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SCREEN SEMI-INVARYANT HALF-LIGHTLIKE SUBMANIFOLDS OF A SEMI-RIEMANNIAN PRODUCT MANIFOLD WITH QUARTER-SYMMETRIC CONNECTION

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Abstract. In this paper, we study half-lightlike submanifolds of a semi-Riemannian product manifold. We introduce a classes half-lightlike submanifolds of called screen semi-invariant half-lightlike submanifolds. We defined some special distribution of screen semi-invariant half-lightlike submanifold. We give some equivalent conditions for integrability of distributions with respect to the Levi-Civita connection of semi-Riemannian manifolds and quarter-symmetric non-metric connection of semi-Riemannian manifolds and some results.

1. Introduction

The theory of degenerate submanifolds of semi-Riemannian manifolds is one of a important topics of diferential geometry. The geometry of lightlike submanifolds a semi-Riemannian manifold was presented in [7] (see also [8]) by K.L. Duggal and A. Bejancu. Differential Geometry of Lightlike Submanifolds was presented in [17] by K. L. Duggal and B. Sahin. In [12],[13], [14], [15], K. L. Duggal and B. Sahin introduced and studied geometry of classes of lightlike submanifolds in indefinite Kaehler and indefinite Sasakian manifolds which is an umbrella of CR-lightlike, SCR-lightlike, Screen real GCR-lightlie submanifolds. In [16], M. Atceken and E. Kilic introduced semi-invariant lightlike submanifolds of a semi-Riemannian product manifold. In [18], E. Kilic and B. Sahin introduced radical anti-invariant lightlike submanifolds of a semi-Riemannian product and gave some examples and results for lightlike submanifolds. In [19] E. Kilic and O. Bahadir studied lightlike hypersurfaces of a semi-Riemannian product manifold with respect to quarter symmetric non-metric connection. In [20] O. Bahadir give some equivalent conditions for integrability of distributions with respect to Levi Civita connection of semi-Riemannian manifolds and some results.

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In this paper, we study half-lightlike submanifolds of a semi-Riemannian product manifold. In Section 2, we give some basic concepts. In Section 3, we introduce screen semi-invariant half-lightlike submanifolds. We defined some special distribution of screen semi-invariant half-lightlike submanifold. In Section 4, we consider half-lightlike submanifolds of a semi-Riemannian product manifold with quarter symmetric non-metric connection determined by the product structure. We compute some results with respect to the quarter-symmetric non-metric connection.

2. Half-lightlike submanifolds

Let $(\widetilde{M}, \widetilde{g})$ be an $(m+2)$-dimensional $(m > 1)$ semi-Riemannian manifold of index $q \geq 1$ and M a submanifold of codimension 2 of $\widetilde{M}$. If $\widetilde{g}$ is degenerate on the tangent bundle TM on M, then M is called a lightlike submanifold of $\widetilde{M}$ [17]. Denote by g the induced degenerate metric tensor of $\widetilde{g}$ on M. Then there exists locally (or globally) a vector field $\xi \in \Gamma(TM)$, $\xi \neq 0$, such that $g(\xi, X) = 0$ for any $X \in \Gamma(TM)$. For any tangent space T_xM, $(x \in M)$, we consider

$$T_xM^{\perp} = \{u \in T_x\widetilde{M} : \widetilde{g}(u, v) = 0, \forall v \in T_xM\}, \tag{2.1}$$

a degenerate 2-dimensional orthogonal (but not complementary) subspace of $T_x\widetilde{M}$. The radical subspace $Rad\ T_xM = T_xM \cap T_xM^{\perp}$ depends on the point $x \in M$. If the mapping

$$Rad\ TM : x \in M \longrightarrow Rad\ T_xM \tag{2.2}$$

defines a radical distribution on M of rank $r > 0$, then the submanifold M is called $r-$lightlike submanifold. If $r = 1$, then M is called half-lightlike submanifold of $\widetilde{M}$ [17]. Then there exist $\xi, u \in T_xM^{\perp}$ such that

$$\widetilde{g}(\xi, v) = 0, \quad \widetilde{g}(u, u) \neq 0, \forall v \in T_xM^{\perp}. \tag{2.3}$$

