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SOME CHARACTERIZATIONS OF EULER SPIRALS IN E_1^3

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Abstract. In this study, some characterizations of Euler spirals in E_1^3 have been presented by using their main property that their curvatures are linear. Moreover, discussing some properties of Bertrand curves and helices, the relationship between these special curves in E_1^3 have been investigated with different theorems and examples. The approach we used in this paper is useful in understanding the role of Euler spirals in E_1^3 in differential geometry.

1. INTRODUCTION

In three dimensional Euchlidean space E^3, Euler spirals are well-known as the curves whose curvatures evolves linearly along the curve. It is also called Clothoid or Cornu spiral whose curvature is equal to its arclength.

The equations of Euler spirals were written by Bernoulli first, in 1694. He didn't compute these curves numerically. In 1744, Euler rediscovered the curve's equations, described their properties, and derived a series expansion to the curve's integrals. Later, in 1781, he also computed the spiral's end points. The curves were re-discovered in 1890 for the third time by Talbot, who used them to design railway tracks [1].

A new type of Euler spirals in E^2 and in E^3 are given in [1] with their properties. They prove that their curve satisfies properties that characterize fair and appealing curves and reduces to the 2D Euler spiral in the planar case. Furthermore, they require that their curve conforms with the definition of a 2D Euler spiral. Smiliarly, these curves are presented in [6] as the ratio of two rational linear functions and have been defined in E^3 as generalized Euler spirals with some various characterizations. On the other hand, linear relation between principal curvatures of spacelike surfaces in Minkowski space is studied in [3].

In this paper, we present the timelike and spacelike Euler spirals in Minkowski space E_1^3. At first, we give the basic concepts and theorems about the study then we deal with these spirals whose curvatures and torsion are linear. Here, we seek

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that if any timelike Euler spirals in E_1^3 is regular or not. Next, we investigate in which conditions the timelike Euler spirals can be Bertrand curve. Additionally, by using the definition of logarithmic spiral having a linear radius of curvature and a radius of torsion from [1,2,6], we obtain the spacelike logarithmic spiral.

We believe that this study gives us a link and relation between the classical differential surface theory and Euler spirals in Minkowski space E_1^3.

2. PRELIMINARIES

Now, we recall the basic concepts and important theorems on classical differential geometry about the study, then we use them in the next sections to give our approach. References [1,2,5,7], contain these concepts.

Let

$$\begin{aligned} \alpha : I &\to E^3 \\ s &\mapsto \alpha(s) \end{aligned}$$

be non-null curve and $\{T, N, B\}$ frame of α. T, N, B are the unit tangent, principal normal and binormal vectors respectively. Let κ and τ be the curvatures of the curve α.

We consider a regular curve α parametrized by the length-arc. We call

$$\overrightarrow{T}(s) = \frac{d\alpha}{ds}$$

the tangent vector at s. In particular, $\langle T(s), T'(s)\rangle = 0$. We will assume that $T'(s) \neq 0$. In this study we investigate the curve α in two different cases timelike and spacelike.

We suppose that α is a timelike curve. Then $T'(s) \neq 0$ is a spacelike vector independent with $T(s)$. We define the curvature of α at s as $\kappa(s) = |T'(s)|$ .The normal vector $N(s)$ is defined by

$$N(s) = \frac{T'(s)}{\kappa(s)} = \frac{\alpha''(s)}{|\alpha''(s)|}.$$

Moreover $\kappa(s) = \langle T'(s), N(s)\rangle$. We call the binormal vector $B(s)$ as

$$B(s) = T(s) \times N(s)$$

The vector $B(s)$ is unitary and spacelike. For each s, $\{T, N, B\}$ is an orthonormal base of E_1^3 which is called the Frenet trihedron of α. We define the torsion of α. We define the torsion of α at s as

$$\tau(s) = \langle N'(s), B(s)\rangle \ \ [5].$$

By differentiation each one of the vector functions of the Frenet trihedron and putting in relation with the same Frenet base, we obtain the Frenet equations, namely,

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

On the other hand, Let α be a spacelike curve. These are three possibilities depending on the casual character of $T'(s)$

1. The vector $T'(s)$ is spacelike. Again, we write $\kappa(s) = |T'(s)|$, $N(s) = T'(s)/\kappa(s)$ and $B(s) = T(s) \times N(s)$. The vectors N and B are called the normal vector and the binormal vector respectively. The curvature of α is defined by κ. The equations are

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

The torsion of α is defined by $\tau = -\langle N', B\rangle$.

