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DEVELOPABLE BERTRAND OFFSETS OF TRAJECTORY SPACELIKE RULED SURFACES

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Abstract. In this study, some characterizations for developable Bertrand offsets of a spacelike ruled surface are introduced. It is shown that if there exist more than one developable Bertrand offsets of a developable spacelike ruled surface, then the striction curve of reference surface is a general helix in the Minkowski 3-space R_1^3.

1. Introduction

Ruled surfaces are the surfaces generated by moving a straight line continuously in the space and are one of the most important topics of differential geometry. In a spatial motion, the trajectories of oriented lines embedded in a moving space or in a moving rigid body are generally called trajectory ruled surfaces [7]. Trajectory ruled surfaces and their offsets are used in many areas of sciences. These surfaces are used to study design problems in spatial mechanisms or space kinematics, Computer Aided Geometric Design (CAGD), geometric modeling and model-based manufacturing of mechanical products [2,3,7,11,14]. The well-known offset of ruled surfaces is Bertrand offset which is a generalization of the notion of Bertrand curve to the ruled surfaces. These offsets have been introduced by Ravani and Ku [11]. The corresponding characterizations of timelike and spacelike ruled surfaces in the Minkowski 3-space R_1^3 have been given by Kasap and Kuruoğlu [4]. Furthermore, Küçük has studied developable timelike ruled surfaces and Bertrand trajectory ruled surface offsets [6,7].

The classification of ruled surfaces in R_1^3 has been introduced by Kim and Yoon [5]. Using this classification, Önder and Uğurlu have introduced the Frenet frames of timelike and spacelike ruled surfaces [9,12].

In this paper, we give characterizations for developable Bertrand offsets of spacelike ruled surfaces in R_1^3. We introduce some theorems and results between curvatures of striction curves of offset surfaces.

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2. Preliminaries

Let consider the standard flat metric defined by

$$\langle\,,\rangle = -dx_1^2 + dx_2^2 + dx_3^2.$$

The real vector space R^3 provided with this metric is called the Minkowski 3-space and denoted by R_1^3. Then, (x_1, x_2, x_3) is a standard rectangular coordinate system of R_1^3. In this space, there exist three types of vectors $\vec{v} \in R_1^3$ such that $\vec{v}$ is spacelike if $\langle \vec{v}, \vec{v}\rangle > 0$ or $\vec{v} = 0$, timelike if $\langle \vec{v}, \vec{v}\rangle < 0$ and null (lightlike) if $\langle \vec{v}, \vec{v}\rangle = 0$ and $\vec{v} \neq 0$. Similarly, Lorentzian casual character of an arbitrary curve $\vec{\alpha} = \vec{\alpha}(s)$ is determinate with its velocity vector $\vec{\alpha'}(s)$. Then, $\vec{\alpha} = \vec{\alpha}(s)$ is locally spacelike, timelike or null (lightlike), if all of its velocity vectors $\vec{\alpha'}(s)$ are spacelike, timelike or null (lightlike), respectively [8]. The norm of the vector $\vec{v} = (v_1, v_2, v_3) \in R_1^3$ is given by

$$\|\vec{v}\| = \sqrt{|\langle \vec{v},\ \vec{v}\rangle|}.$$

For any vectors $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ in R_1^3, Lorentzian cross product is defined by

$$\vec{x} \times \vec{y} = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2y_3 - x_3y_2,\ x_1y_3 - x_3y_1,\ x_2y_1 - x_1y_2).$$

The Lorentzian sphere and hyperbolic sphere of radius r and center origin 0 in R_1^3 are given by

$$S_1^2 = \left\{\vec{x} = (x_1, x_2, x_3) \in R_1^3 : \quad \langle \vec{x}, \vec{x}\rangle = r^2\right\},$$

and

$$H_0^2 = \left\{\vec{x} = (x_1, x_2, x_3) \in R_1^3 : \quad \langle \vec{x}, \vec{x}\rangle = -r^2\right\},$$

respectively (See [13] for details).

For two spacelike vectors $\vec{x}$ and $\vec{y}$ that span a spacelike vector subspace, the real number $\theta \geq 0$ given by $< \vec{x}, \vec{y} >= \|\vec{x}\|\,\|\vec{y}\|\ \cos\theta$ is called *spacelike angle* between the vectors x and y [10].

Analogue to the curves, the characterization of a surface in R_1^3 is determinate by its normal vector. Then, a surface is called a timelike surface if its normal vector is spacelike and called a spacelike surface if its normal vector is timelike [1]. Now, we consider ruled surfaces in R_1^3 and we give a brief summary of theory of ruled surfaces in R_1^3. The more information can be found in [9,12].