Furthermore, $\xi \in Rad\ T_xM$, and

$$\widetilde{g}(\xi, X) = \widetilde{g}(\xi, v) = 0, \forall X \in \Gamma(TM),\ v \in \Gamma(TM^{\perp}). \tag{2.4}$$

Thus, $Rad\,TM$ is locally (or globally) spanned by ξ. By denote the complementary vector bundle $S(TM)$ of $Rad\,TM$ in TM which is called screen bundle of M. Thus we have the following decomposition

$$TM = Rad\ TM \perp S(TM), \tag{2.5}$$

where $\perp$ denotes the orthogonal-direct sum. In this paper, we assume that M is half-lightlike. Then there exists complementary non-degenerate distribution $S(TM^{\perp})$ of $Rad\ TM$ in $TM^{\perp}$ such that

$$TM^{\perp} = Rad\ TM \perp S(TM^{\perp}). \tag{2.6}$$

Choose $u \in S(TM^{\perp}$ as a unit vector field with $\widetilde{g}(u, u) = \epsilon = \pm 1$. Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\widetilde{M}$. We note that ξ and u belong to $S(TM)^{\perp}$. Thus we have

$$S(TM)^{\perp} = S(TM^{\perp}) \perp S(TM^{\perp})^{\perp},$$

where $S(TM^{\perp})^{\perp}$ is the orthogonal complementary to $S(TM^{\perp})$ in $S(TM)^{\perp}$. For any null section ξ of $Rad\ TM$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a uniquely determined null vector field $N \in \Gamma(ltr(TM))$ satisfying

$$(2.7) \qquad \widetilde{g}(\xi, N) = 1,\ \widetilde{g}(N, N) = \widetilde{g}(N, X) = \widetilde{g}(N, u) = 0, \forall X \in \Gamma(TM),$$

where N, $ltr(TM)$ and $tr(TM) = S(TM^{\perp}) \perp ltr(TM)$ are called the lightlike transversal vector field, lightlike transversal vector bundle and transversal vector bundle of M with respect to $S(TM)$, respectively. Then we have the following decomposition:

$$(2.8) T\widetilde{M} = TM \oplus tr(TM) = S(TM) \perp \{Rad\ TM \oplus ltr(TM)\} \perp S(TM^{\perp}).$$

Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\widetilde{M}$ and P the projection of TM on $S(TM)$ with respect to the decomposition (2.5). Thus, for any $X \in \Gamma(TM)$, we can write $X = PX + \eta(X)\xi$, where η is a local differential 1-form on M given by $\eta(X) = \widetilde{g}(X, N)$. Then the Gauss and Weingarten formulas are given by

$$\begin{aligned}
(2.9) \qquad \widetilde{\nabla}_X Y &= \nabla_X Y + D_1(X,Y)N + D_2(X,Y)u, \\
(2.10) \qquad \widetilde{\nabla}_X U &= -A_U X + \nabla^t_X U, \\
(2.11) \qquad \widetilde{\nabla}_X N &= -A_N X + p_1(X)N + p_2(X)u, \\
(2.12) \qquad \widetilde{\nabla}_X u &= -A_u X + \varepsilon_1(X)N + \varepsilon_2(X)u, \\
(2.13) \qquad \nabla_X PY &= \nabla^*_X PY + E(X, PY)\xi, \\
(2.14) \qquad \nabla_X \xi &= -A^*_\xi X - p_1(X)\xi,
\end{aligned}$$

for any $X, Y \in \Gamma(TM)$, $u \in s(TM^{\perp})$, $U \in \Gamma(tr(TM))$, where ∇, ∇^* and ∇^t are induced linear connections on M, $S(TM)$ and $tr(TM)$, respectively, D_1 and D_2 are called the lightlike second fundamental and screen second fundemental form of M respectively, E is called the local second fundamental form on $S(TM)$. A_U, A_N, A^*_ξ and A_u are linear operators on TM and τ, ρ and ϕ are $1-$forms on TM. We note that, the induced connection ∇ is torsion-free but it is not metric connection on M and satisfies

$$(2.15) \qquad (\nabla_X g)(Y, Z) = D_1(X, Y)\eta(Z) + D_1(X, Z)\eta(Y),$$

for any $X, Y, Z \in \Gamma(TM)$. However the connection ∇^* on $S(TM)$ is metric. From the above statements, we have

