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ON APPROXIMATION PROPERTIES OF TWO VARIABLES OF MODIFIED KANTOROVICH-TYPE OPERATORS

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Abstract. In the present paper, we introduce certain modification of Szász-Mirakyan-Kantorovich-type operators in polynomial weighted spaces of continuous functions of two variables. Then we research some approximation properties of these operators. We give some inequalities for the operators by means of the weighted modulus of continuity and also obtain a Voronovskaya-type theorem. Furthermore, in the paper we show that our operators give better degree of approximation of functions belonging to weighted spaces than classical Szász-Mirakyan operators.

1. Introduction

In 1930, Kantorovich [7] introduced the following operators for $f \in L_1[0,1]$ and $x \in [0,1]$:

$$K_n(f;x) := (n+1)\sum_{k=0}^{\infty}\binom{n}{k}x^k(1-x)^{n-k}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(s)\,ds,\ n \in N. \tag{1.1}$$

In many papers various modifications of operators $K_n(f)$ were introduced and many authors studied their approximation properties in different function spaces (see [4, 5, 9, 12, 13, 14, 15, 18]).

In papers [1, 2, 8, 11, 14, 15, 16, 17, 21] Szász-Mirakyan operators

$$S_n(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right),\ x \in R_0 = [0,\infty), \tag{1.2}$$

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were studied for $f \in C_p$ where C_p with fixed $p \in N_0 := \{0, 1, 2, ...\}$ denotes the polynomial weighted space generated by the weight function

$$\omega_0(x) := 1, \ \omega_p(x) := (1 + x^p)^{-1}, \ \ p \geq 1, \tag{1.3}$$

i.e. the space C_p is the set of all real-valued functions f continuous on R_0 for $\omega_p f$ is uniformly continuous and bounded on R_0. The norm in C_p is also defined by

$$||f||_p := ||f(.)||_p := \sup_{x \in R_0} \omega_p(x) |f(x)|. \tag{1.4}$$

The degree of approximation of $f \in C_p$ by the operators (1.2) were studied and it was proved that

$$\lim_{n \to \infty} S_n(f; x) = f(x) \tag{1.5}$$

for every $f \in C_p$, $p \in N_0$ and $x \in R_0$. Moreover, the convergence in (1.5) is uniform on every interval $[x_1, x_2]$, $x_2 > x_1 \geq 0$.

In [19] Szász-Mirakyan-Kantorovich operators were defined as

$$T_n(f; x) := ne^{-nx} \sum_{k=0}^{\infty} \binom{n}{k} \frac{(nx)^k}{k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt \tag{1.6}$$

for $x \in R_0$, $p \in N_0$ and $f \in L_1[0, \infty)$(see also some modified analogues of these operators [3, 6, 10, 15, 21]).

In 2003, Walczak [20] introduced modification of the operators (1.2) with two variables. In the paper he considered the space $C_{p,q}$, associated with the weight function

$$\omega_{p,q}(x, y) := \omega_p(x)\omega_q(y), \ p, q \geq 1, \ (x, y) \in R_0^2 = R_0 \times R_0, \tag{1.7}$$

and composed of all real-valued functions f continuous on R_0, for $w_{p,q}f$ is uniformly continuous and bounded on R_0^2. The norm on $C_{p,q}$ is defined as

$$||f||_{p,q} := ||f(., .)||_{p,q} := \sup_{(x,y) \in R_0^2} \omega_{p,q}(x, y) |f(x, y)|. \tag{1.8}$$

Similarly, the modulus of continuity of $f \in C_{p,q}$ is defined as usual by the formula

$$\omega(f; C_{p,q}; t, s) = \omega_{p,q}(x, y) := \sup_{0 \leq h \leq t, \ 0 \leq \delta \leq s} ||\Delta_{h,\delta} f(., .)||_{p,q}, \ \forall t, s \geq 0, \tag{1.9}$$

where $\Delta_{h,\delta} f(x, y) := f(x + h, y + \delta) - f(x, y)$ for $(x + h, y + \delta) \in R_0^2$. In addition $C^1_{p,q}$ is the set of all functions $f \in C_{p,q}$, which whose first partial derivatives belong also to $C_{p,q}$. From (1.9) it follows that

$$\lim_{t,s \to 0^+} \omega(f; C_{p,q}; t, s) = 0$$

for every $f \in C_{p,q}$ and $p,q \in N_0$. In [20] Walczak introduced a modified Szász-Mirakyan operators on $C_{p,q}$ for m, n, r, $s \in N$, $\alpha > 0$ and $(x,y) \in R_0^2$

