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INEQUALITIES OF HERMITE-HADAMARD TYPE FOR φ-CONVEX FUNCTIONS

S. S. DRAGOMIR[1,2]

Abstract. Some inequalities of Hermite-Hadamard type for φ-convex functions defined on real intervals are given.

1. Introduction

We recall here some concepts of convexity that are well known in the literature. Let I be an interval in $\mathbb{R}$.

Definition 1.1 ([37])**.** We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$ we have

$$f(tx + (1 - t)y) \leq \frac{1}{t} f(x) + \frac{1}{1 - t} f(y). \tag{1.1}$$

Some further properties of this class of functions can be found in [28], [29], [31], [43], [46] and [47]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

Definition 1.2 ([31])**.** We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0, 1]$ we have

$$f(tx + (1 - t)y) \leq f(x) + f(y). \tag{1.2}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$f(tx + (1 - t)y) \leq \max\{f(x), f(y)\} \tag{1.3}$$

for all $x, y \in I$ and $t \in [0, 1]$.

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For some results on P-functions see [31] and [44] while for quasi convex functions, the reader can consult [30].

Definition 1.3 ([7])**.** Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

For some properties of this class of functions see [1], [2], [7], [8], [26], [27], [38], [40] and [49].

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}$, $(0,1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 1.4 ([52])**.** Let $h : J \to [0,\infty)$ with h not identical to 0. We say that $f : I \to [0,\infty)$ is an h-convex function if for all $x, y \in I$ we have

$$f(tx + (1-t)y) \le h(t) f(x) + h(1-t) f(y) \tag{1.4}$$

for all $t \in (0,1)$.

For some results concerning this class of functions see [52], [6], [41], [50], [48] and [51].

We can introduce now another class of functions.

Definition 1.5. We say that the function $f : I \to [0,\infty)$ is of s-Godunova-Levin type, with $s \in [0,1]$, if

$$f(tx + (1-t)y) \le \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{1.5}$$

for all $t \in (0,1)$ and $x, y \in I$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(I)$ the class of s-Godunova-Levin functions defined on I, then we obviously have

$$P(I) = Q_0(I) \subseteq Q_{s_1}(I) \subseteq Q_{s_2}(I) \subseteq Q_1(I) = Q(I)$$

for $0 \le s_1 \le s_2 \le 1$.

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a) f\left(\frac{a+b}{2}\right) < \int_a^b f(x)dx < (b-a)\frac{f(a)+f(b)}{2}, \quad a, b \in \mathbb{R}. \tag{1.6}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [42]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [5]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [42]. Since (1.6) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

For related results, see [10]-[19], [22]-[25], [32]-[35] and [45].

The following inequality of Hermite-Hadamard type holds [48]

Theorem 1.1. *Assume that the function $f : I \to [0,\infty)$ is an h-convex function with $h \in L[0,1]$. Let $y, x \in I$ with $y \neq x$ and assume that the mapping $[0,1] \ni t \mapsto f[(1-t)x + ty]$ is Lebesgue integrable on $[0,1]$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(u)\,du \leq [f(x) + f(y)]\int_0^1 h(t)\,dt. \tag{1.7}$$

If we write (1.7) for $h(t) = t$, then we get the classical Hermite-Hadamard inequality for convex functions

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(u)\,du \leq \frac{f(x)+f(y)}{2}. \tag{1.8}$$

If we write (1.7) for the case of P-type functions $f : I \to [0,\infty)$, i.e., $h(t) = 1, t \in [0,1]$, then we get the inequality

$$\frac{1}{2} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(u)\,du \leq f(x) + f(y), \tag{1.9}$$

that has been obtained for functions of real variable in [31].

If f is Breckner s-convex on I, for $s \in (0,1)$, then by taking $h(t) = t^s$ in (1.7) we get

$$2^{s-1} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(u)\,du \leq \frac{f(x)+f(y)}{s+1}, \tag{1.10}$$

that was obtained for functions of a real variable in [26].