2. The vector $T'(s)$ is timelike. The normal vector is $N = T'/\kappa$, where $\kappa(s) = \sqrt{-\langle T'(s), T'(s)\rangle}$ is the curvature of α. The binormal vector is $B(s) = T(s) \times N(s)$, which is spacelike vector. Now, the Frenet equations

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

The torsion of α is $\tau = -\langle N', B\rangle$.

3. The vector $T'(s)$ is lightlike for any s (recall that $T'(s) \neq 0$ and it is not proportional to $T(s)$). We define the normal vector as $N(s) = T'(s)$, which is independent linear with $T(s)$. Let B be the unique lightlike vector such that $\langle N, B\rangle = 1$ and orthogonal to T. The vector $B(s)$ is called the binormal vector of α at s. The Frenet equations are

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \tau & 0 \\ -1 & 0 & -\tau \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

The function τ is called the torsion of α. There is not a definition of the curvature of α [5].

It is well-known that a planar curve of E_1^3 is included in a affine plane and also this plane is a vector plane. In [5], planar curves with constant curvature is studied. If the curve α is a planar curve in E^2 parametrized by the length-arc and ν is a fixed unitary direction, we call $\theta(s)$ the angle that makes $T(s)$ and ν, that is

$$\cos(\theta(s)) = \langle T(s), \nu\rangle \ \ [5].$$

It can be proved that the curve α is $|\theta'(s)|$.

Harary and Tall define the Euler spirals in E^3 the curve having both its curvature and torsion evolve linearly along the curve. Furthermore, they require that their curve conforms with the definition of a *Cornu spiral*.

Here, these curves whose curvatures and torsion evolve linearly are called Euler spirals in E_1^3. Thus for some constants $a, b, c, d \in \mathbb{R}$,

$$\begin{aligned} \kappa(s) &= as + b \\ \tau(s) &= cs + d \end{aligned} \tag{2.1}$$

Next, we define logarithmic spiral having a linear radius of curvature and a linear radius of torsion from [1,2]. They seek a spiral that has both a linear radius of curvature and a linear radius of torsion in the arc-length parametrization s :

$$\kappa(s) = \frac{1}{as+b}$$
$$\tau(s) = \frac{1}{cs+d}$$

where a, b, c and d are constants.

Accordingly, the Euler spirals in E_1^2 and in E_1^3 that satisfy equation above by a set of differential equations is the spacelike curve α for which the following conditions hold:

$$\begin{aligned}
\frac{d\overrightarrow{T}(s)}{ds} &= \left(\frac{1}{(as+b)}\right)\overrightarrow{N}(s),\\
\frac{d\overrightarrow{N}(s)}{ds} &= \left(\frac{1}{(as+b)}\right)\overrightarrow{T}(s) + \left(\frac{1}{(cs+d)}\right)\overrightarrow{B}(s)\\
\frac{d\overrightarrow{B}(s)}{ds} &= -\left(\frac{1}{(cs+d)}\right)\overrightarrow{N}(s).
\end{aligned}$$

In addition to these, we want to give our definition that Euler spirals in E_1^3 whose ratio between its curvature and torsion evolve linearly is called generalized Euler spirals in E_1^3. Thus for some constants $a, b, c, d \in \mathbb{R}$,

$$\frac{\kappa}{\tau} = \frac{as+b}{cs+d}$$

Theorem 2.1. *Let α be a timelike curve parametrized by the length-arc and included in a timelike plane. Let ν be a unit fixed vector of the plane pointing to the future. Let $\phi(s)$ be the hyperbolic angle between $T(s)$ and ν. Then $\kappa(s) = |\phi'(s)|$* [5].