$$\begin{aligned}
(2.16) \quad & D_1(X, PY) = g(A^*_\xi X, PY), \quad g(A^*_\xi X, N) = 0, \quad D_1(X, \xi) = 0, \\
& \widetilde{g}(A_N X, N) = 0, \\
(2.17) \quad & E(X, PY) = g(A_N X, PY), \\
& \epsilon D_2(X, Y) = g(A_u X, Y) - \varepsilon_1(X)\eta(Y), \\
(2.18) \quad & \epsilon\rho(X) = \widetilde{g}(A_u X, N),\ p_1(X) = -\eta(\nabla_X \xi), \quad p_2(X) = \epsilon\eta(A_u X), \\
& \varepsilon_1(X) = -\epsilon D_2(X, \xi),
\end{aligned}$$

for any $X, Y \in \Gamma(TM)$. From (2.17) and (2.18), A^*_ξ and A_N are $\Gamma(S(TM))-$valued shape operators related to D_1 and E, respectively and $A^*_\xi \xi = 0$.

Using torsion free linear connection $\nabla$and (2.13) we have

$$\begin{aligned}[X, Y] &= \{\nabla^*_X PY - \nabla^*_Y PX + \eta(X)A^*_\xi Y - \eta(Y)A^*_\xi X\} \\ &+\{E(X, PY) - E(Y, PX) + X(\eta(Y)) \\ &-Y(\eta(X)) + \eta(X)p_1(Y) - \eta(Y)p_1(X)\}\xi.\end{aligned}$$

The last equation and (2.17)

$$\begin{aligned}g(\nabla^*_X PY, PZ) - g(\nabla^*_X PZ, PY) - g([X, Y], PZ) \\ = \eta(Y)D_1(X, PZ) - \eta(X)D_1(Y, PZ), \\ 2d\eta(X, Y) = E(Y, PX) - E(X, PY) \\ +p_1(X)\eta(Y) - p_1(Y)\eta(X).\end{aligned} \tag{2.19}$$

From the second equation (2.19) we have

$$\eta([PX, PY]) = E(PX, PY) - E(PY, PX). \tag{2.20}$$

From (2.18) and (2.20), we have the following theorem.

Theorem 2.1. *Let M be a half-lightlike submanifold of a semi-Riemannian manifold $\widetilde{M}$. Then the following assertions are equivalent:.*
(1) *The screen distribution $S(TM)$ is integrable.*
(2) *The second fundamental form of $S(TM)$ is symmetric on $\Gamma(s(TM)$.*
(3) *The shape operator A_N of the immersion of M in $\widetilde{M}$ is symmetric with respect to g on $\Gamma(s(TM)$.*

Next by using (2.14), (2.15), (2.17) and (2.18) we obtain

Theorem 2.2. *Let M be a half-lightlike submanifold of a semi-Riemannian manifold $\widetilde{M}$. Then the following assertions are equivalent:*
(1)*The induced connection ∇ on M is a metric connection.*
(2) *D_1 vanishes identically on M.*
(3) *A^*_ξ vanishes identically on M.*
(4) *ξ is a Killing vector field.*
(5) *$TM^\perp$ is a parallel distribution with respect to ∇.*

Theorem 2.3. *Let (M, g) be a proper totally umbilical half-lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}(c), \widetilde{g})$ of constant sectional curvature c. Then the following assertions are equivalent:*
(i) *The screen distribution $s(TM)$ is integrable.*
(ii) *Each $1-$ form p_1 is closed on $s(TM)$, i.e., $dp_1 = 0$*
(iii) *Each $1-$ form p_2 induced by $s(TM)$ satisfies*

$$2dp_2(X, Y) = p_1(X)p_2(Y) - p_2(X)p_1(Y), \quad \forall X, Y \in \Gamma(TM).$$

For basic information on the geometry of lightlike submanifolds, we refer to [7], [17].

