponential	.013	.269	.182	.087	.153	.173	.180	.156	.203
		.024	.578	.329	.107	.249	.353	.408	.392	.503
		.031	.747	.426	.119	.305	.463	.565	.577	.689

All selected distributions are in standard forms. ** True α's.

Table 6. Empirical power comparison for $n = 10$.

Trend	Distribution*	θ	Test Statistics: Q .036**	D .043**	d .035**	$T_{2,8}$ .021**	$T_{3,7}$ .022**	$T_{4,6}$ .015**	$T_{5,5}$ .029**	$T_{6,4}$ .045**
None	Any Dist.	0	.036	.042	.035	.020	.022	.015	.029	.044
Linear	Normal	.148	.257	.216	.150	.132	.147	.113	.173	.222
		.254	.556	.424	.292	.304	.341	.278	.370	.430
		.327	.750	.575	.406	.448	.497	.419	.519	.572
	Logistic	.248	.254	.203	.142	.129	.148	.114	.174	.222
		.442	.565	.417	.291	.316	.357	.289	.378	.436
		.578	.758	.573	.412	.471	.521	.436	.527	.580
	Exponential	.100	.265	.246	.142	.149	.188	.153	.193	.197
		.185	.546	.453	.249	.323	.428	.377	.410	.394
		.265	.755	.624	.362	.500	.637	.574	.587	.567
Concave	Normal	.139	.246	.213	.163	.137	.145	.106	.158	.196
		.248	.558	.444	.342	.338	.357	.275	.349	.386
		.319	.747	.595	.473	.496	.519	.413	.485	.510
	Logistic	.238	.249	.205	.158	.138	.150	.111	.163	.200
		.425	.555	.426	.332	.342	.363	.278	.348	.386
		.561	.752	.591	.474	.514	.537	.425	.493	.518
	Exponential	.091	.250	.249	.160	.157	.182	.135	.166	.169
		.181	.549	.479	.297	.362	.440	.367	.387	.353
		.258	.748	.647	.419	.537	.642	.562	.560	.510
Convex	Normal	.264	.247	.209	.124	.110	.131	.106	.175	.243
		.469	.553	.424	.241	.253	.303	.265	.386	.497
		.615	.752	.581	.339	.378	.444	.401	.545	.665
	Logistic	.452	.251	.202	.119	.111	.137	.112	.180	.249
		.805	.551	.410	.234	.258	.311	.269	.387	.493
		1.095	.763	.585	.347	.403	.471	.420	.557	.670
	Exponential	.185	.256	.225	.110	.117	.167	.152	.209	.230
		.357	.561	.443	.200	.277	.408	.383	.446	.479
		.498	.757	.607	.290	.428	.591	.547	.603	.648

*All selected distributions are in standard forms. ** True α's.

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