Theorem 2.2. *Let the curve α be a timelike planar Euler spirals in E_1^3. Then α is a Bertrand curve and there are two constants A and B; the curvature κ and the torsion τ such that*

$$A\kappa + B\tau = 1 \quad [5].$$

3. PLANAR EULER SPIRALS IN E_1^3

In this section, we will discuss the main properties of timelike and spacelike Euler spirals. At that time, the relationship between Euler spirals, Bertrand curves, regular curves and helices are given with some theorems and cases.

3.1. **The Timelike Planar Euler Spirals in E_1^3.** Let $\phi : I \to \mathbb{R}$ be the function. We assume that the plane is timelike and the plane P be timelike, that is $P = \langle E_2, E_3\rangle$ and then let parametrized curve by the arc-length be the curve β. Thus, this curve can be given as

$$\beta(s) = y(s)E_2 + z(s)E_3$$

with

$$y'(s)^2 - z'(s)^2 = -1.$$

From that,

$$\beta(s) = \left(\int_{s_0}^{s} \cosh\phi(t)dt, \int_{s_0}^{s} \sinh\phi(t)dt\right)$$

with the linear curavature

$$\phi(s) = \int_{s_0}^{s} \kappa(u)du.$$

This approach shows that a plane curve with any given smooth function as its signed curvature can be found. But simple curvature can lead to these curves.

Thus, the function $\phi \in \mathbb{R}$ such that

$$y'(s) = \sinh \phi(t)dt,$$
$$z'(s) = \cosh \phi(t)dt$$

taking the derivative of this curve,

$$\beta'(s) = (\sinh \phi(s), \cosh \phi(s))$$

and giving the dot product

$$\langle \beta', \beta' \rangle = \sinh \phi(s)^2 - \cosh \phi(s)^2 = -1$$

the timelike Euler spiral can be computed with the property

$$\kappa(s) = |\beta''(s)| = |\phi'(s)| = s.$$

Therefore, taking the signed curvature be $\kappa(s) = s$ and $s_0 = 0$, the timelike Euler spiral β can be obtained as

$$\beta(s) = \left(\int_0^s \cosh \frac{s^2}{2} ds, \int_0^s \sinh \frac{s^2}{2} ds \right)$$

Example 3.1. Let the signed curvature be $\kappa(s) = as + b$. Taking $s_0 = 0$, we get

$$\phi'(s) = as + b \text{ and}$$
$$\phi(s) = \frac{a}{2}s^2 + bs.$$

then the timelike planar curve β can be given as

$$\beta(s) = \left(\int_0^s \cosh(\frac{a}{2}s^2 + bs) ds, \int_0^s \sinh(\frac{a}{2}s^2 + bs) ds \right).$$

Example 3.2. Let the signed curvature be $\kappa(s) = \frac{1}{s}$. Taking $s_0 = 0$, we get the timelike planar curve as

$$\beta(s) = \left(\int_0^s \cosh(\ln s) ds, \int_0^s \sinh(\ln s) ds \right)$$

Theorem 3.1. *Any timelike planar Euler spirals in E_1^3 is regular.*

Proof. Assume that the curve β is timelike planar Euler spiral and is written by

$$\beta(s) = \left(\int_{s_0}^s \cosh \phi(s) ds, \int_{s_0}^s \sinh \phi(s) ds \right)$$

where

$$\beta'(s) = (x'(s), y'(s), z'(s)) = (0, y'(s), z'(s))$$

and

$$\langle \beta'(s), \beta'(s) \rangle = y'(s)^2 - z'(s)^2 \langle 0,$$

In particular ${}_{s_0}^{s} \sinh \phi(s) \neq 0$ that is, β is a regular curve. $\square$