If $f : I \to [0,\infty)$ is of s-Godunova-Levin type, with $s \in [0,1)$, then

$$\frac{1}{2^{s+1}} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(u)\,du \leq \frac{f(x)+f(y)}{1-s}. \tag{1.11}$$

We notice that for $s = 1$ the first inequality in (1.11) still holds, i.e.

$$\frac{1}{4} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]\,dt. \tag{1.12}$$

The case for functions of real variables was obtained for the first time in [31].

2. φ-Convex Functions

We introduce the following class of h-convex functions.

Definition 2.1. Let $\varphi : (0,1) \to (0,\infty)$ a measurable function. We say that the function $f : I \to [0,\infty)$ is a φ-convex function on the interval I if for all $x, y \in I$ we have

$$f(tx + (1-t)y) \leq t\varphi(t) f(x) + (1-t)\varphi(1-t) f(y) \tag{2.1}$$

for all $t \in (0,1)$.

If we denote $\ell(t) = t$, the identity function, then it is obvious that f is h-convex with $h = \ell\varphi$. Also, all the examples from the introduction can be seen as φ-convex functions with appropriate choices of φ.

If we take $\varphi(t) = \frac{1}{t^{s+1}}$ with $s \in [0,1]$ then we get the class of s-Godunova-Levin functions. Also, if we put $\varphi(t) = t^{s-1}$ with $s \in (0,1)$, then we get the concept of Breckner s-convexity. We notice that for all these examples we have

$$\varphi_+(0) := \lim_{t \to 0+} \varphi(t) = \infty.$$

The case of convex functions, i.e. when $\varphi(t) = 1$ is the only example from above for which $\varphi_+(0)$ is finite, namely $\varphi_+(0) = 1$.

Consider the family of functions, for $p > 1$ and $k > 0$

$$\delta(p,k) : [0,1] \to \mathbb{R}_+,\ \delta(p,k)(t) = k(1-t)^p + 1. \tag{2.2}$$

We observe that $\delta_+(p,k)(0) = \delta(p,k)(0) = k+1$, $\delta(p,k)$ is strictly decreasing on $[0,1]$ and $\delta(p,k)(t) \geq \delta(p,k)(1) = 1$.

Definition 2.2. We say that the function $f : I \to [0,\infty)$ is a $\delta(p,k)$-convex function on the interval I if for all $x, y \in I$ we have

$$f(tx + (1-t)y) \leq t\left[k(1-t)^p + 1\right] f(x) + (1-t)\left(kt^p + 1\right) f(y) \tag{2.3}$$

for all $t \in (0,1)$.

It is obvious that any nonnegative convex function is a $\delta^{(p,k)}$-convex function for any $p > 1$ and $k > 0$.

For $m > 0$ we consider the family of functions

$$\eta(m) : [0,1] \to \mathbb{R}_+, \eta(m)(t) := \exp\left[m(1-t)\right].$$

We observe that $\eta_+(m)(0) = \eta(m)(0) = \exp(m)$, $\eta(m)$ is strictly decreasing on $[0,1]$ and $\eta(m)(t) \geq \eta(m)(1) = 1$.

Definition 2.3. We say that the function $f : I \to [0,\infty)$ is a $\eta(m)$-convex function on the interval I if for all $x, y \in I$ we have

$$f(tx + (1-t)y) \leq t\exp\left[m(1-t)\right] f(x) + (1-t)\exp(mt) f(y) \tag{2.4}$$

for all $t \in (0,1)$.

It is obvious that any nonnegative convex function is a $\eta(m)$-convex function for any $m > 0$.

There are many other examples one can consider. In fact any continuos function $\varphi : [0,1] \to [1,\infty)$ can generate a class of φ-convex function that contains the class of nonnegative convex functions.

Utilising Theorem 1.1 we can state the following result.

Theorem 2.1. *Assume that the function $f : I \to [0,\infty)$ is a φ-convex function with $\ell\varphi \in L[0,1]$. Let $y, x \in I$ with $y \neq x$ and assume that the mapping $[0,1] \ni t \mapsto f[(1-t)x + ty]$ is Lebesgue integrable on $[0,1]$. Then*

$$\frac{1}{\varphi\left(\frac{1}{2}\right)} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(u)\,du \leq \left[f(x) + f(y)\right]\int_0^1 t\varphi(t)\,dt. \tag{2.5}$$

The proof follows from (1.7) by taking $h(t) = t\varphi(t)$, $t \in (0,1)$.