Let I be an open interval in the real line R. Let $\vec{k} = \vec{k}(s)$ be a curve in R_1^3 defined on I and $\vec{q} = \vec{q}(s)$ be a unit direction vector of an oriented line in R_1^3. Then the parametric representation of a ruled surface N is given by

$$\varphi(s, v) = \vec{k}(s) + v\,\vec{q}(s). \tag{2.1}$$

Different positions of the straight lines i.e., the parametric s-curve of surface is called ruling. In (2.1), if we take $v = 0$, then we obtain the curve $\vec{k} = \vec{k}(s)$ which is called base curve or generating curve of the surface. Of course, there exist much more regular curves on the surface. But one of these curves have an important role and is called striction curve and denoted by $\vec{c} = \vec{c}(s)$. The striction curve is the focus of striction point which is the foot of common normal between two consecutive rulings. The parametrization of the striction curve $\vec{c} = \vec{c}(s)$ is given by

$$\vec{c}(s) = \vec{k}(s) - \frac{\left\langle d\vec{q}/ds,\ d\vec{k}/ds \right\rangle}{\langle d\vec{q}/ds,\ d\vec{q}/ds \rangle} \vec{q}(s). \tag{2.2}$$

From (2.2) it is clear that the base curve of the ruled surface is its striction curve if and only if $\left\langle d\vec{q}/ds,\ d\vec{k}/ds \right\rangle = 0$. Furthermore, the generator $\vec{q}$ of a developable ruled surface is tangent of its striction curve [11].

The distribution parameter (or drall) of a ruled surface is given by

$$\delta_\varphi = \frac{\left| d\vec{k}/ds,\ \vec{q},\ d\vec{q}/ds \right|}{\langle d\vec{q}/ds,\ d\vec{q}/ds \rangle}. \tag{2.3}$$

The geometric interpretation of (2.3) can be given as follows: If $\left| d\vec{k}/ds,\ \vec{q},\ d\vec{q}/ds \right| = 0$, then the timelike normal vectors of spacelike surface are collinear at all points of the same ruling which means that the tangent plane does not change and contacts the surface along a ruling which is called a *torsal* ruling. If $\left| d\vec{k}/ds,\ \vec{q},\ d\vec{q}/ds \right| \neq 0$, then the tangent planes of the surface N are different at all points of the same ruling, such a ruling is called nontorsal.

Definition 2.1. ([12]) A spacelike ruled surface whose all rulings are torsal is called a *developable spacelike ruled surface*. The remaining spacelike ruled surfaces are called skew spacelike ruled surfaces. Then, it is clear that a spacelike ruled surface is developable if and only if distribution parameter δ_φ is zero at all points of the surface.

For the unit normal vector $\vec{m}$ of spacelike ruled surface N, we can write $\vec{m} = \frac{\vec{\varphi}_s \times \vec{\varphi}_v}{\|\vec{\varphi}_s \times \vec{\varphi}_v\|}$. So, at the points of a nontorsal ruling $s = s_1$ we have

$$\vec{a} = \lim_{v \to \infty} \vec{m}(s_1, v) = \frac{(d\vec{q}/ds) \times \vec{q}}{\|d\vec{q}/ds\|}, \tag{2.4}$$

which is called *central tangent*. The timelike vector $\vec{h}$ defined by $\vec{h} = \vec{a} \times \vec{q}$ is called *central normal*. Since the vectors $\vec{q},\ d\vec{q}/ds$ and $\vec{a}$ are orthogonal, the unit vector $\vec{h}$ of the central normal is given by

$$\vec{h} = \frac{d\vec{q}/ds}{\|d\vec{q}/ds\|}. \tag{2.5}$$

Then, the orthonormal system $\left\{C;\ \vec{q},\ \vec{h}(timelike),\ \vec{a}\right\}$ is called Frenet frame of spacelike ruled surface N where C is the striction point.

Let now assume that the spacelike ruled surface N has non-null Frenet vectors and non-null derivatives of Frenet vectors. Then, for the vectors $\vec{q}, \vec{h}$ and $\vec{a}$ we have following Frenet formulae

$$\begin{bmatrix} dq/ds \\ dh/ds \\ da/ds \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 \\ \kappa_1 & 0 & \kappa_2 \\ 0 & \kappa_2 & 0 \end{bmatrix} \begin{bmatrix} q \\ h \\ a \end{bmatrix}, \tag{2.6}$$

where s is arc length parameter of striction curve and κ_1, κ_2 are first and second curvatures of the surface. If the surface is developable, then κ_1, κ_2 coincide with the curvature and torsion of striction curve. From (2.6), the Darboux vector of the Frenet frame $\left\{O; \vec{q}, \vec{h}, \vec{a}\right\}$ can be given by $\vec{w} = -\kappa_2 \vec{q} + \kappa_1 \vec{a}$. Thus, for the derivatives in (2.6) we can write

$$d\vec{q}/ds = \vec{w} \times \vec{q}, \quad d\vec{h}/ds = \vec{w} \times \vec{h}, \quad d\vec{a}/ds = \vec{w} \times \vec{a}, \tag{2.7}$$

and also we have

$$\vec{q} \times \vec{h} = -\vec{a}, \quad \vec{h} \times \vec{a} = -\vec{q}, \quad \vec{a} \times \vec{q} = \vec{h}. \tag{2.8}$$

(For details [12]).