3.2. **The Spacelike Planar Euler Spirals in E_1^3.** Let $\phi : I \to \mathbb{R}$ be the function. We suppose that the plane is timelike and is given by $\{x = 0\}$. We are going to find the spacelike planar curve α; at this time, the curve α is written as

$$\alpha(s) = x(s)E_1 + z(s)E_3 \text{ and}$$
$$\alpha(s) = (x(s), 0, z(s))$$

with

$$x'(s)^2 - z'(s)^2 = 1.$$

Thus, the spacelike planar Euler spiral is given by

$$\alpha(s) = \left(\int_{s_0}^{s} \sinh\phi(t)dt, \int_{s_0}^{s} \cosh\phi(t)dt \right)$$

where

$$\phi(s) = \int_{s_0}^{s} \kappa(u)du$$

with the curvature κ. Here, as it is known the curvature κ is linear and the function $\phi \in \mathbb{R}$ such that

$$x'(s) = \cosh\phi(s) \text{ and}$$
$$z'(s) = \sinh\phi(s).$$

Then, the tangent vector α is

$$\alpha'(s) = (x'(s), z'(s)) \text{ and}$$
$$\alpha'(s) = (\cosh\phi(s), \sinh\phi(s)).$$

It is clear that

$$\langle \alpha'(s), \alpha'(s) \rangle = \cosh\phi(s)^2 - \sinh\phi(s)^2 = 1.$$

Let the signed curvature be $\kappa(s) = s$. Thus, the spacelike planar Euler spiral in E_1^3 can be found as

$$\alpha(s) = \left(\int_{0}^{s} \sinh\frac{s^2}{2}ds, \int_{0}^{s} \cosh\frac{s^2}{2}ds \right).$$

Example 3.3. If we take

$$\kappa(s) = as + b$$

then

$$\phi'(s) = as + b$$

and

$$\phi(s) = a\frac{s^2}{2}ds + bs$$

So,

$$\alpha(s) = \left(\int_{0}^{s} \sinh(\frac{a}{2}s^2 + bs)ds, \int_{0}^{s} \cosh(\frac{a}{2}s^2 + bs)ds \right)$$

Example 3.4. If the curvature κ is constant as $\kappa = a$, then $\phi(s) = as + b$. Thus, the spacelike planar curve is obtained as

$$\alpha(s) = \left(\int_0^s \sinh(as + b)ds, \int_0^s \cosh(as + b)ds \right).$$

Example 3.5. If the curvature κ is given as

$$\kappa = \frac{1}{s},$$

then

$$\phi(s) = \ln s.$$

Therefore, the spacelike planar curve is

$$\alpha(s) = \left(\int_0^s \sinh(\ln s)ds, \int_0^s \cosh(\ln s)ds \right).$$

4. EULER SPIRALS IN E_1^3

In this section, we study some characterizations of Euler spirals in E_1^3 by giving some theorems and definitions from [4, 5, 6].

Proposition 4.1. *If the curvature τ is zero then $\kappa = as + b$ and also the curve is planar cornu spiral in E_1^3.*

Proof. If $\tau = 0$ and the curvature is linear, then the ratio

$$\frac{\tau}{\kappa} = 0$$

Therefore, we see that the curve is planar Euler spiral in E_1^3. □

Proposition 4.2. *If the curvatures are*

$$\tau = as + b$$
$$\kappa = c$$

then the Euler spirals are rectifying curves in E_1^3.

Proof. If we take the ratio

$$\frac{\tau}{\kappa} = \frac{as + b}{c}$$

where λ_1 and λ_2 , with $\lambda_1 \neq 0$ are constants, then [4]

$$\frac{\tau}{\kappa} = \lambda_1 s + \lambda_2.$$

It shows us that the Euler spirals are rectifying curves in E_1^3. It can be easily seen from [3] that rectifying curves have very simple characterization in terms of the ratio $\frac{\tau}{\kappa}$. □

Proposition 4.3. *Euler spirals in E_1^3 are Bertrand curves.*

Proof. From the definition of Euler spiral in E_1^3 and the equations above, we can take

$$\begin{aligned}\tau(s) &= c_1 s + c_2\\ \kappa(s) &= d_1 s + d_2,\ \text{with } c_1 \neq 0 \text{ and } d_1 \neq 0.\end{aligned}$$

Here,

$$s = \frac{1}{c_1}(\tau - c_2)$$

and then,

$$\begin{aligned}\kappa &= \frac{d_1}{c_1}(\tau - c_2) + d_2\\ c_1\kappa &= d_1(\tau - c_2) + c_1 d_2\\ c_1\kappa - d_1\tau &= c_3\\ \frac{c_1}{c_3}\kappa - \frac{d_1}{c_3}\tau &= 1\end{aligned}$$