Let ($\widetilde{M}$ be an $n-$ dimensional diferentiable manifold with a tensor field F of type $(1, 1)$ on $\widetilde{M}$ such that $F^2 = I$. Then M is called an almost product manifold with almost product structure F. If we put $\pi = \frac{1}{2}(I + F)$, $\sigma = \frac{1}{2}(I - F)$ then we have

$$\pi + \sigma = I,\ \pi^2 = \pi,\ \sigma^2 = \sigma,\ \pi\sigma = \sigma\pi = 0,\ F = \pi - \sigma.$$

Thus π and σ define two complementary distributions and the eigenvalue of F are ∓ 1. If an almost product manifold $\widetilde{M}$ admits a semi-Riemannian metric $\widetilde{g}$ such that

$$\widetilde{g}(FX, FY) = \widetilde{g}(X, Y),\ \widetilde{g}(FX, Y) = \widetilde{g}(X, FY), \forall X, Y \in \Gamma(\widetilde{M}),$$

then $(\widetilde{M}, \widetilde{g})$ is called semi-Riemannian almost product manifold. If, for any X, Y vector fields on $\widetilde{M}$, $(\widetilde{\nabla}_X F)Y = 0$, that is

$$\widetilde{\nabla}_X FY = F\widetilde{\nabla}_X Y,$$

then M is called an semi-Riemannian product manifold, where $\widetilde{\nabla}$ is the Levi-Civita connection on $\widetilde{M}$.

3. Screen Semi-Invariant Lightlike Submanifolds

Let (M, g) be a half-lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g})$ For any $X \in \Gamma(TM)$ we can write

$$FX = fX + wX, \tag{3.1}$$

where f and w are the projections on of $\Gamma(T\widetilde{M})$ onto TM and $trTM$, respectively, that is, fX and wX are tangent and transversal components of FX. From (2.8) and (3.1), we can write

$$FX = fX + w_1(X)N + w_2(X)u, \tag{3.2}$$

where $w_1(X) = \widetilde{g}(FX, \xi)$, $w_2(X) = \epsilon\widetilde{g}(FX, u)$.

Definition 3.1. Let (M, g) be a half-lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g})$. If $FRad\ TM \subset S(TM)$, $Fltr(TM) \subset S(TM)$ and $F(S(TM^{\perp})) \subset S(TM)$ then we say that M is a screen semi-invaryant (SSI) half-lightlike submanifold.

If $FS(TM) = S(TM)$, then we say that M is a screen invaryant half-lightlike submanifold.

Now, let M be a screen semi-invariant half-lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g})$. If we set $L_1 = FRad\ TM$, $L_2 = Fltr(TM)$ and $L_3 = F(S(TM^{\perp}))$, then we can write

$$S(TM) = L_0 \perp \{L_1 \oplus L_2\} \perp L_3, \tag{3.3}$$

where L_0 is a $(m-4)-$dimensional distribution. Hence we have the following decompositions:

$$TM = L_0 \perp \{L_1 \oplus L_2\} \perp L_3 \perp Rad\ TM, \tag{3.4}$$

$$T\widetilde{M} = L_0 \perp \{L_1 \oplus L_2\} \perp L_3 \perp S(TM^{\perp}) \perp \{Rad\ TM \oplus ltr(TM)\}. \tag{3.5}$$

Let (M, g) be a screen semi-invariant half-lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g})$. If we set

$$L = L_0 \perp L_1 \perp Rad\ TM \qquad L^{\perp} = L_2 \perp L_3,$$

then we can write

$$TM = L \oplus L^{\perp}.$$

We note that the distribution L is a invariant distribution and the distribution $L^{\perp}$ is anti-invariant distribution with respect to F on M.

4. Quarter-symmetric Non-metric Connections

Let (M, g, F) be a semi-Riemannian product manifold and $\widetilde{\nabla}$ be the Levi-Civita connection on M. If we set

$$\widetilde{D}_X Y = \widetilde{\nabla}_X Y + \pi(Y)FX \tag{4.1}$$

for any $X, Y \in \Gamma(T\widetilde{M})$, then $\widetilde{D}$ is a linear connection on $\widetilde{M}$, where u is a 1-form on $\widetilde{M}$ with U as associated vector field, that is

$$\pi(X) = \widetilde{g}(X, U).$$

The torsion tensor of $\widetilde{D}$ on $\widetilde{M}$ denoted by $\widetilde{T}$. Then we obtain

