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SPHERICAL PRODUCT SURFACES IN THE GALILEAN SPACE

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Abstract. In the present paper, we consider the spherical product surfaces in a Galilean 3-space $\mathbb{G}_3$. We derive a classification result for such surfaces of constant curvature in $\mathbb{G}_3$. Moreover, we analyze some special curves on these surfaces in $\mathbb{G}_3$.

1. Introduction

The tight embeddings of product spaces were investigated by N.H. Kuiper (see [17]) and he introduced a different tight embedding in the (n_1+n_2-1)−dimensional Euclidean space $\mathbb{R}^{n_1+n_2-1}$ as follows: Let

$$\begin{aligned} c_1 &: M^m \longrightarrow \mathbb{R}^{n_1}, \\ c_1(u_1,...,u_m) &= (f_1(u_1,...,u_m),...,f_{n_1}(u_1,...,u_m)) \end{aligned}$$

be a tight embedding of a m−dimensional manifold M^m satisfying Morse equality and

$$\begin{aligned} c_2 &: \mathbb{S}^{n_2-1} \longrightarrow \mathbb{R}^{n_2}, \\ c_1(v_1,...,v_{n_2-1}) &= (g_1(v_1,...,v_{n_2-1}),...,g_{n_2}(v_1,...,v_{n_2-1})) \end{aligned}$$

the standard embedding of (n_2-1)−sphere in $\mathbb{R}^{n_2}$, where $u=(u_1,...,u_m)$ and $v=(v_1,...,v_{n_2-1})$ are the local coordinate systems on M^m and $\mathbb{S}^{n_2-1}$, respectively. Then a new *tight embedding* is given by

$$\begin{aligned} \mathbf{x} = c_1 \otimes c_2 : M^m \times \mathbb{S}^{n_2-1} \longrightarrow \mathbb{R}^{n_1+n_2-1}, \\ (u,v) \longmapsto (f_1(u),...,f_{n_1-1}(u), f_{n_1}(u)g_1(v),...,f_{n_1}(u)g_{n_2}(v)). \end{aligned}$$

Such embeddings are obtained from c_1 by rotating $\mathbb{R}^{n_1}$ about $\mathbb{R}^{n_1-1}$ in $\mathbb{R}^{n_1+n_2-1}$ (cf. [4]).

B. Bulca et al. [6, 7] called such embeddings *rotational embeddings* and considered the spherical product surfaces in Euclidean spaces, which are a special type

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of the rotational embeddings as taking $m = 1, n_1 = 2, 3$ and $n_2 = 2$ in above definition.

The surfaces of revolution in $\mathbb{R}^3$ can be considered as simplest models of spherical product surfaces as well as the quadrics and the superquadrics [5].

On the other hand, the Galilean geometry is one model of the real Cayley-Klein geometries which has projective signature $(0, 0, +, +)$. In particular, the Galilean plane $\mathbb{G}_2$ is one of three Cayley-Klein planes (including Euclidean and Lorentzian planes) with a parabolic measure of distance. This projective-metric plane has an absolute figure $\{f, P\}$ for an absolute (ideal) line f and an absolute point P on f.

Many kind of surfaces in the (pseudo-) Galilean 3-space $\mathbb{G}_3$ (further details of $\mathbb{G}_3$ see Section 2) have been studied in [3], [8]-[10], [15, 16], [22]-[28] such as ruled surfaces, translation surfaces, tubular surfaces, etc.

In the present paper, we consider the spherical product surfaces of two Galilean plane curves in $\mathbb{G}_3$. We obtain several classifications for the spherical product surfaces of constant curvature in $\mathbb{G}_3$. Then some special curves on such surfaces are also analyzed.

2. Preliminaries

For later use, we provide a brief review of Galilean geometry from [12, 13], [18]-[28].