Thus, we obtain

$$\lambda\kappa + \mu\tau = 1$$

from **Theorem.2** that Euler spirals are Bertrand curves in E_1^3. □

Theorem 4.1. *Let M and M_r be parallel surfaces in E_1^3 and also let the curve α be a geodesic timelike Euler spiral on the surface M such that the curvatures*

$$\begin{aligned}\kappa(s) &= c_1 s + c_2\\ \tau(s) &= d_1 s + d_2,\ \textit{with } c_1 \neq 0 \textit{ and } d_1 \neq 0.\end{aligned}$$

In this case, the Bertrand pair of the curve α is on the surface M_r. Here,

$$r = \frac{c_1}{c_1 d_2 - d_1 c_2}.$$

Proof. From the proposition before, the Euler spiral is Bertrand curve. Let β be the Bertrand pair of the curve of α as:

$$\beta(s) = \alpha(s) + \nu N(s).$$

From the property that α is geodesic on M

$$N(s) = n(s).$$

Therefore,

$$\beta(s) = \alpha(s) + \nu n(s).$$

From the **Proposition 12,** we can give

$$r = \frac{c_1}{c_1 d_2 - d_1 c_2}.$$

Thus, if we take $r = \nu$ then we have $\beta(s) \in M_r$. □

Theorem 4.2. *Let M be a surface in E_1^3 and $\alpha : I \to M$ be non-null curve (timelike or spacelike). If the Darboux curve*

$$W(s) = \varepsilon\tau T + \kappa B$$

is geodesic curve on the surface M, then the curve α is Euler spiral in E_1^3. Here, if the curve α is timelike then $\varepsilon = -1$ and if the curve α is spacelike then $\varepsilon = 1$.

Proof. Let α be timelike. Thus,

$$W(s) = -\tau T - \kappa B$$

then we have

$$W'(s) = -\tau' T - \kappa' B \tag{4.1}$$

$$W''(s) = -\tau'' T - \kappa'' B - (\kappa\tau' - \tau\kappa')N \tag{4.2}$$

and also

$$W''(s) = \lambda(s)n(s).$$

Here, $n(s)$ is the unit normal vector field of the surface M. Darboux curve is geodesic on surface M, therefore we have

$$W''(s) = \lambda(s)N(s).$$

and then it can be easily given that $n = N$. If we take

$$\tau'' = 0,\ \kappa'' = 0$$

and also

$$\kappa = as + b$$
$$\tau = cs + d$$

then the curve α is generalized Euler spiral in E_1^3. $\square$

On the other hand, let α be spacelike. Thus,

$$W(s) = \tau T - \kappa B.$$

Here, if the vector $T'(s)$ is spacelike or timelike then we have

$$W'(s) = \tau' T - \kappa' B$$
$$W''(s) = \tau'' T - \kappa'' B + (\kappa\tau' - \tau\kappa')N$$

and also it can be seen that

$$W''(s) = \lambda(s)n(s).$$

Smiliarly, $n(s)$ is the unit normal vector field of the surface M. Darboux curve is geodesic on surface M, therefore we have

$$W''(s) = \lambda(s)N(s).$$

and because of that it can be easily given $n = N$. If we take

$$\tau'' = 0,\ \kappa'' = 0$$

and also

$$\kappa = as + b$$
$$\tau = cs + d$$

then the curve α is generalized Euler spiral in E_1^3.

5. GENERALIZED EULER SPIRALS IN E_1^3

In this section, we investigate generalized Euler spirals in E_1^3 by using the definitions in above.

Theorem 5.1. *In E_1^3, all logarithmic spirals are generalized Euler spirals.*

Proof. As it is known that in all logarithmic spirals, the curvatures are linear as:

$$\kappa(s) = \frac{1}{as+b}$$
$$\tau(s) = \frac{1}{cs+d}$$

In that case, it is clear that the ratio between the curvatures can be given as:

$$\frac{\kappa}{\tau} = \frac{cs+d}{as+b}$$

Thus, it can be easily seen all logarithmic spirals are generalized euler spirals.