3. Developable Bertrand Offsets of Spacelike Ruled Surfaces

Let φ and φ^* be two spacelike ruled surfaces given by the parametrizations

$$\varphi(s, v) = \vec{c}(s) + v\,\vec{q}(s), \qquad \langle \vec{q}, \vec{q} \rangle = 1, \tag{3.1}$$

$$\varphi^*(s, v) = \vec{c}^*(s) + v\,\vec{q}^*(s), \quad \langle \vec{q}^*, \vec{q}^* \rangle = 1, \tag{3.2}$$

respectively, where $(\vec{c})$ (resp. $(\vec{c}^*)$) is striction curve of φ (resp. φ^*). Let the Frenet frames of surfaces φ and φ^* be $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$ and $\left\{\vec{q}^*, \vec{h}^*, \vec{a}^*\right\}$, respectively. The ruled surface φ^* is said to be Bertrand offset of φ, if there exists a one to one correspondence between their rulings such that they have common central normal along the striction lines $(\vec{c})$ and $(\vec{c}^*)$. In this case, (φ, φ^*) is called a pair of Bertrand offsets of spacelike ruled surfaces. By definition, we have

$$\vec{h}^* = \vec{h}, \tag{3.3}$$

and so, we can write

$$\begin{pmatrix} \vec{q}^* \\ \vec{h}^* \\ \vec{a}^* \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \vec{q} \\ \vec{h} \\ \vec{a} \end{pmatrix}, \tag{3.4}$$

where θ is spacelike angle between the rulings $\vec{q}$ and $\vec{q}^*$ and called offset angle.

By definition, the striction curve of φ^* is given by

$$\vec{c}^*(s) = \vec{c}(s) + \lambda \vec{h}(s), \tag{3.5}$$

where λ is called offset distance. Then the parametrization of φ^* is obtained as follows

$$\begin{aligned}\varphi^*(s,v) &= \vec{c}^*(s) + v\,\vec{q}^*(s) \\ &= \left(\vec{c}(s) + \lambda\vec{h}(s)\right) + v\left(\cos\theta\vec{q} + \sin\theta\vec{a}\right).\end{aligned} \tag{3.6}$$

Theorem 3.1. *([4]) The offset angle θ and offset distance λ are constants.*

In general, the Bertrand offset of a developable ruled surface is not a developable surface.

Assume now that spacelike ruled surface φ is developable and (φ, φ^*) be a pair of Bertrand offsets of trajectory spacelike ruled surfaces. From (2.3), (3.4) and (3.6) the distribution parameter of φ^* is

$$\delta_{\varphi^*} = \frac{(\lambda\kappa_2)\cos\theta - (1+\lambda\kappa_1)\sin\theta}{\kappa_1\cos\theta + \kappa_2\sin\theta}, \tag{3.7}$$

which gives us that φ^* is developable if and only if

$$(\lambda\kappa_2)\cos\theta - (1+\lambda\kappa_1)\sin\theta = 0, \tag{3.8}$$

holds. So, we may give the following theorem and corollaries.

Theorem 3.2. *Let (φ, φ^*) be a pair of Bertrand offsets of spacelike ruled surfaces and φ be developable. Then φ^* is developable if and only if*

$$(\lambda\kappa_2)\cos\theta - (1+\lambda\kappa_1)\sin\theta = 0$$

is satisfied.

From Theorem 3.2 we obtain the following special cases.

Corollary 3.1. *i) If $\theta = 0$, then $\lambda = 0$ or $\kappa_2 = 0$.*
(ii) *If $\lambda = 0$, then $\theta = 0$, i.e., rulings are congruent.*
(iii) *If $\kappa_2 = 0, \lambda \neq 0$, then $\kappa_1 = -\frac{1}{\lambda} = const$ or $\theta = 0$.*
(iv) *If $\theta = \frac{\pi}{2}$, $\lambda \neq 0$, then $\kappa_1 = -\frac{1}{\lambda} = const$.*

Let now assume that both φ and φ^* are developable spacelike ruled surfaces and (φ, φ^*) is a pair of Bertrand offsets. Then, from (2.6) and (3.5), we can write

$$\vec{q}^*\frac{ds^*}{ds} = (1+\lambda\kappa_1)\vec{q} + \lambda\kappa_2\vec{a}. \tag{3.9}$$