The Galilean 3-space $\mathbb{G}_3$ can be defined in three-dimensional real projective space $P_3(\mathbb{R})$ and its *absolute figure* is an ordered triple $\{\omega, f, I\}$, where ω is the ideal (absolute) plane, f a line in ω and I is the fixed elliptic involution of the points of f. The homogeneous coordinates in $\mathbb{G}_3$ is introduced in such a way that the ideal plane ω is given by $x_0 = 0$, the ideal line f by $x_0 = x_1 = 0$ and the elliptic involution by

$$(0 : 0 : x_2 : x_3) \longrightarrow (0 : 0 : x_3 : -x_2).$$

By means of the affine coordinates defined by $(x_0 : x_1 : x_2 : x_3) = (1 : x : y : z)$, the *similarity group* H_8 of $\mathbb{G}_3$ has the following form

$$\begin{aligned} \bar{x} &= a + bx \\ \bar{y} &= c + dx + r(\cos\theta)\, y + r(\sin\theta)\, z \\ \bar{z} &= e + fx + r(-\sin\theta)\, y + r(\cos\theta)\, z, \end{aligned}$$

where a, b, c, d, e, f, r and θ are real numbers. In particular, for $b = r = 1$, the group becomes the *group of isometries* (*proper motions*), $B_6 \subset H_8$, of $\mathbb{G}_3$.

A plane is called *Euclidean* if it contains f, otherwise it is called *isotropic*, i.e., the planes $x = const.$ are Euclidean, in particular the plane ω. Other planes are isotropic.

We introduce the metric relations with respect to the absolute figure. The *Galilean distance* between the points $P_i = (u_i, v_i, w_i)$ $(i = 1, 2)$ is given by

$$d(P_1, P_2) = \begin{cases} |u_2 - u_1|, & \text{if } u_1 \neq 0 \text{ or } u_2 \neq 0, \\ \sqrt{(v_2 - v_1)^2 + (w_2 - w_1)^2}, & \text{if } u_1 = 0 \text{ and } u_2 = 0. \end{cases}$$

The *Galilean scalar product* between two vectors $\mathbf{X} = (x_1, x_2, x_3)$ and $\mathbf{Y} = (y_1, y_2, y_3)$ is given by

$$\mathbf{X} \cdot \mathbf{Y} = \begin{cases} x_1 y_1, & \text{if } x_1 \neq 0 \text{ or } y_1 \neq 0, \\ x_2 y_2 + x_3 y_3, & \text{if } x_1 = 0 \text{ and } y_1 = 0. \end{cases}$$

In this sense, the *Galilean norm* of a vector $\mathbf{X}$ is $\|\mathbf{X}\| = \sqrt{\mathbf{X}\cdot\mathbf{X}}$. A vector $\mathbf{X} = (x_1, x_2, x_3)$ is called *isotropic* if $x_1 = 0$, otherwise it is called *non-isotropic*.

The *cross product* in the sense of Galilean space is

$$\mathbf{X}\times_{\mathbb{G}}\mathbf{Y} = \left(0, -\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}\right).$$

Let D be an open subset of $\mathbb{R}^2$ and M^2 a surface in $\mathbb{G}_3$ parametrized by

$$\mathbf{r} : D \longrightarrow \mathbb{G}_3, \ \ (u_1, u_2) \longmapsto (r_1(u_1, u_2), r_2(u_1, u_2), r_3(u_1, u_2)),$$

where r_k is a smooth real-valued function on D, $1 \le k \le 3$. Denote

$$(r_k)_{u_i} = \partial r_k/\partial u_i \text{ and } (r_k)_{u_i u_j} = \partial^2 r_k/\partial u_i \partial u_j, \ 1 \le k \le 3 \text{ and } 1 \le i, j \le 2.$$

Then such a surface is *admissible* (i.e., without Euclidean tangent planes) if and only if $(r_1)_{u_i} \neq 0$ for some $i = 1, 2$.

Let us introduce

$$g_i = (r_1)_{u_i}, \ h_{ij} = (r_2)_{u_i}(r_2)_{u_j} + (r_3)_{u_i}(r_3)_{u_j}, \ i, j = 1, 2.$$

Hence the first fundamental form of M^2 is

$$\mathbf{I} = ds_1^2 + \varepsilon ds_2^2,$$

where

$$ds_1^2 = (g_1 du_1 + g_2 du_2)^2, \ \ ds_2^2 = h_{11} du_1^2 + 2h_{12} du_1 du_2 + h_{22} du_2^2$$

and

$$\varepsilon = \begin{cases} 0 & \text{if the direction } du_1 : du_2 \text{ is non-isotropic,} \\ 1 & \text{if the direction } du_1 : du_2 \text{ is isotropic.} \end{cases}$$