$$A_{m,n}^{*}(f;r,s;\alpha;x,y) = \frac{1}{g((m^\alpha x+1)^2;r)g((n^\alpha y+1)^2;s)} \times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(m^\alpha x+1)^{2j}}{(j+r)!}\frac{(n^\alpha y+1)^{2k}}{(k+s)!} f\left(\frac{j+r}{m^\alpha x+1}, \frac{k+s}{n^\alpha y+1}\right), \tag{1.10}$$

where

$$g(t;r) := \sum_{k=0}^{\infty} \frac{t^k}{(k+r)!}, \quad t \in R_0 \tag{1.11}$$

i.e.

$$g(0;r) = \frac{1}{r!},\ g(t;r) = \frac{1}{t^r}\left(e^t - \sum_{j=0}^{r-1}\frac{t^j}{j!}\right),\ t>0.$$

If $f \in C_{p,q}$ and $f(x,y) = f_1(x) f_2(y)$, then

$$A_{m,n}^{*}(f;r,s;\alpha;x,y) = A_m^{*}(f_1;r;\alpha;x)\, A_n^{*}(f_2;s;\alpha;y) \tag{1.12}$$

for all $(x,y) \in R_0^2$ and m, $n,r,s \in N$.

Also he gave the theorems on the degree of approximation of functions from polynomial and exponential weighted spaces by the operators (1.10). In his work degree of these operators for approximation is similar but in some cases it is better than for aproximation in [19].

The purpose of this paper is to introduce a modified Kantorovich-type of (1.10) with two variables and also study convergence properties of the operators for functions on $C_{p,q}$ and $C_{p,q}^2$ by using the methods in [6, 20, 21].

2. Auxiliary Results

In the sequel we shall need several lemmas, which are necessary to prove the main theorems. Firstly we will give the moments of the operators. For this purpose we introduce the following class of operators on $C_{p,q}$.

Definition 1. Let $m,n,r,s \in N$ and $p,q \in N_0$ and $(m^\alpha),(n^\alpha)$ be positive sequences such that $\lim\limits_{m\to\infty} m^\alpha = \lim\limits_{n\to\infty} n^\alpha = \infty$ for $\alpha > 0$. Then for $f \in C_{p,q}$ we define the modified Szász-Mirakyan-Kantorovich operators as

$$A_{m,n}(f;r,s,\alpha;x,y) = A_{m,n}(f;x,y) := \frac{mn}{g((m^\alpha x+1)^2;r)g((n^\alpha y+1)^2;s)} \times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(m^\alpha x+1)^{2j}}{(j+r)!}\frac{(n^\alpha y+1)^{2k}}{(k+s)!} \int_{\frac{j+r}{m}}^{\frac{j+r+1}{m}}\int_{\frac{k+s}{n}}^{\frac{k+s+1}{n}} f\left(\frac{t}{m^\alpha x+1}, \frac{u}{n^\alpha y+1}\right) dt du, \tag{2.1}$$

where (1.11) holds.

In this paper we use short notation

$$g\left((m^{\alpha}x+1)^2;r\right)g\left((n^{\alpha}y+1)^2;s\right)=g_{m,r}\left(x\right)g_{n,s}(y).$$

Also, we will denote by M_k, $k=1,2,...$, the suitable positive constants depending only on the parameters p,q and r,s.

It is known that $A_{m,n}$ are positive linear operators acting from $C_{p,q}$ to $C_{p,q}$ a nd we have

$$A_{m,n}\left(1;x,y\right)=1 \tag{2.2}$$

for $p,q\in R_0$, $m,n,r,s\in N$, $\alpha>0$ and $(x,y)\in R_0^2$.

Other moments of $A_{m,n}(t^k;r;s;x,y)$ can be obtained easily for $k=1,2$. From (2.2) and (1.12) we get the following lemmas:

Lemma 1. *Let $m,n,r,s\in N$ be fixed numbers. Then for all $(x,y)\in R_0^2$, $f\in C_{p,q}$ and $\alpha>0$ we have*

$$A_{m,n}\left((t-x)\,;r,s;.,.\right)=\frac{1}{m^{\alpha}}+\frac{1}{2m^{\alpha}(m^{\alpha}x+1)}+\frac{1}{m^{\alpha}(m^{\alpha}x+1)(r-1)!g_{m,r}(x)}, \tag{2.3}$$

$$\begin{aligned}A_{m,n}\left((t-x)^2\,;r,s;.,.\right)&=\frac{2}{m^{2\alpha}}+\frac{(r-1)!g_{m,r}(x)+3(r+1)+3(m^{\alpha}x+1)^2}{6m^{2\alpha}(m^{\alpha}x+1)^2(r-1)!g_{m,r}(x)}\\&-\frac{1}{2m^{2\alpha}(m^{\alpha}x+1)}-\frac{(m^{\alpha}x+1)^3(r-1)!-m^{2\alpha}(m^{\alpha}x+1)^2}{m^{2\alpha}(m^{\alpha}x+1)^4(r-1)!}+\frac{(m^{\alpha}x+1)^2(r-1)!}{m^{2\alpha}(m^{\alpha}x+1)^4}.\end{aligned} \tag{2.4}$$

Lemma 2. *Let $m,n,r,s\in N$ be fixed numbers. Then there exist $\beta_{s,j}\left(r\right)=r^{j-1}$ and $\beta_{\nu,j}\left(s\right)=s^{j-1}$ depending only j,r,s such that*

$$\begin{aligned}&A_{m,n}\left(t^{u+1}+z^{\nu+1};x,y\right)\\&=\left(x+\tfrac{1}{m^{\alpha}}\right)^{u+1}\left\{\sum_{j=1}^{u+1}\frac{1}{(m^{\alpha}x+1)^{2(j-1)}}\left(\varphi_{u,j}+\frac{\gamma_j}{(m^{\alpha}x+1)^2}+\frac{\beta_{u,j}(r)}{(m^{\alpha}x+1)^2(r-1)!g_{m,r}(x)}\right)\right\}\\&+\left(y+\tfrac{1}{n^{\alpha}}\right)^{\nu+1}\left\{\sum_{j=1}^{\nu+1}\frac{1}{(n^{\alpha}y+1)^{2(j-1)}}\left(\varphi_{\nu,j}+\frac{\gamma_j}{(n^{\alpha}y+1)^2}+\frac{\beta_{\nu,j}(s)}{(n^{\alpha}y+1)^2(s-1)!g_{n,s}(y)}\right)\right\}\end{aligned} \tag{2.5}$$

for all $f\in C_{p,q}$, $\alpha>0$, $1\leq j\leq r$, $1\leq j\leq s$ and $(x,y)\in R_0^2$. Also $\beta_{u,1}\left(.\right),\beta_{\nu,1}\left(.\right)$ and $\varphi_{u,j}$, $\varphi_{\nu,j}$, γ_j are positive constants and the others are equal to one.

Lemma 3. *Let $p,q\in N_0$ and $m,n,r,s\in N$ be fixed numbers. Then for given positive constants M_2,M_3 we have*

$$\left|\left|A_{m,n}\left(\frac{1}{\omega_{p,q}\left(t,z\right)};r,s;.,.\right)\right|\right|_{p,q}\leq M_2,\ m,n\in N \tag{2.6}$$

and for all $f\in C_{p,q}$ we obtain

$$\left|\left|A_{m,n}\left(f;r,s;.,.\right)\right|\right|_{p,q}\leq M_3\left|\left|f\right|\right|_{p,q},\ m,n\in N. \tag{2.7}$$

Lemma 4. *Let $p, q \in N_0$ and $m, n, r, s \in N$ be fixed numbers. Then for given positive constants M_4, M_5 we have*

$$\left\| A_{m,n}\left(\frac{(t-.)^2}{\omega_{p,q}(t,z)}; r, s; ., .\right)\right\|_{p,q} \leq \frac{M_4}{m^{2\alpha}} + \frac{M_5}{n^{2\alpha}},\ m, n \in N \tag{2.8}$$

for all $f \in C_{p,q}$.

The methods used to prove the above Lemmas are similar to modified Szász-Mirakyan operators for f in [14, 15, 20]. Thus their proofs are very obvious.

3. Approximation Behaviour of Operators

Our first main result is the following theorem for approximation behaviour of $A_{m,n}$.