Remark 2.1. We notice that, since $\int_0^1 t\varphi(t)\,dt$ can be seen as the expectation of a random variable X with the density function φ, the inequality (2.5) provides a connection to Probability Theory and motivates the introduction of φ-convex function as a natural concept, having available many examples of density functions φ that arise in applications.

We have the following particular cases:

Corollary 2.1. *Assume that the function $f : I \to [0, \infty)$ is a a $\delta(p,k)$-convex function on the interval I with $p > 1$ and $k > 0$. Let $y, x \in I$ with $y \neq x$ and assume that the mapping $[0,1] \ni t \mapsto f[(1-t)x + ty]$ is Lebesgue integrable on $[0,1]$. Then*

$$\frac{2^p}{k+2^p} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(u)\,du \leq [f(x)+f(y)]\left[\frac{1}{2}+\frac{k}{(p+1)(p+2)}\right]. \tag{2.6}$$

Proof. For $\varphi(t) = k(1-t)^p + 1$ we have $\varphi\left(\frac{1}{2}\right) = \frac{k+2^p}{2^p}$ and

$$\int_0^1 t\varphi(t)\,dt = \int_0^1 (1-t)\varphi(1-t)\,dt = \int_0^1 (1-t)(kt^p+1)\,dt = k\int_0^1 \left(t^p - t^{p+1}\right)dt + \frac{1}{2} = \frac{k}{(p+1)(p+2)} + \frac{1}{2},$$

and utilizing (2.5) we get (2.6). □

and

Corollary 2.2. *Assume that the function $f : I \to [0,\infty)$ is a $\eta(m)$-convex function on the interval I with $m > 0$. Let $y, x \in I$ with $y \neq x$ and assume that the mapping $[0,1] \ni t \mapsto f[(1-t)x + ty]$ is Lebesgue integrable on $[0,1]$. Then*

$$e^{-\frac{m}{2}} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(u)\,du \leq \frac{e^m - m - 1}{m^2}[f(x)+f(y)]. \tag{2.7}$$

Proof. For $\varphi(t) = \exp[m(1-t)]$ we have $\varphi\left(\frac{1}{2}\right) = e^{\frac{m}{2}}$ and

$$\int_0^1 t\varphi(t)\,dt = \int_0^1 (1-t)\varphi(1-t)\,dt = \int_0^1 (1-t)e^{mt}dt = \frac{1}{m}\int_0^1 (1-t)\,d\left(e^{mt}\right) = \frac{1}{m}\left[(1-t)e^{mt}\Big|_0^1 + \int_0^1 e^{mt}dt\right] = \frac{1}{m}\left[-1+\frac{1}{m}(e^m-1)\right] = \frac{e^m-m-1}{m^2}$$

and utilizing (2.5) we get (2.7). □

3. Some Results for Differentiable Functions

If we assume that the function $f : I \to [0,\infty)$ is differentiable on the interior of I denoted by $\mathring{I}$ then we have the following *"gradient inequality"* that will play an essential role in the following.

Theorem 3.1. *Let $\varphi : (0,1) \to (0,\infty)$ a measurable function and such that the right limit $\varphi_+(0)$ exists and is finite, the left limit $\varphi_-(1) = 1$ and the left derivative in 1 denoted $\varphi'_-(1)$ exists and is finite. If the function $f : I \to [0,\infty)$ is differentiable on $\mathring{I}$ and φ-convex, then*

$$\varphi_+(0) f(x) - \left[\varphi'_-(1)+1\right] f(y) \geq f'(y)(x-y) \tag{3.1}$$

for any $x, y \in \mathring{I}$ with $x \neq y$.