Define the function w as

$$w = \sqrt{\left((r_1)_{u_2}(r_3)_{u_1} - (r_1)_{u_1}(r_3)_{u_2}\right)^2 + \left((r_1)_{u_1}(r_2)_{u_2} - (r_1)_{u_2}(r_2)_{u_1}\right)^2}.$$

Thus a side tangential vector $\mathbf{S}$ in the tangent plane of M^2 is defined by

$$\mathbf{S} = \frac{1}{w}\left(0, (r_1)_{u_2}(r_2)_{u_1} - (r_1)_{u_1}(r_2)_{u_2}, (r_1)_{u_2}(r_3)_{u_1} - (r_1)_{u_1}(r_3)_{u_2}\right). \tag{2.1}$$

The unit normal vector field $\mathbf{U}$ of M^2 is an isotropic vector field given by

$$\mathbf{U} = \frac{1}{w}\left(0, (r_1)_{u_2}(r_3)_{u_1} - (r_1)_{u_1}(r_3)_{u_2}, (r_1)_{u_1}(r_2)_{u_2} - (r_1)_{u_2}(r_2)_{u_1}\right). \tag{2.2}$$

In the sequel, the second fundamental form $\mathbf{II}$ of M^2 is

$$\mathbf{II} = L_{11} du_1^2 + 2L_{12} du_1 du_2 + L_{22} du_2^2,$$

where

$$\begin{aligned} L_{ij} &= \frac{1}{g_1}\left(g_1\left(0, (r_2)_{u_i u_j}, (r_3)_{u_i u_j}\right) - (g_i)_{u_j}\left(0, (r_2)_{u_1}, (r_3)_{u_1}\right)\right)\cdot\mathbf{U} \\ &= \frac{1}{g_2}\left(g_2\left(0, (r_2)_{u_i u_j}, (r_3)_{u_i u_j}\right) - (g_i)_{u_j}\left(0, (r_2)_{u_2}, (r_3)_{u_2}\right)\right)\cdot\mathbf{U}. \end{aligned}$$

A surface is called *totally geodesic* if its second fundamental form is identically zero.

The *third fundamental form* of M^2 is

$$\mathbf{III} = P_{11} du_1^2 + 2P_{12} du_1 du_2 + P_{22} du_2^2,$$

where

$$P_{11} = \mathbf{U}_{u_1}\cdot\mathbf{U}_{u_1}, \ P_{12} = \mathbf{U}_{u_1}\cdot\mathbf{U}_{u_2}, \ P_{22} = \mathbf{U}_{u_2}\cdot\mathbf{U}_{u_2}. \tag{2.3}$$

The *Gaussian curvature* K and the *mean curvature* H of M^2 are of the form

$$K = \frac{L_{11}L_{22} - L_{12}^2}{w^2} \text{ and } H = \frac{g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}}{2w^2}. \tag{2.4}$$

A surface in $\mathbb{G}_3$ is said to be *minimal* (resp. *flat*) if its mean curvature (resp. Gaussian curvature) vanishes.

3. Spherical product surfaces of constant curvature in $\mathbb{G}_3$

Let $c_i : I_i \subset \mathbb{R} \longrightarrow \mathbb{G}_2$, $i = 1, 2$, be two Galilean plane curves given by

$$c_1(u) = (p_1(u), p_2(u)) \text{ and } c_2(v) = (q_1(v), q_2(v)),$$

where p_i and q_i $(i = 1, 2)$ are respectively smooth real-valued non-constant functions on the intervals I_1 and I_2. Thus the *spherical product surface* M^2 of the two plane curves in $\mathbb{G}_3$ is defined by

$$\mathbf{r} := c_1 \otimes c_2 : I_1 \times I_2 \longrightarrow \mathbb{G}_3, \quad (u, v) \longmapsto (p_1(u), p_2(u) q_1(v), p_2(u) q_2(v)). \tag{3.1}$$

We call the curves c_1 and c_2 *generating curves.* Denote $p_i' = \frac{dp_i}{du}, q_i' = \frac{dq_i}{dv}$, etc. Since p_i and q_i are non-constant, M^2 is always admissible.