$$\widetilde{T}(X, Y) = \pi(Y)FX - \pi(X)FY, \tag{4.2}$$

and

$$(\widetilde{D}_X \widetilde{g})(Y, Z) = -\pi(Y)\widetilde{g}(FX, Z) - \pi(Z)\widetilde{g}(FX, Y), \tag{4.3}$$

for any $X, Y \in \Gamma(T\widetilde{M})$. Thus $\widetilde{D}$ is a quarter-symmetric non-metric connection on $\widetilde{M}$. From (4.1) we have

$$(\widetilde{D}_X F)Y = \pi(FY)FX - \pi(Y)X. \tag{4.4}$$

Replacing X by FX and Y by FY in (4.4) we obtain

$$(\widetilde{D}_{FX} F)FY = \pi(Y)X - \pi(FY)FX. \tag{4.5}$$

Thus we have

$$(\widetilde{D}_X F)Y + (\overline{D}_{FX} F)FY = 0. \tag{4.6}$$

If we set

$${}'F(X, Y) = \widetilde{g}(FX, Y) \tag{4.7}$$

for any $X, Y \in \Gamma(T\overline{M})$, from (4.1) we get

$$(\widetilde{D}_X\, {}'F)(Y, Z) = (\widetilde{\nabla}_X\, {}'F)(Y, Z) - \pi(Y)\widetilde{g}(X, Z) - \pi(Z)\widetilde{g}(X, Y). \tag{4.8}$$

From (4.1) the curvature tensor $\widetilde{R}^D$ of the quarter-symmetric non-metric connection $\widetilde{D}$ is given by

$$\widetilde{R}^D(X, Y)Z = \widetilde{R}(X, Y)Z + \widetilde{\lambda}(X, Z)FY - \widetilde{\lambda}(Y, Z)FX, \tag{4.9}$$

for any $X, Y, Z \in \Gamma(T\widetilde{M})$, where $\widetilde{\lambda}$ is a $(0, 2)$-tensor given by $\widetilde{\lambda}(X, Z) = (\widetilde{\nabla}_X \pi)(Z) - \pi(Z)\pi(FX)$. If we set $\widetilde{R}^D(X, Y, Z, W) = \widetilde{g}(\overline{R}^D(X, Y)Z, W)$, then, from (4.9), we obtain

$$\widetilde{R}^D(X, Y, Z, W) = -\widetilde{R}^D(Y, X, Z, W).$$

We note that the Riemannian curvature tensor $\widetilde{R}^D$ of $\widetilde{D}$ does not satisfy the other curvature-like properties. But, from (4.9), we have

$$\begin{aligned}\widetilde{R}^D(X, Y)Z + \widetilde{R}^D(Y, Z)X + \widetilde{R}^D(Z, X)Y &= (\widetilde{\lambda}(Z, Y) - \widetilde{\lambda}(Y, Z))FX \\ &+ (\widetilde{\lambda}(X, Z) - \widetilde{\lambda}(Z, X))FY \\ &+ (\widetilde{\lambda}(Y, X) - \widetilde{\lambda}(X, Y))FZ.\end{aligned}$$

Thus we have the following proposition.

Proposition 4.1. *Let M be a half-lightlike submanifold of a semi-Riemannian product manifold $\widetilde{M}$. Then the first Bianchi identity of the quarter-symmetric non-metric connection $\widetilde{D}$ on M is provided if and only if $\widetilde{\lambda}$ is symmetric.*

Let M be a half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M}, \widetilde{g}\right)$ with quarter-symmetric non-metric connection $\widetilde{D}$. Then the Gauss and Weingarten formulas with respect to $\widetilde{D}$ are given by, respectively,

$$\widetilde{D}_X Y = D_X Y + \widetilde{D}_1(X,Y)N + \widetilde{D}_2(X,Y)u, \tag{4.10}$$

$$\widetilde{D}_X N = -\widetilde{A}_N X + \widetilde{p}_1(X)N + \widetilde{p}_2(X)u, \tag{4.11}$$

$$\widetilde{D}_X u = -\widetilde{A}_u X + \widetilde{\varepsilon}_1(X)N + \widetilde{\varepsilon}_2(X)u. \tag{4.12}$$