Proof. Since f is φ-convex on I, then

$$t\varphi(t) f(x) + (1-t)\varphi(1-t) f(y) \geq f(tx + (1-t)y)$$

for any $t \in (0,1)$ and for any $x, y \in \mathring{I}$, which is equivalent to

$$t\varphi(t) f(x) + [(1-t)\varphi(1-t) - 1] f(y) \geq f(tx + (1-t)y) - f(y)$$

and by dividing by $t > 0$ we get

$$\varphi(t) f(x) + \left[\frac{(1-t)\varphi(1-t) - 1}{t}\right] f(y) \geq \frac{f(tx + (1-t)y) - f(y)}{t} \tag{3.2}$$

for any $t \in (0,1)$.

Now, since f is differentiable on $y \in \mathring{I}$, then we have

$$\begin{aligned} \lim_{t\to 0+} \frac{f(tx + (1-t)y) - f(y)}{t} &= \lim_{t\to 0+} \frac{f(y + t(x-y)) - f(y)}{t} \\ &= (x-y) \lim_{t\to 0+} \frac{f(y + t(x-y)) - f(y)}{t(x-y)} \\ &= (x-y) f'(y) \end{aligned} \tag{3.3}$$

for any $x \in \mathring{I}$ with $x \neq y$.

Also since $\varphi_-(1) = 1$ and $\varphi'_-(1)$ exists and is finite, we have

$$\begin{aligned} \lim_{t\to 0+} \frac{(1-t)\varphi(1-t) - 1}{t} &= \lim_{s\to 1-} \frac{s\varphi(s) - 1}{1-s} = -\lim_{s\to 1-} \frac{s\varphi(s) - 1}{s-1} \\ &= -\lim_{s\to 1-} \frac{s(\varphi(s) - \varphi(1)) + s - 1}{s-1} \\ &= -\varphi'_-(1) - 1. \end{aligned} \tag{3.4}$$

Taking the limit over $t \to 0+$ in (3.2) and utilizing (3.3) and (3.4) we get the desired result (3.1). □

Remark 3.1. If we assume that

$$\varphi_+(0) - \varphi_-(1) \geq \varphi'_-(1), \tag{3.5}$$

then the inequality (3.1) also holds for $x = y$.

There are numerous examples of such functions, for instance, if , as above. we take $\varphi(t) = k(1-t)^p + 1$, $t \in [0,1]$ $(p > 1, k > 0)$ then $\varphi_+(0) = k+1$, $\varphi_-(1) = 1$ and $\varphi'_-(1) = 0$, which satisfy the condition (3.5).

If we take $\varphi(t) = \exp[m(1-t)]$ $(m > 0)$, then $\varphi_+(0) = \exp m$, $\varphi_-(1) = 1$ and $\varphi'_-(1) = -m$. We have

$$\varphi_+(0) - \varphi_-(1) - \varphi'_-(1) = e^m - 1 + m > 0$$

for $m > 0$.

The following result holds:

Theorem 3.2. *Let $\varphi : (0,1) \to (0,\infty)$ a measurable function and such that the right limit $\varphi_+(0)$ exists and is finite, the left limit $\varphi_-(1) = 1$ and the left derivative in* 1 *denoted $\varphi'_-(1)$ exists and is finite. Assume also that $\varphi'_-(1) > -1$. If the function $f : I \to [0,\infty)$ is differentiable on $\mathring{I}$ and φ-convex, then*

$$\frac{\varphi_+(0)}{\varphi'_-(1) + 1} \cdot \frac{f(x) + f(y)}{2} \geq \frac{1}{y-x} \int_x^y f(u)\, du \geq \frac{\varphi'_-(1) + 1}{\varphi_+(0)} f\left(\frac{x+y}{2}\right) \tag{3.6}$$

for any $x, y \in I$.

Proof. Assume that $y > x$ with $x, y \in I$. From (3.1) we get

$$\varphi_+(0) f(u) - \left[\varphi'_-(1) + 1\right] f\left(\frac{x+y}{2}\right) \geq f'\left(\frac{x+y}{2}\right)\left(x - \frac{x+y}{2}\right)$$

for any $u \in [x, y]$ with $u \neq \frac{x+y}{2}$.