From (3.4) we have

$$\vec{q}^* = \cos\theta\vec{q} + \sin\theta\vec{a}. \tag{3.10}$$

Then, from (3.9) and (3.10), we have the following relationships

$$\begin{cases} \cos\theta = (1+\lambda\kappa_1)\frac{ds}{ds^*}, \\ \sin\theta = \lambda\kappa_2\frac{ds}{ds^*}. \end{cases} \tag{3.11}$$

Since, there exits a reciprocal relationship between the curves $\vec{c}$ and $\vec{c}^*$, from (3.11) we may write

$$\begin{cases} \cos\theta = (1-\lambda\kappa_1^*)\frac{ds^*}{ds}, \\ \sin\theta = \lambda\kappa_2^*\frac{ds^*}{ds}, \end{cases} \tag{3.12}$$

and if we take

$$\frac{\cos\theta}{\sin\theta}\lambda = a = const.$$

from (3.11), we have

$$a\kappa_2 - \lambda\kappa_1 = 1. \tag{3.13}$$

Similarly, from (3.12), we have

$$a\kappa_2^* + \lambda\kappa_1^* = 1. \tag{3.14}$$

Then Eq. (3.13) and (3.14) give the following theorem and corollary.

Theorem 3.3. *Let (φ, φ^*) be a pair of developable Bertrand offsets of spacelike ruled surfaces. Then there exists the following relationship between curvatures and torsions of striction curves,*

$$\frac{\kappa_1^* + \kappa_1}{\kappa_2^* - \kappa_2} = \frac{a}{\lambda} = const.$$

Corollary 3.2. *The striction curves $\vec{c}$ and $\vec{c}^*$ are planar curves in R_1^3 if and only if $\kappa_1 = -\kappa_1^*$ holds.*

Now, assume that φ^* and φ^{**} are two developable Bertrand offsets of the same developable spacelike ruled surface φ. Since φ^* is a Bertrand offset of developable spacelike surface φ, from (3.13) and (3.14), there exist two real constants $a,\ \lambda$ such that

$$\begin{cases} a\kappa_2 - \lambda\kappa_1 = 1, \\ a\kappa_2^* + \lambda\kappa_1^* = 1. \end{cases} \tag{3.15}$$

Similarly, since φ^{**} is another developable Bertrand offset of φ, there are two real constants b, c such that

$$\begin{cases} b\kappa_2 - c\kappa_1 = 1, \\ b\kappa_2^{**} + c\kappa_1^{**} = 1. \end{cases} \tag{3.16}$$

where κ_1^{**} and κ_2^{**} are curvature and torsion of striction curve $\vec{c}^{**}$ of developable spacelike ruled surface φ^{**}, respectively. From (3.15) and (3.16) we have

$$\frac{\kappa_1}{\kappa_2} = \frac{b-a}{c-\lambda} = const., \tag{3.17}$$

which means that the striction curve $\vec{c}$ is a general helix. Then, we have the followings.

Theorem 3.4. *If there exist more than one developable Bertrand offsets of a developable spacelike ruled surface φ, then the striction curve $\vec{c}$ of φ is a general helix.*

Also, we can give the following corollary from (3.11) and (3.12).

Corollary 3.3. *There are the following relationships between curvatures of the striction curves $\vec{c}$ and $\vec{c}^*$,*

$$\left\{\begin{array}{l}\sin^2(\theta) = \lambda^2\kappa_2\kappa_2^* = const. \\ \cos^2(\theta) = (1-\lambda\kappa_1^*)(1+\lambda\kappa_1) = const.\end{array}\right. \tag{3.18}$$

Also from (3.18), the following corollary may be given.

Corollary 3.4. ***i)*** *If $\theta = 0,\ \ \lambda \neq 0$, then $\kappa_2 = 0$ or $\kappa_2^* = 0$ and $\frac{\kappa_1-\kappa_1^*}{\kappa_1\kappa_1^*} = \lambda = const.$*
ii) *If $\theta = \frac{\pi}{2}$, then we have $\lambda \neq 0, \kappa_2\kappa_2^* = \frac{1}{\lambda^2}$ and $\kappa_1^* = \frac{1}{\lambda}$ or $\kappa_1 = -\frac{1}{\lambda}$.*

4. Conclusions

Some conditions characterizing developable Bertrand offsets of spacelike ruled surfaces are given. It is shown that the striction curve of the reference spacelike ruled surface is a general helix if there are more than one developable Bertrand offset of a developable spacelike ruled surface in R_1^3. Furthermore, some relationships between curvatures and torsions of the striction curves of Bertrand offsets of spacelike ruled surface are found.

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