It follows from (2.1), (2.2) and (3.1) that the side tangent vector field $\mathbf{S}$ is

$$\mathbf{S} = \frac{1}{\sqrt{(q_1')^2 + (q_2')^2}} (0, -q_1', -q_2') \tag{3.2}$$

and the unit normal vector field $\mathbf{U}$ becomes

$$\mathbf{U} = \frac{1}{\sqrt{(q_1')^2 + (q_2')^2}} (0, -q_2', q_1'). \tag{3.3}$$

Remark 3.1. The equality (3.3) immediately implies from (2.3) that a spherical product surface in $\mathbb{G}_3$ has degenerate third fundamental form, i.e., $P_{11}P_{22} - P_{12}^2 = 0$.

For the coefficients of the first fundamental form, we have $g_1 = p_1'$ and $g_2 = 0$. Also the coefficients of the second fundamental form are

$$L_{11} = -\frac{(p_1')(q_1)^2}{\sqrt{(q_1')^2 + (q_2')^2}} \alpha' \beta', \; L_{12} = 0, \; L_{22} = \frac{p_2 (q_1')^2}{\sqrt{(q_1')^2 + (q_2')^2}} \gamma', \tag{3.4}$$

where

$$\alpha = \frac{p_2'}{p_1'}, \; \beta = \frac{q_2}{q_1}, \; \gamma = \frac{q_2'}{q_1'}. \tag{3.5}$$

Remark 3.2. It is easy to see that when c_2 is a line passing through the origin, then $\beta = const.$ and hence the spherical product surface is totally geodesic.

Therefore, the next results classify the spherical product surfaces in $\mathbb{G}_3$ with constant mean curvature and null Gaussian curvature.

Theorem 3.1. *There does not exist a spherical product surface in* $\mathbb{G}_3$ *with constant mean curvature except isotropic planes.*

Proof. Let M^2 be a spherical product surface given by (3.1) in $\mathbb{G}_3$ with constant mean curvature H_0. From (2.4), we have

$$2H_0 = \frac{(q_1')^2}{p_2\left((q_1')^2 + (q_2')^2\right)^{\frac{3}{2}}}\gamma'. \tag{3.6}$$

Then differentiating of (3.6) with respect to u yields that

$$0 = \frac{p_2'(q_1')^2}{-(p_2)^2\left((q_1')^2 + (q_2')^2\right)^{\frac{3}{2}}}\gamma'. \tag{3.7}$$

Since the functions p_i and q_i are non-constant functions, it follows from (3.7) that $\gamma' = 0$ and thus $H_0 = 0$. Considering $\gamma = const.$ in (3.5), then it turns to

$$q_2 = \lambda_1 q_1 + \lambda_2,\ \lambda_1 \neq 0, \tag{3.8}$$

which implies that c_2 is a line. Moreover, from (3.3), we have the constant unit normal vector field $\mathbf{U}$ as

$$\mathbf{U} = \frac{1}{\sqrt{1 + (\lambda_1)^2}}(0, -\lambda_1, 1),\ \lambda_1 \neq 0. \tag{3.9}$$

This means that the spherical product surface is an open part of an isotropic plane, which proves the theorem. □

Theorem 3.2. *A spherical product surface of the curves c_1 and c_2 in $\mathbb{G}_3$ is flat if and only if either it is an isotropic plane or the generating curve c_1 is a line.*

Proof. Assume that M^2 is a flat spherical product surface of the curves c_1 and c_2 in $\mathbb{G}_3$. For the Gaussian curvature K, by using (2.4), we get

$$0 = K = \frac{(q_1)^2(q_1')^2}{p_1'p_2\left((q_1')^2 + (q_2')^2\right)^2}\alpha'\beta'\gamma'.$$

Thus three cases occur:

Case (A) $\alpha = const.$ Then, we deduce

$$p_1 = \lambda_3 p_2 + \lambda_4,\ \lambda_3 \neq 0,$$

which implies that c_1 is a line.

Case (B) $\beta = const.$ Hence $\frac{q_2}{q_1} = const.$ for all $v \in I_2$ and the generating curve c_2 is a line passing through the origin. This gives that M^2 is a totally geodesic surface and an open part of an isotropic plane.

Case (C) $\gamma = const.$ This case was already analyzed via (3.8) and in this case M^2 is an open part of an isotropic plane.

Therefore the proof is completed. □

By using Theorem 3.1 and Theorem 3.2, we have the following classification result.