Integrating this inequality over u on $[x, y]$ we get

$$\begin{aligned}&\varphi_+(0) \int_x^y f(u)\,du - \left[\varphi'_-(1) + 1\right](y - x) f\left(\frac{x+y}{2}\right)\\&\geq f'\left(\frac{x+y}{2}\right) \int_x^y \left(u - \frac{x+y}{2}\right) du = 0\end{aligned}$$

which implies (3.6).

The case $y < x$ goes likewise and the proof of the second inequality in (3.6) is completed.

Assume that $y > x$ with $x, y \in I$. From (3.1) we get

$$\begin{aligned}&\varphi_+(0) f(x) - \left[\varphi'_-(1) + 1\right] f((1-t)x + ty)\\&\geq f'((1-t)x + ty)(x - (1-t)x - ty)\\&= t f'((1-t)x + ty)(x - y)\end{aligned} \tag{3.7}$$

for any $t \in (0, 1)$ and

$$\begin{aligned}&\varphi_+(0) f(y) - \left[\varphi'_-(1) + 1\right] f((1-t)x + ty)\\&\geq f'((1-t)x + ty)(y - (1-t)x - ty)\\&= (1-t) f'((1-t)x + ty)(y - x)\end{aligned} \tag{3.8}$$

for any $t \in (0, 1)$.

Now, if we multiply (3.7) by $1 - t$, (3.8) by t and add the obtained inequalities, then we get

$$\varphi_+(0)\left[(1-t) f(x) + t f(y)\right] \geq \left[\varphi'_-(1) + 1\right] f((1-t)x + ty) \tag{3.9}$$

for any $t \in (0, 1)$, that is of interest in itself as well.

Now, if we integrate this inequality on $[0, 1]$ we get

$$\begin{aligned}&\varphi_+(0)\left[f(x)\int_0^1 (1-t)\,dt + f(y)\int_0^1 t dt\right]\\&\geq \left[\varphi'_-(1) + 1\right]\int_0^1 f((1-t)x + ty)\,dt.\end{aligned} \tag{3.10}$$

Since

$$\int_0^1 (1-t)\,dt = \int_0^1 t dt = \frac{1}{2}$$

and

$$\int_0^1 f((1-t)x + ty)\,dt = \frac{1}{y-x}\int_x^y f(u)\,du,$$

then by (3.11) we get the desired inequality (3.7). □

Remark 3.2. Since the function f takes nonnegative values, then the second inequality in (3.6) and the inequality (3.10) are trivially satisfied if $\varphi'_-(1) + 1 \le 0$, so we must assume that $\varphi'_-(1) + 1 > 0$.

This condition is satisfied for the function $\varphi(t) = k(1-t)^p + 1$, $t \in [0,1]$ $(p > 1, k > 0)$. If $\varphi(t) = \exp[m(1-t)]$ $(m > 0)$ then the condition $\varphi'_-(1)+1 = 1-m > 0$ is satisfied only for $m \in (0,1)$.

Now, if we write the inequality (3.6) for $\varphi(t) = k(1-t)^p + 1$,we get

$$(k+1)\frac{f(x)+f(y)}{2} \ge \frac{1}{y-x}\int_x^y f(u)\,du \ge \frac{1}{k+1} f\left(\frac{x+y}{2}\right) \tag{3.11}$$

From (2.6) we also have

$$\begin{aligned}[f(x)+f(y)]\left[\frac{1}{2}+\frac{k}{(p+1)(p+2)}\right] &\ge \frac{1}{y-x}\int_x^y f(u)\,du \\ &\ge \frac{2^p}{k+2^p} f\left(\frac{x+y}{2}\right).\end{aligned} \tag{3.12}$$

Since

$$\frac{2^p}{k+2^p} - \frac{1}{k+1} = \frac{2^p k + 2^p - k - 2^p}{(k+2^p)(k+1)} = \frac{(2^p-1)k}{(k+2^p)(k+1)} \ge 0$$

and

$$\frac{k+1}{2} - \frac{1}{2} - \frac{k}{(p+1)(p+2)} = \frac{k}{2} - \frac{k}{(p+1)(p+2)} \ge 0$$

it follows that the inequality (3.12) is better than (3.11).