for any $X, Y \in \Gamma(TM)$, where $D_X Y,\ \widetilde{A}_N X,\ \widetilde{A}_u X \in \Gamma(TM)$, $\widetilde{D}_1(X,Y) = \widetilde{g}(\widetilde{D}_X Y, \xi)$, $\widetilde{D}_2(X,Y) = \epsilon \widetilde{g}(\widetilde{D}_X Y, u)$, $\widetilde{p}_1(X) = \widetilde{g}(\widetilde{D}_X N, \xi)$, $\widetilde{p}_2(X) = \epsilon \widetilde{g}(\widetilde{D}_X N, u)$, $\widetilde{\varepsilon}_1(X) = \widetilde{g}(\widetilde{D}_X u, \xi)$, $\widetilde{\varepsilon}_2(X) = \epsilon \widetilde{g}(\widetilde{D}_X u, u)$. Here, $\widetilde{D}_1$ and $\widetilde{D}_2$ the lightlike second fundamental form and the screen second fundamental form of M with respect to $\widetilde{D}$ respectively. Both $\widetilde{A}_N$ and $\widetilde{A}_u$ are linear operators on $\Gamma(TM)$. From (2.9), (2.11), (2.12), (4.1), (4.10), (4.11) and (4.12) we obtain

$$D_X Y = \nabla_X Y + \pi(Y) f X, \tag{4.13}$$

$$\widetilde{D}_1(X,Y) = D_1(X,Y) + \pi(Y) w_1(X), \tag{4.14}$$

$$\widetilde{D}_2(X,Y) = D_2(X,Y) + \pi(Y) w_2(X), \tag{4.15}$$

$$\widetilde{A}_N X = A_N X - \pi(N) f X, \tag{4.16}$$

$$\widetilde{p}_1(X) = p_1(X) + \pi(N) w_1(X), \tag{4.17}$$

$$\widetilde{p}_2(X) = p_2(X) + \pi(N) w_2(X), \tag{4.18}$$

$$\widetilde{A}_u X = A_u X - \pi(u) f X, \tag{4.19}$$

$$\widetilde{\varepsilon}_1(X) = \varepsilon_1(X) + \pi(u) w_1(X), \tag{4.20}$$

$$\widetilde{\varepsilon}_2(X) = \varepsilon_2(X) + \pi(u) w_2(X). \tag{4.21}$$

for any $X, Y \in \Gamma(TM)$. From (2.15), (4.1) we get

$$\begin{aligned} (D_x g)(Y,Z) &= D_1(X,Y)\eta(Z) + D_1(X,Z)\eta(Y) \\ &\quad -\pi(Y) g(fX, Z) - \pi(Z) g(fX, Y), \end{aligned} \tag{4.22}$$

On the other hand, the torsion tensor of the induced connection D is

$$T^D(X,Y) = \pi(Y) f X - \pi(X) f Y. \tag{4.23}$$

From last two equations we have the following proposition.

Proposition 4.2. *Let M be a half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M}, \widetilde{g}\right)$ with quarter-symmetric non-metric connection $\overline{D}$. Then the induced connection D is a quarter-symmetric non-metric connection on the half-lightlike submanifold M.*

From (4.2), (4.14) and (4.15) we have the following theorem

For any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(RadTM)$ we can write

$$D_X PY = D_X^* PY + E^*(X, PY)\xi, \tag{4.24}$$

$$D_X \xi = -\widetilde{A}_\xi^* X - \widetilde{p}_1(X)\xi, \tag{4.25}$$

where $D^*_X PY \ \ \widetilde{A}^*_\xi X \in \Gamma(S(TM))$, $E^*(X,PY) = \widetilde{g}(D_X\ PY, N)$ and $\widetilde{p}_1(X) = -\widetilde{g}(D_X\xi, N)$. From (2.13), (2.14), (4.24) and (4.25), we obtain

$$D^*_X PY = \nabla^*_X PY + \pi(PY)PfX, \tag{4.26}$$

$$E^*(X,PY) = E(X,PY) + \pi(PY)\eta(fX), \tag{4.27}$$

$$\widetilde{A}^*_\xi X = A^*_\xi X - \pi(\xi)PfX, \tag{4.28}$$

$$\widetilde{u}_1(X) = u_1(X) + \pi(\xi)\eta(fX). \tag{4.29}$$

Proposition 4.3. *Let M be a half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M},\widetilde{g}\right)$. Then D^* the induced connection is quarter-symmetric non-metric connection on $s(TM)$*