Theorem 1. *Let $f \in C^1_{p,q}$, $\alpha > 0$ be with $p, q \in N_0$ and $r, s \in N$. Then for a given positive constant M_6 we have*

$$\|A_{m,n}(f; ., .) - f(., .)\|_{p,q} \leq M_6 \left\{\frac{1}{m^{\alpha}} \|f'_x\|_{p,q} + \frac{1}{n^{\alpha}} \|f'_y\|_{p,q}\right\},\ m, n \in N. \tag{3.1}$$

Proof. Let $(x, y) \in R_0^2$ be a fixed point. Then for $f \in C^1_{p,q}$ and $(t, z) \in R_0^2$, $t \geq x$, $\alpha > 0$ we get

$$f(t,z) - f(x,y) = \int_x^t f'_u(u,z)\,du + \int_y^z f'_v(x,v)\,dv. \tag{3.2}$$

By linearity of $A_{m,n}$, (3.2) we obtain

$$\begin{aligned} A_{m,n}(f(t,z); x, y) - f(x,y)) &= A_{m,n}\left(\int_x^t f'_u(u,z)\,du; x, y\right) \\ &+ A_{m,n}\left(\int_y^z f'_v(x,v)\,dv; x, y\right) \end{aligned}$$

From (1.4) and (1.5) we have

$$\left|\int_x^t f'_u(u,z)\,du\right| \leq \|f'_x\|_{p,q}\left[\frac{1}{\omega_{p,q}(t,z)} + \frac{1}{\omega_{p,q}(x,z)}\right]|t-x|,\ \ (x,y) \in R_0^2. \tag{3.3}$$

By (3.3) it follows that

$$\begin{aligned} &\omega_{p,q}(x,y)\,|A_{m,n}f(t,z); x, y) - f(x,y)| \\ &\leq \|f'_x\|_{p,q}\,\omega_{p,q}(x,y)\left\{A_{m,n}\left(\tfrac{|t-x|}{\omega_{p,q}(x,z)}; x, y\right) + A_{m,n}\left(\tfrac{|t-x|}{\omega_{p,q}(t,z)}; x, y\right)\right\} \end{aligned} \tag{3.4}$$

for $m,n \in N$. Using the Hölder inequality, by Lemmas 1, 3, 4 and (2.2) we obtain

$$A_{m,n}\left(|t-x|;x,y\right) \leq \left\{A_{m,n}\left((t-x)^2;x,y\right)\right\}^{\frac{1}{2}}\left\{A_{m,n}\left(1;x,y\right)\right\}^{\frac{1}{2}} \leq \tfrac{M_7}{m^{\alpha}}.$$

Applying for the last inequality by (1.9), we get

$$\begin{aligned}\omega_{p,q}(x,y)\,A_{m,n}\left(\tfrac{|t-x|}{\omega_{p,q}(t,z)};x,y\right) &\leq \left\{\omega_{p,q}(x,y)\,A_{m,n}\left(\tfrac{(t-x)^2}{\omega_{p,q}(t,z)};x,y\right)\right\}^{\frac{1}{2}} \\ &\quad\times\left\{\omega_{p,q}(x,y)\,A_{m,n}\left(\tfrac{1}{\omega_{p,q}(t,z)};x,y\right)\right\}^{\frac{1}{2}} \\ &\leq \tfrac{M_8}{m^{\alpha}}\end{aligned} \tag{3.5}$$

for every $(x,y) \in R_0^2$ implying

$$\omega_{p,q}(x,y)\left|A_{m,n}\left(\int\limits_x^t f'_u(u,z)\,du;x,y\right)\right| \leq \tfrac{M_9}{m^{\alpha}}\left|\left|f'_x\right|\right|_{p,q},\ m,n \in N. \tag{3.6}$$

Analogously we have

$$w_{p,q}(x,y)\left|A_{m,n}\left(\int\limits_y^z f'_v(x,v)\,dv;x,y\right)\right| \leq \tfrac{M_{10}}{n^{\alpha}}\left|\left|f'_y\right|\right|_{p,q},\ m,n \in N. \tag{3.7}$$

We combine (3.6) and (3.7) and derive from (3.3) that (3.1) is satisfied.

Now, we compute the rate of convergence of $A_{m,n}$ by means of the weighted modulus of continuity given by (1.9). □

Theorem 2. *Let $f \in C^1_{p,q}$ and $p,q \in N_0$, $r,s \in N$ and $\alpha > 0$. Then there exists a positive constant M_{11} such that*

$$\left|\left|A_{m,n}(f;r,s;.,.) - f(.,.)\right|\right|_{p,q} \leq M_{11}\,\omega_1\left(f;C_{p,q};\frac{1}{m^{\alpha}},\frac{1}{n^{\alpha}}\right),\ m,n \in N. \tag{3.8}$$