Now, consider the family of functions

$$\vartheta(k,p,q) := kt^p(1-t)^q + 1$$

where $k > 0, p > 0$ and $q > 1$.

Definition 3.1. We say that the function $f : I \to [0,\infty)$ is a $\vartheta(k,p,q)$-convex function on the interval I if for all $x, y \in I$ we have

$$f(tx+(1-t)y) \le t\left[kt^p(1-t)^q+1\right]f(x) + (1-t)\left[k(1-t)^p t^q + 1\right]f(y) \tag{3.13}$$

for all $t \in (0,1)$.

We observe that this class contains the class of nonnegative convex functions for any $k > 0, p > 0$ and $q > 1$.

Corollary 3.1. *If the function $f : I \to [0,\infty)$ is differentiable on $\mathring{I}$ and $\vartheta(k,p,q)$-convex with $k > 0, p > 0$ and $q > 1$ then*

$$\frac{f(x)+f(y)}{2} \ge \frac{1}{y-x}\int_x^y f(u)\,du \ge f\left(\frac{x+y}{2}\right) \tag{3.14}$$

for any $x, y \in I$.

If we write the inequality (2.5) for $\varphi = \vartheta(k,p,q)$, then we get

$$\begin{aligned}\frac{1}{k\left(\frac{1}{2}\right)^{p+q}+1} f\left(\frac{x+y}{2}\right) &\le \frac{1}{y-x}\int_x^y f(u)\,du \\ &\le [f(x)+f(y)]\left[k\beta(p+2,q+1)+\frac{1}{2}\right]\end{aligned} \tag{3.15}$$

where

$$\beta(u,v) := \int_a^1 t^{u-1}(1-t)^{v-1}, u,v>0$$

is Euler's Beta function.

Since

$$\frac{1}{k\left(\frac{1}{2}\right)^{p+q}+1} < 1 \text{ and } k\beta(p+2,q+1)+\frac{1}{2} > \frac{1}{2},$$

it follows that the inequality (3.14) is better than (3.15).

Now, more generally, assume that

$$\varphi(g,q):[0,1]\to[1,\infty),\ \ \varphi(g,q)(t)=g(t)(1-t)^q+1$$

where $g:[0,1]\to[0,\infty)$ is continuous and $q>1$.

We then have

$$\varphi_+(g,q)(0)=g(0)+1,\ \varphi_-(g,q)(1)=1,\ \varphi'_-(g,q)(1)=0$$

and

$$\varphi\left(\frac{1}{2}\right)=g\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^q+1,\ \ \int_0^1 t\varphi(t)\,dt=\int_0^1 t(1-t)^q g(t)\,dt+\frac{1}{2}.$$

If we apply Theorem 2.1 to the function $\varphi(g,q)$ we have

$$\begin{aligned}(3.16)\qquad [f(x)+f(y)]\left[\int_0^1 t(1-t)^q g(t)\,dt+\frac{1}{2}\right] &\geq \frac{1}{y-x}\int_x^y f(u)\,du\\ &\geq \frac{1}{g\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^q+1}f\left(\frac{x+y}{2}\right).\end{aligned}$$

If we apply Theorem 3.2 to the same function $\varphi(g,q)$ we also have

$$\begin{aligned}(3.17)\qquad (g(0)+1)\frac{f(x)+f(y)}{2} &\geq \frac{1}{y-x}\int_x^y f(u)\,du\\ &\geq \frac{1}{g(0)+1}f\left(\frac{x+y}{2}\right).\end{aligned}$$

Consider the difference

$$\begin{aligned}\Delta_1 &:= \frac{1}{g(0)+1}-\frac{1}{g\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^q+1}\\ &= \frac{g\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^q-g(0)}{[g(0)+1]\left[g\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^q+1\right]}\end{aligned}$$

and the difference

$$\begin{aligned}\Delta_2 &:= \int_0^1 t(1-t)^q g(t)\,dt+\frac{1}{2}-\frac{g(0)+1}{2}\\ &= \int_0^1 t(1-t)^q g(t)\,dt-\frac{1}{2}g(0).\end{aligned}$$

We observe that if $\Delta_1,\Delta_2 \geq (\leq)\,0$ then the double inequality (3.17) is better (worse) than (3.17).