Proof. *For any $X, Y, Z \in \Gamma(s(TM))$, we know that ∇^* is metric connection. Thus from (4.26), we get*

$$(D^*_X g)(Y,Z) = -\pi(Y)g(PfX,Z) - \pi(Z)g(Y,PfX). \tag{4.30}$$

Let T^{D^*} be torsion tensor with respect to D^*. From (4.26), we obtain

$$T^{D^*}(X,Y) = \pi(Y)PfX - \pi(X)PfY. \tag{4.31}$$

Then from (4.30) and (4.31), we have proof.

We know that $\widetilde{\nabla}F = 0$. From (4.1) and (4.13) we obtain

$$(\widetilde{D}_X F)Y = \pi(FY)FX - \pi(Y)X, \tag{4.32}$$

and

$$(D_X f)Y = (\nabla_X f)Y + \pi(fY)fX - \pi(Y)f^2X. \tag{4.33}$$

From (4.32) and (4.33) we have the following propositions.

Proposition 4.4. *Let M be a half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M},\widetilde{g}\right)$. F is not parallel with respect to quarter-symmetric non-metric connection $\widetilde{D}$.*

Proposition 4.5. *Let M be a half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M},\widetilde{g}\right)$. f is not parallel with respect to quarter-symmetric non-metric connection D.*

From (4.14) we have

$$\begin{aligned}\widetilde{D}_1(X,Y) - \widetilde{D}_1(Y,X) &= D_1(X,Y) - D_1(Y,X) + g(\pi(Y)FX - \pi(X)FY, \xi)\\ &= g(\widetilde{T}(X,Y),\xi).\end{aligned} \tag{4.34}$$

Similarly from (4.15) we obtain

$$\widetilde{D}_2(X,Y) - \widetilde{D}_2(Y,X) = g(\widetilde{T}(X,Y),u). \tag{4.35}$$

From the (4.34) and (4.35) we have the following theorems

Theorem 4.1. *Let M be a half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M},\widetilde{g}\right)$. Then the lightlike second fundemental form $\widetilde{D}_1$ of quarter symmetric non-metric connection is symmetric if and only if there is no $ltrTM$ component of the torsion $\widetilde{T}$.*

Theorem 4.2. *Let M be a half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M}, \widetilde{g}\right)$. Then the screen second fundemental form $\widetilde{D}_2$ of quarter symmetric non-metric connection $\widetilde{D}$ is symmetric if and only if there is no $s(TM^{\perp})$ component of the torsion $\widetilde{T}$.*

Theorem 4.3. *Let M be a screen semi-invariant half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M}, \widetilde{g}\right)$. Then the second fundemental form of $s(TM)$ is symmetric with respect to quarter symmetric non-metric connection if and only if there is no $RadTM$ component of the torsion tesor T^D.*

Proof. *For any $X, Y \in \Gamma(s(TM))$, since E is symmetric, from (4.27) we obtain*

$$E^*(X,Y) - E^*(Y,X) = \pi(Y)\eta(fX) - \pi(X)\eta(fY) = g(T^D(X,Y), N).$$

Thus proof is completed.

Lemma 4.1. *Let M be a half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M}, \widetilde{g}\right)$. Then we have the following equation;*

$$\widetilde{D}_i(X,Y) = D_i(X,Y),\ i \in \{1,2\},\ \forall X \in \Gamma(L_0) \text{ and } Y \in \Gamma(TM)$$

.

Proof. *For any $X \in \Gamma(L_0)$, we know that $wX = 0$. Then from (4.14) and (4.15) proof is completed.*

From the above lemma we have the following theorem.