Theorem 5.2. *Euler spirals are generalized Euler spirals in E_1^3.*

Proof. It is clear from the property of curvature, torsion and the ratio that are linear as:

$$\kappa(s) = as+b$$
$$\tau(s) = cs+d$$

and then

$$\frac{\kappa}{\tau} = \frac{as+b}{cs+d}$$

That shows us Euler spirals in E_1^3 are generalized Euler spirals in E_1^3. □

□

Proposition 5.1. *All generalized Euler spirals in E_1^3 that have the property*

$$\frac{\tau}{\kappa} = d_1 s + d_2$$

are rectifying curves.

Proof. If the curvatures $\kappa(s)$ and $\tau(s)$ are taken as

$$\kappa(s) = c$$
$$\tau(s) = d_1 s + d_2 \text{ with } d_1 \neq 0$$

then, from [4]

$$\frac{\tau}{\kappa} = \frac{d_1 s + d_2}{c} = \lambda_1 s + \lambda_2$$

where λ_1 and λ_2 are constants. This gives us that if the curve α is generalized Euler spiral in E_1^3 then it is also in rectifying plane. □

Result.1 General helices are generalized Euler spirals in E_1^3.

Proof. It can be seen from the property of curvatures that are linear and the ratio is also constant as it is shown:

$$\frac{\tau}{\kappa} = \lambda$$

□

Theorem 5.3. *Let*

$$\alpha : I \to E_1^3 \tag{5.1}$$
$$s \mapsto \alpha(s)$$

be non-null curve (spacelike or timelike) and let κ and τ be the curvatures of the Frenet vectors of the curve α. For $a, b, c, d, \lambda \in \mathbb{R}$, let take the curve β as

$$\beta(s) = \alpha(s) + (as + b)T + (cs + d)B + \lambda N. \tag{5.2}$$

In this case, the curve α is generalized Euler spiral in E_1^3 which has the property

$$\frac{\kappa}{\tau} = \varepsilon \frac{cs + d}{as + b}, \qquad \varepsilon = \mp 1$$

if and only if the curves β and (T) are the involute-evolute pair. Here, the curve (T) is the tangent indicatrix of α.

Proof. The tangent of the curve β is

$$\beta'(s) = ((1 - \lambda)\kappa + a)T + (c + \lambda\tau)B + (\kappa(as + b) - \tau(cs + d))N. \tag{5.3}$$

The tangent of the curve (T) is

$$\frac{dT}{ds_T} = \varepsilon N.$$

Here, s_T is the arc parameter of the curve (T).

$$\langle \beta', N \rangle = \kappa(as + b) - \tau(cs + d) \tag{5.4}$$

If the curves β and (T) are the involute-evolute pair then

$$\langle \beta', N \rangle = 0.$$

From (7), it can be easily obtained

$$\frac{\kappa}{\tau} = \frac{cs + d}{as + b}$$

This means that the curve α is generalized Euler spiral in E_1^3.

On the other hand, if the curve α is generalized Euler spiral in E_1^3 which has the property

$$\frac{\kappa}{\tau} = \frac{cs + d}{as + b},$$

then from (7)

$$\langle \beta', N \rangle = 0.$$

This means that β and (T) are the involute-evolute pair in E_1^3. □

Result.2 From the hypothesis of the theorem above, the curve α is generalized Euler spiral in E_1^3 which has the property

$$\frac{\kappa}{\tau} = \frac{cs + d}{as + b}$$

if and only if β and (B) are the involute-evolute pair. Here, (B) is the binormal of the curve α in E_1^3.

Proof. The tangent of the curve (B) is

$$\frac{dB}{ds_B} = -N.$$

It can be easily seen from [5] for (B).

Theorem 5.4. *Let the ruled surface Φ be*

$$\begin{aligned} \Phi : I \times \mathbb{R} &\longrightarrow E_1^3 \\ (s,v) \to \Phi(s,v) &= \alpha(s) + v[(as+b)T + (cs+d)B] \end{aligned} \tag{5.5}$$

and the curve $\alpha : I \to M$ be non-null curve. The ruled surface Φ is developable if and only if the curve α is generalized Euler spiral in E_1^3 which has the property

$$\frac{\kappa}{\tau} = \varepsilon \frac{cs+d}{as+b}, \qquad \varepsilon = \mp 1.$$

Here, α is the base curve and T, B are the tangent and binormal of the curve α, respectively.