If we take $g(0)=0$, then (3.17) is better than (3.16) for any $q>1$.

If we take $g(t) = kt + 1$, $k > 0$ then

$$\Delta_1 = \frac{g\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^q - g(0)}{[g(0)+1]\left[g\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^q + 1\right]} = \frac{k\left(\frac{1}{2}\right)^{q+1}}{k\left(\frac{1}{2}\right)^{q+1} + 1} > 0$$

showing that the second inequality in (3.17) is better than the same inequality in (3.16) for any $k > 0$ and $q > 1$.

We also have

$$\begin{aligned}\Delta_2 &= \int_0^1 t(1-t)^q g(t)\,dt - \frac{1}{2}g(0) = \int_0^1 t(1-t)^q (kt+1)\,dt - \frac{1}{2} \\ &= k\int_0^1 t^2(1-t)^q\,dt + \int_0^1 t(1-t)^q\,dt - \frac{1}{2} \\ &= k\beta(3, q+1) + \beta(2, q+1) - \frac{1}{2}.\end{aligned}$$

If we take

$$\begin{aligned}k &> \frac{\frac{1}{2} - \beta(2, q+1)}{\beta(3, q+1)} = \frac{\frac{1}{2} - \frac{1}{(q+1)(q+2)}}{\beta(3, q+1)} \\ &= \frac{(q+1)(q+2) - 2}{2(q+1)(q+2)\beta(3, q+1)}\ (> 0)\end{aligned}$$

then $\Delta_2 > 0$ showing that the first inequality in (3.17) is better than the first inequality in (3.16).

If we take

$$0 < k < \frac{(q+1)(q+2) - 2}{2(q+1)(q+2)\beta(3, q+1)}$$

then $\Delta_2 < 0$ showing that the first inequality in (3.17) is worse than the first inequality in (3.16).

Conclusion 1. *The inequalities (2.5) and (3.6) are not comparable, meaning that some time one is better then the other, depending on the φ-convex function involved.*

4. Some Related Results

If we apply Theorem 2.1 on the subintervals $\left[x, \frac{x+y}{2}\right]$ and $\left[\frac{x+y}{2}, y\right]$ (provided $x < y$) and add the corresponding inequalities we get:

Proposition 4.1. *Assume that the function $f : I \to [0, \infty)$ is a φ-convex function with $\ell\varphi \in L[0,1]$. Let $y, x \in I$ with $y \neq x$ and assume that the mappings $[0,1] \ni t \mapsto f\left[(1-t)x + t\frac{x+y}{2}\right], f\left[(1-t)\frac{x+y}{2} + ty\right]$ are Lebesgue integrable on $[0,1]$. Then*

$$\begin{aligned}&\frac{1}{\varphi\left(\frac{1}{2}\right)}\left[f\left(\frac{3x+y}{4}\right) + f\left(\frac{x+3y}{4}\right)\right] \\ &\leq \frac{1}{y-x}\int_x^y f(u)\,du \leq \left[f\left(\frac{x+y}{2}\right) + \frac{f(x)+f(y)}{2}\right]\int_0^1 t\varphi(t)\,dt.\end{aligned} \tag{4.1}$$

Also, by Theorem 3.2 we have

Proposition 4.2. *Let $\varphi : (0,1) \to (0, \infty)$ a measurable function and such that the right limit $\varphi_+(0)$ exists and is finite, the left limit $\varphi_-(1) = 1$ and the left* *derivative in* 1 *denoted* $\varphi'_-(1)$ *exists and is finite. Assume also that* $\varphi'_-(1) > -1$. *If the function* $f : I \to [0, \infty)$ *is differentiable on* $\mathring{I}$ *and* φ*-convex, then*

$$\begin{aligned}(4.2)\qquad &\frac{\varphi'_-(1)+1}{\varphi_+(0)}\left[f\left(\frac{3x+y}{4}\right)+f\left(\frac{x+3y}{4}\right)\right]\\ &\le \frac{1}{y-x}\int_x^y f(u)\,du \le \left[f\left(\frac{x+y}{2}\right)+\frac{f(x)+f(y)}{2}\right]\frac{\varphi_+(0)}{\varphi'_-(1)+1}\end{aligned}$$

for any $x, y \in I$.