Theorem 4.4. *Let M be a half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M}, \widetilde{g}\right)$. Then M is L_0- totally geodesic with respect to quarter symmetric non-metric connection if and only if M is L_0- totally geodesic with respect to connection ∇.*

Theorem 4.5. *Let M be a screen semi-invariant half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M}, \widetilde{g}\right)$. Then the following equivalent;*

(i) *$L^{\perp}$ is integrable.*

(ii) *$\widetilde{A}_{FY}X = \widetilde{A}_{FX}Y$, $X, Y \in \Gamma(L^{\perp})$*

(iii) *E_1^* second fundemental form of $s(TM)$ with quarter symmetric non-metric connection is symmetric on $L^{\perp}$.*

Proof. *For any $X, Y \in \Gamma(L^{\perp})$ we obtain*

$$\begin{aligned} g([X,Y], FN) &= g(F[X,Y], N) \\ &= g(\widetilde{\nabla}_X FY - \widetilde{\nabla}_Y FX, N) \\ &= g(A_{FX}Y - A_{FY}X, N). \end{aligned}$$

and for any $Z \in \Gamma(L_0)$ we get

$$\begin{aligned} g([X,Y], Z) &= g(F[X,Y], FZ) \\ &= g(\widetilde{\nabla}_X FY - \widetilde{\nabla}_Y FX, FZ) \\ &= g(A_{FX}Y - A_{FY}X, FZ). \end{aligned}$$

From (4.16) ve (4.19) we know that

$$\widetilde{A}_{FY}X = A_{FY}X.$$

Thus we get $(i) \Leftrightarrow (ii)$.

From (4.27) we know that $E_1^*(X,Y) = E_1(X,Y)$ *and since teorem (2.1), we get* $(i) \Leftrightarrow (iii)$.

For any $X, Y, Z \in \Gamma(L^{\perp})$ from (2.15) and (4.22) we obtain

$$(\nabla_X g)(Y,Z) = 0. \tag{4.36}$$

and

$$(D_X g)(Y,Z) \quad = 0.$$

Thus we have the following proposition

Proposition 4.6. *Let* M *be a screen semi-invariant half-lightlike submanifold of a semi-Riemannian product manifold* $\left(\widetilde{M}, \widetilde{g}\right)$. *Then we have*

$$\nabla_X g = 0 \text{ and } D_X g = 0, \text{ for any } X, Y \in \Gamma(L^{\perp}).$$

Corollary 4.1. *Let* M *be a screen semi-invariant half-lightlike submanifold of a semi-Riemannian product manifold* $\left(\widetilde{M}, \widetilde{g}\right)$. *Then the following assertions are equivalent:*

(i) $\widetilde{D}_i(X,Y) = D_i(X,Y), i = 1,2,\ X, Y \in \Gamma(L)$

(ii) $\widetilde{D}_1$ *and* $\widetilde{D}_2$ *is symmetric on* L.

(iii) *If* M *is* $L-$ *totally geodesic then* M *is* $L-$ *totally geodesic with respect to quarter symmetric non-metric connection.*

(iv) *If* M *is* $L-$ *totally umbilic then* M *is* $L-$ *totally umbilic with respect to quarter symmetric non-metric connection.*

Proof. *For any* $X, Y \in \Gamma(L)$
since $w_1(X) = 0 = w_2(X)$, *we obtain*

$$\widetilde{D}_1(X,Y) = D_1(X,Y),$$

$$\widetilde{D}_2(X,Y) = D_2(X,Y).$$

Thus proof is completed.

Theorem 4.6. *Let* M *be a mixed geodesic semi-invariant half-lightlike submanifold of a screen semi-Riemannian product manifold* $\left(\widetilde{M}, \widetilde{g}\right)$. *Then for any* $X \in \Gamma(L)$ *and* $Y \in \Gamma(L^{\perp})$ *we have*

$$\widetilde{D}_i(X,Y) = 0,\ i = 1,2.$$

Proof. *For any* $X \in \Gamma(L)$ *and* $Y \in \Gamma(L^{\perp})$ *we obtain*

$$\widetilde{D}_1(X,Y) = \widetilde{g}(\widetilde{D}_X Y, \xi) = \widetilde{g}(\widetilde{\nabla}_X Y, \xi) = D_1(X,Y),$$

and

$$\widetilde{D}_2(X,Y) = \widetilde{g}(\widetilde{D}_X Y, u) = \widetilde{g}(\widetilde{\nabla}_X Y, u) = D_2(X,Y).$$

thus proof is completed.

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