Corollary 3.1. ***(Classification)*** *For a spherical product surface M^2 of the curves c_1 and c_2 in $\mathbb{G}_3$, the following statements hold:*

(A) If c_1 is a line, then M^2 is flat but not minimal,

(B) If c_2 is a line passing through the origin, then M^2 is a totally geodesic surface and an open part of an isotropic plane,

(C) If c_2 is a line of the form $y = mx + n$, $m, n \neq 0$, then M^2 is an open part of an isotropic plane,

(D) There does not exist a spherical product surface with constant mean curvature except isotropic planes.

Example 3.1. Let us consider the spherical product surface of the Euclidean ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and the line $y = 0.5x + 2.5$. Thus we parametrize the surface being flat but not minimal as follows

$$\mathbf{r}(u, v) = (u - 3, (0.5u + 1)(2\sin v), (0.5u + 1)(3\cos v)), \ 0 \leq u, v \leq 2\pi.$$

We plot it as in Fig. 1.

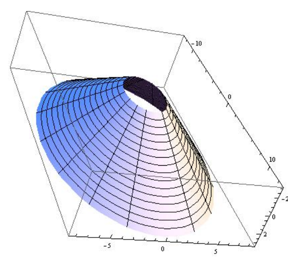

FIGURE 1. The flat spherical product surface of an Euclidean ellipse and a line, $K = 0$.

4. Curves on spherical product surfaces in $\mathbb{G}_3$

There exist a frame field, also called the *Darboux frame field,* for the curves lying on surfaces apart from the Frenet frame field. For details, see [11, 14]. Let γ be a curve lying on the surface M^2 with unit normal vector field $\mathbf{U}$. By taking $\mathbf{T} = \gamma_* \left(\frac{d}{dt}\right)$ one can get a new frame field $\{\mathbf{T}, \mathbf{T} \times \mathbf{U}, \mathbf{U}\}$ which is the Darboux frame field of γ with respect to M^2.

On the other hand, the second derivative $\ddot{\gamma}$ of the curve γ on M^2 has a component perpendicular to M^2 and a component tangent to M^2, i.e.,

$$\ddot{\gamma} = \tan(\ddot{\gamma}) + \mathrm{nor}(\ddot{\gamma}), \tag{4.1}$$

where the dot " $\cdot$ " denotes the derivative with respect to the parameter of the curve. The norms $\|\tan(\ddot{\gamma})\|$ and $\|\mathrm{nor}(\ddot{\gamma})\|$ are called the *geodesic curvature* and the *normal curvature* of γ on M^2, respectively. The curve γ is called *geodesic* (resp. *asymptotic line*) if and only if its geodesic curvature κ_g (resp. normal curvature κ_n) vanishes.

Let us consider the spherical product surface $\mathbf{r} = c_1 \otimes c_2$ in $\mathbb{G}_3$ given by (3.1). As in the previous section, put

$$c_1(u) = (p_1(u), p_2(u)) \text{ and } c_2(v) = (q_1(v), q_2(v)).$$

The geodesic curvatures of the $u-$parameter curves and $v-$parameter curves on $\mathbf{r} = c_1 \otimes c_2$ are respectively given by (see [10])

$$\kappa_g^u = \mathbf{S} \cdot \mathbf{r}_{uu} = \begin{cases} 0, & \text{if } p_1 \text{ is non-linear} \\ \frac{-p_2''(q_1 q_1' + q_2 q_2')}{\sqrt{(q_1')^2 + (q_2')^2}}, & \text{if } p_1 \text{ is linear} \end{cases} \tag{4.1}$$

and

$$\kappa_g^v = \mathbf{S}\cdot\mathbf{r}_{vv} = \frac{-p_2\left(q_1'q_1'' + q_2'q_2''\right)}{\sqrt{\left(q_1'\right)^2 + \left(q_2'\right)^2}}. \tag{4.2}$$

By considering (4.1) and (4.2), we derive the following result.

Theorem 4.1. *Let M^2 be a spherical product surface of the curves $c_1(u) = (p_1(u), p_2(u))$ and $c_2(v) = (q_1(v), q_2(v))$ in $\mathbb{G}_3$. Then we have*

(A) If p_1 is a non-linear function, then the $u-$parameter curves are geodesic lines. Otherwise (when p_1 is a linear function) the $u-$ parameter curves are geodesic lines if and only if either

(A.1) p_2 is a linear function, or

(A.2) c_2 is an Euclidean circle.