Proof. Let $f_{h,\delta}$ be the Steklov means of function $f \in C^1_{p,q}$ defined by the formula

$$f_{h,\delta}(x,y) := \frac{1}{h\delta}\int\limits_0^h du\int\limits_0^{\delta} f(x+u,y+v)\,dv,\quad (x,y) \in R_0^2,\ h,\delta > 0. \tag{3.9}$$

From (3.9) we get

$$\frac{\partial}{\partial x} f_{h,\delta}(x,y) = (f_{h,\delta})'_x(x,y) = \frac{1}{h\delta}\int_0^h \Delta_{h,0} f(x+u,y)\,du,$$

$$\frac{\partial}{\partial y} f_{h,\delta}(x,y) = (f_{h,\delta})'_y(x,y) = \frac{1}{h\delta}\int_0^\delta \Delta_{o,\delta} f(x,y+v)\,dv,$$

which imply $f_{h,\delta}(x,y) \in C^1_{p,q}$ for every fixed $h,\delta > 0$. Also we have

$$\left\| f_{h,\delta} - f \right\|_{p,q} \le \omega\left(f; C_{p,q}; h,\delta\right), \tag{3.10}$$

$$\left|\left|(f_{h,\delta})'_x\right|\right|_{p,q} \le 2h^{-1}\omega\left(f; C_{p,q}; h,\delta\right), \tag{3.11}$$

$$\left|\left|(f_{h,\delta})'_y\right|\right|_{p,q} \le 2\delta^{-1}\omega\left(f; C_{p,q}; h,\delta\right). \tag{3.12}$$

Hence by the last inequalities we can write

$$\begin{aligned}
&\omega_{p,q}(x,y)\left|(A_{m,n}(f;r;x,y) - f(x,y))\right| \\
&\le \omega_{p,q}(x,y)\left\{\left|A_{m,n}\left(f(t,z)) - f_{h,\delta}(t,z);x,y\right)\right| + \left|A_{m,n}\left(f_{h,\delta}(t,z);x,y\right) - f_{h,\delta}(x,y)\right|\right. \\
&\left. + \left|f_{h,\delta}(x,y) - f(x,y)\right|\right\} := L_1 + L_2 + L_3
\end{aligned} \tag{3.13}$$

for every $m,n \in N$, $h,\delta > 0$ and $(x,y) \in R_0^2$. For L_1 and L_3, by using Lemma 3 and (3.10), we get

$$\begin{aligned}
\|L_1\|_{p,q} &\le M_{12}\|f - f_{h,\delta}\|_{p,q} \le M_{12}\omega\left(f; C_{p,q}; h,\delta\right), \\
\|L_3\|_{p,q} &\le \omega\left(f; C_{p,q}; h,\delta\right).
\end{aligned}$$

Similarly, by Theorem 1 and (3.11),(3.12) we have

$$\begin{aligned}
\|L_2\|_{p,q} &\le M_{13}\left\{\tfrac{1}{m^\alpha}\left\|(f_{h,\delta})'_x\right\|_{p,q} + \tfrac{1}{n^\alpha}\left\|(f_{h,\delta})'_y\right\|_{p,q}\right\} \\
&\le 2M_{14}\omega\left(f; C_{p,q}; h,\delta\right)\left(\tfrac{1}{m^\alpha h} + \tfrac{1}{n^\alpha \delta}\right),\ h,\delta > 0,\ m,n \in N.
\end{aligned} \tag{3.14}$$

Hence, from (3.14) for (3.13) it follows that

$$\|A_{m,n}(f; r,s,\alpha; .,.) - f(.,.)\|_{p,q} \le M_{15}\left(1 + \frac{1}{m^\alpha h} + \frac{1}{n^\alpha \delta}\right)\omega\left(f; C_{p,q}; h,\delta\right).$$

Now, for fixed $m,n \in N$, substitution of $h = \frac{1}{m^\alpha}$ and $\delta = \frac{1}{n^\alpha}$ in the last inequality, we obtain the desired result of (3.8). This completes the proof of Theorem 2. □

The following corollories are immediate consequences of Theorem 1 and 2.

Corollary 1. *For every fixed numbers $r,s \in N$, $p,q \in N_0$ and $f \in C_{p,q}$, we have*

$$\lim_{m,n\to\infty} \left|\left|A_{m,n}\left(f;r,s;.,.\right)-f\left(.,.\right)\right|\right|_{p,q}=0. \tag{3.15}$$

Corollary 2. *For every fixed numbers $r,s \in N$, $p,q \in N_0$ and $f \in C^1_{p,q}$, we have*

$$\left|\left|A_{m,n}\left(f;r,s;.,.\right)-f\left(.,.\right)\right|\right|_{p,q}=o\left(\frac{1}{m^\alpha},\frac{1}{n^\alpha}\right) \tag{3.16}$$

as $m,n\to\infty$.