□

Proof. For the directrix of the surface

$$X(s) = (as+b)T + (cs+d)B,$$

and also for

$$X'(s) = aT + [(as+b)\kappa \pm (cs+d)\tau]N + cB,$$

we can easily give that

$$\det(T, X, X') = \begin{vmatrix} 1 & 0 & 0 \\ as+b & 0 & cs+d \\ a & (as+b)\kappa \pm \tau(cs+d) & c \end{vmatrix}$$

In this case, the ruled surface is developable if and only if $\det(T, X, X') = 0$ then

$$(cs+d)(as+b)\kappa - (cs+d)\tau = 0$$

then for $cs + d \neq 0$

$$(as+b)\kappa \pm (cs+d)\tau = 0.$$

Thus, the curve α is generalized Euler spiral in E_1^3 which has the property

$$\frac{\tau}{\kappa} = \frac{as+b}{cs+d}.$$

□

Theorem 5.5. *Let $\alpha : I \to M$ be non-null curve. If the curve*

$$U(s) = \frac{\varepsilon}{\kappa} T - \frac{1}{\tau} B$$

is a geodesic curve then the curve α is a logarithmic spiral in E_1^3.

Proof. Let α be timelike. Thus we have

$$U(s) = -\frac{1}{\kappa} T - \frac{1}{\tau} B$$

then

$$U' = \left(-\frac{1}{\kappa}\right)' T - \left(\frac{1}{\tau}\right)' B \tag{5.6}$$

$$U'' = \left(-\frac{1}{\kappa}\right)'' T - \left(\frac{1}{\tau}\right)'' B - \left[\left(\frac{1}{\kappa}\right)' \kappa - \left(\frac{1}{\tau}\right)' \tau\right] N. \tag{5.7}$$

$U(s)$ is geodesic on surface M, therefore

$$U'' = \mu_1(s)n(s).$$

Here, $n(s)$ is the unit normal vector field of the surface M. And also $n = N$, then we have

$$U'' = \mu_1(s)N(s).$$

Also, from (10) it can be easily seen that

$$\frac{1}{\kappa} = as + b$$
$$\frac{1}{\tau} = cs + d$$

Thus, it is clear that the curve α is a logarithmic spiral in E_1^3.

On the other hand, let α be spacelike. Thus we have

$$U(s) = \frac{1}{\kappa}T - \frac{1}{\tau}B$$

then

$$U' = \left(\frac{1}{\kappa}\right)' T - \left(\frac{1}{\tau}\right)' B \tag{5.8}$$

$$U'' = \left(\frac{1}{\kappa}\right)'' T - \left(\frac{1}{\tau}\right)'' B + \left[\left(\frac{1}{\kappa}\right)' \kappa - \left(\frac{1}{\tau}\right)' \tau\right] N. \tag{5.9}$$

$U(s)$ is geodesic on surface M, therefore

$$U'' = \mu_2(s)n(s).$$

Here, $n(s)$ is the unit normal vector field of the surface M. And also $n = N$, then we have

$$U'' = \mu_2(s)N(s).$$

Also, from (12) it can be easily seen that

$$\frac{1}{\kappa} = as + b$$
$$\frac{1}{\tau} = cs + d$$

□

6. CONCLUSIONS

In this study, at the beginning, planar Euler spirals in E_1^3 have been defined and then generalized Euler spirals in E_1^3 have been introduced by using their important properties. Depending on these, some different characterizations of Euler spirals in E_1^3 are expressed by giving theorems, propositions with their results and proofs. At this time, it is obtained that Euler spirals are generalized Euler spirals in E_1^3. Additionally, we show that all logarithmic spirals are generalized Euler spirals in E_1^3. Moreover, many different approaches about generalized Euler spirals in E_1^3 are presented in this paper.

We hope that this study will gain different interpretation to the other studies in this field.

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