(B) The $v-$ parameter curves are geodesic lines if and only if c_2 is curve satisfying the equation

$$q_1 = \pm\int\sqrt{\lambda_2 - \left(q_2'\right)^2}dv.$$

Proof. From (4.1), the statement (A) of the theorem is clear. Now let assume that p_1 is a linear function. Then, by (4.1), we deduce that the $u-$parameter curves are geodesic lines (i.e. κ_g^u vanishes) if and only if either p_2 is a linear function (this implies the statement (A.1) of the theorem) or

$$q_1q_1' + q_2q_2' = 0. \tag{4.3}$$

From (4.3), we conclude $q_1^2 + q_2^2 = \lambda_1$ for some constant $\lambda_1 > 0$. It means that c_2 is an Euclidean circle with radius $\sqrt{\lambda_1}$ and centered at origin. This proves the statement (A.2) of the theorem.

If κ_g^v is equivalently zero, then we have from (4.2) that $q_1'q_1'' + q_2'q_2'' = 0$, i.e.,

$$q_1 = \pm\int\sqrt{\lambda_2 - \left(q_2'\right)^2}dv,$$

which completes the proof. □

The normal curvatures of the parameter curves on $\mathbf{r} = c_1 \otimes c_2$ (see [10]) are respectively given by

$$\kappa_n^u = \mathbf{U}\cdot\mathbf{r}_{uu} = \begin{cases} 0, & \text{if } p_1 \text{ is non-linear} \\ \frac{-p_2''\left(q_1q_2' - q_1'q_2\right)}{\sqrt{\left(q_1'\right)^2+\left(q_2'\right)^2}}, & \text{if } p_1 \text{ is linear} \end{cases} \tag{4.4}$$

and

$$\kappa_n^v = \mathbf{U}\cdot\mathbf{r}_{vv} = \frac{p_2\left(q_1'q_2'' - q_1''q_2'\right)}{\sqrt{\left(q_1'\right)^2 + \left(q_2'\right)^2}}. \tag{4.5}$$

Theorem 4.2. *Let M^2 be a spherical product surface of the curves $c_1(u) = (p_1(u), p_2(u))$ and $c_2(v) = (q_1(v), q_2(v))$ in $\mathbb{G}_3$. Then we have the following:*

(A) If p_1 is a non-linear function, then the $u-$parameter curves are asymptotic lines. Otherwise (when p_1 is a linear function) the $u-$ parameter curves are asymptotic lines if and only if either

(A.1) p_2 is a linear function, or

(A.2) M^2 is a totally geodesic surface.

(B) The $v-$ parameter curves are asymptotic lines if and only if M^2 is an open part of an isotropic plane.

Proof. From (4.4), the statement (A) of the theorem is obvious. If p_1 is a linear function, then by (4.4) we derive that the $u-$parameter curves are asymptotic lines if and only if either p_2 is a linear function (it gives the proof of the statement (A.1) of the theorem), or

$$q_1 q_2' - q_1' q_2 = 0. \tag{4.6}$$

It follows from (4.6) that $q_2 = \lambda_1 q_1$ for nonzero constant λ_1. Considering Remark 3.2 implies that M^2 is totally geodesic surface, which proves the statement (A.2).

Also, in case when $v-$parameter curves are asymptotic lines, from (4.5), the following satisfies

$$q_2 = \lambda_2 q_1 + \lambda_3, \ \lambda_2 \neq 0. \tag{4.7}$$

From (3.3), the equality (4.7) implies the statement (B) of the theorem.

Thus the proof is completed. □

A curve γ on a regular surface M^2 is called a *principal curve* if and only if the its velocity vector field always points in a principal direction. Moreover, a surface M^2 is called a *principal surface* if and only if its parameter curves are principal curves (cf. [14]).

A principal curve γ on a surface in $\mathbb{G}_3$ is determined by the following formula

$$\det\left(\dot{\gamma}, \mathbf{U}, \dot{\mathbf{U}}\right) = 0, \tag{4.8}$$

where $\mathbf{U}$ is the unit normal vector field of the surface (see [10]). Considering (3.1), (3.3) and (4.8), we immediately derive

$$\det(\mathbf{r}_u, \mathbf{U}, \mathbf{U}_u) = 0 \text{ and } \det(\mathbf{r}_v, \mathbf{U}, \mathbf{U}_v) = 0,$$

which yields the following.

Corollary 4.1. *The spherical product surfaces in $\mathbb{G}_3$ are principal ones.*

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