Now we will prove the following Voronovskaya-type theorem.

Theorem 3. *Let $f \in C^2_{p,q}$ be with given $p,q \in N_0$ and $r,s \in N$. Then for every $(x,y) \in R^2_0$*

$$\lim_{n\to\infty} n^\alpha \left\{A_{n,n}\left(f;r,s;x,y\right)-f\left(x,y\right)\right\}=\frac{x}{2}f''_{xx}\left(x,y\right)+\frac{y}{2}f''_{yy}\left(x,y\right). \tag{3.17}$$

Proof. Let (x,y) be a fixed point in R^2_0. Then, by the Taylor formula we can write

$$\begin{aligned} f\left(t,z\right)= \quad & f\left(\,x,y\right)+f'_x\left(x,y\right)\left(t-x\right)+f'_y\left(x,y\right)\left(z-y\right)+\tfrac{1}{2}\left\{f''_{xx}\left(x,y\right)\left(t-x\right)^2\right. \\ & \left.+2f''_{xy}\left(x,y\right)\left(t-x\right)\left(z-y\right)+f''_{yy}\left(x,y\right)\left(z-y\right)^2\right\} \\ & +\varepsilon\left(t,z;x,y\right)\left\{\left(t-x\right)^4+\left(z-y\right)^4\right\}^{\frac{1}{2}} \end{aligned}$$

for $f \in C^2_{p,q}$, $(t,z) \in R^2_0$ where $\varepsilon\left(.,.;x,y\right)\equiv\varepsilon\left(.;.\right)\in C^1_{p,q}$ is function such that

$$\lim_{(t,z)\to(x_0,y_0)} \varepsilon\left(t,z;x,y\right)=0.$$

Applying (2.1) to the last equality, we get

$$\begin{aligned} A_{n,n}\left(f;x,y\right)-f\left(x,y\right)= \quad & f'_x\left(x,y\right)A_{n,n}\left(\left(t-x\right);x,y\right)+f'_y\left(x,y\right)A_{n,n}\left(\left(z-y\right);x,y\right) \\ & +\tfrac{1}{2}\left\{f(x,y)''_{xx}\left(x,y\right)A_{n,n}\left(\left(t-x\right)^2,x,y\right)\right. \\ & +2f''_{xy}\left(x,y\right)A_{n,n}(\left(t-x\right)\left(z-y\right);x,y) \\ & \left.+f''_{yy}\left(x,y\right)A_{n,n}(\left(z-y\right)^2;x,y)\right\} \\ & +A_{n,n}\left(\varepsilon\left(t,z\right)\sqrt{\left(t-x\right)^4+\left(z-y\right)^4};x,y)\right) \\ & := L_1+L_2+L_3+L_4+L_5+L_6. \end{aligned}$$

From (3.2),(3.3) and Lemma 1, the limit of the L_1, L_2 and L_4 are equal to zero as $n\to\infty$ and

$$\lim_{n\to\infty} n^\alpha L_3 = x, \quad \lim_{n\to\infty} n^\alpha L_5 = y.$$

For the right term in the last equation by the Hölder inequality we obtain

$$|L_6| \leq 2\left\{A_{n,n}(\varepsilon^2(t,z);x,y)\right\}^{\frac{1}{2}}\left\{A_{n,n}\left((t-x)^4+(z-y)^4;x,y\right)\right\}^{\frac{1}{2}}$$

By Corollary 1 and properties of $\varepsilon(.,.)$ we deduce that

$$\lim_{n\to\infty} A_{n,n}\left(\varepsilon^2(t,z);x,y\right) = \varepsilon^2(x,y) = 0.$$

From this, the linearity of $A_{n,n}$ and Lemma 1 we have

$$\lim_{n\to\infty} n^{\alpha} A_{n,n}\left(\sqrt{(t-x)^4+(z-y)^4};x,y\right) = 0.$$

Collecting these results, we immediately obtain the desired result (3.17).

In this paper, Theorem 1, 2 and Corollary 2 show that our operator $A_{m,n}, m,n \in N$, give better degree of approximation of functions $f \in C_{p,q}$ and $f \in C^1_{p,q}$ than classical Szász-Kantorovich operators. □

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