Now we can prove the following result as well:

Theorem 4.1. *Let* $\varphi : (0,1) \to (0,\infty)$ *a measurable function and such that the right limit* $\varphi_+(0)$ *exists and is finite, the left limit* $\varphi_-(1) = 1$ *and the left derivative in* 1 *denoted* $\varphi'_-(1)$ *exists and is finite. Assume also that* $\varphi'_-(1) > -2$. *If the function* $f : I \to [0,\infty)$ *is differentiable on* $\mathring{I}$ *and* φ*-convex, then*

$$\begin{aligned}(4.3)\qquad &\frac{1}{y-x}\int_x^y f(u)\,du\\ &\le \frac{\varphi_+(0)}{\varphi'_-(1)+2}f\left(\frac{x+y}{2}\right)+\frac{1}{\varphi'_-(1)+2}\cdot\frac{f(x)+f(y)}{2}\end{aligned}$$

for any $x, y \in I$.

Proof. Assume that $x < y$. From the inequality (3.1) we have

$$(4.4)\qquad \varphi_+(0)\,f\left(\frac{x+y}{2}\right)-\left[\varphi'_-(1)+1\right]f(u) \ge f'(u)\left(\frac{x+y}{2}-u\right)$$

for any $u \in [x,y]$ with $u \neq \frac{x+y}{2}$.

Integrating over $u \in [x,y]$ and dividing by $y-x$ we have

$$\begin{aligned}(4.5)\qquad &\varphi_+(0)\,f\left(\frac{x+y}{2}\right)-\left[\varphi'_-(1)+1\right]\frac{1}{y-x}\int_x^y f(u)\,du\\ &\ge \frac{1}{y-x}\int_x^y f'(u)\left(\frac{x+y}{2}-u\right)du.\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}\int_x^y f'(u)\left(\frac{x+y}{2}-u\right)du &= \left(\frac{x+y}{2}-u\right)f(u)\Big|_x^y+\int_x^y f(u)\,du\\ &= \int_x^y f(u)\,du-\frac{f(y)+f(x)}{2}(y-x)\end{aligned}$$

and by (4.5) we get

$$\begin{aligned}&\varphi_+(0)\,f\left(\frac{x+y}{2}\right)-\left[\varphi'_-(1)+1\right]\frac{1}{y-x}\int_x^y f(u)\,du\\ &\ge \frac{1}{y-x}\int_x^y f(u)\,du-\frac{f(y)+f(x)}{2},\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\varphi_+(0)\, f\left(\frac{x+y}{2}\right)+\frac{f(y)+f(x)}{2}\\&\geq \frac{1}{y-x}\int_x^y f(u)\,du+\left[\varphi'_-(1)+1\right]\frac{1}{y-x}\int_x^y f(u)\,du\\&=\left[\varphi'_-(1)+2\right]\frac{1}{y-x}\int_x^y f(u)\,du.\end{aligned}$$

Since $\varphi'_-(1)+2>0$, then on dividing by $\varphi'_-(1)+2$ we get the desired result (4.3). □

Remark 4.1. We observe that

$$\frac{\varphi_+(0)}{\varphi'_-(1)+2}<\frac{\varphi_+(0)}{\varphi'_-(1)+1}$$

and if we assume that φ is taken to satisfy the condition

$$\varphi_+(0)>\frac{\varphi'_-(1)+1}{\varphi'_-(1)+2}\in(0,1),$$

then

$$\frac{1}{\varphi'_-(1)+2}<\frac{\varphi_+(0)}{\varphi'_-(1)+1}$$

and the inequality (4.3) is better than the second inequality in (4.2).

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia., [2]School of Computational & Applied Mathematics,, University of the Witwatersrand, Private Bag 3,, Johannesburg 2050, South Africa

E-mail address: sever.dragomir@vu.edu.au