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EXISTENCE OF MILD SOLUTIONS FOR AN IMPULSIVE FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH NON-LOCAL CONDITION

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Abstract. In this paper we are interested in studying the existence of solutions for a controlled impulsive fractional evolution equations. We use several tools such as fractional calculus, fixed point theorems and the theory of semigroup. We first give some preliminaries and notations, the second part of the work we provide an existence result for our problem and in the final section, we give some examples to show the importance of our results.

1. Introduction

The theory of non-integer calculus has been introduced in 1695 by Leibniz and L'Hopital. Fractional calculus caracterize the memory in the evolution process and it is an important tool in describing real-world phenomena and it is used in many fields, such as physics, biology and economics.

Fractional differential equations with instanteneous impulses have been introduced as a new exciting and interesting branch. But, this type of equations cannot describe the dynamics of evolution process in many areas of research, such as pharmacotherapy. That is why, Hernandez and O'Regan [5] introduced a new model which is the impulsive differential equations with non-instantaneous impulses.

In [8] the authors studied the following non-instantaneous model

$$(1.1)\quad \begin{cases} {}^cD^{\alpha}x(t) = Ax(t) + f(t, x(t)) + B(t)u(t),\ t \in \cup_{i=0}^{N}(s_i, t_{i+1}],\ u \in U_{ad} \\ x(t) = g_i(t, x(t)),\ t \in (t_i, s_i], i = 1, 2, ..., N, \\ x(s_i^+) = x(s_i^-), i = 1, 2, ..., N, \\ x(0) = x_0 \in X \end{cases}$$

Where ${}^cD^{\alpha}$ denotes the Caputo fractional derivatives of order $\alpha \in (0, 1)$ with the lower limit zero,
$A : D(A) \subset X \longrightarrow X$ is the generator of a C_0-semigroup of bounded operators

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$T(t)_{t\geq 0}$ on a Banach space $(X, \|\ .\ \|)$, $x_0 \in X$,$0 = t_0 = s_0 < t_1 < s_1 < t_2 < s_2 < ... < t_m \leq s_m < t_{m+1} = T$ are fixed numbers, $g_i \in C(J \times X; X)$.

The symbols $x(s_i^+)$ and$x(s_i^-)$ represents the limits of $x(t)$ at $t = s_i$.

Motivated by the work of S. Liu and J. Wang in [8], we study the impulsive differential equation

$$(1.2)\quad \begin{cases} {}^cD^{\alpha}x(t) = Ax(t) + f(t, x(t), Fx(t), Bx(t)) + \int_0^t q(t-s)k(s, x(s))ds + \\ C(t)u(t),\ t \in (s_i, t_{i+1}],\ i = 0, 1, 2, ..., m,\ u \in U_{ad}, \\ {}^cD^{\beta}x(t) = g_i(t, x(t)),\ t \in (t_i, s_i],\ i = 1, 2, ..., m, \\ x(0) = x_0 + h(x). \end{cases}$$

Where ${}^cD^{\alpha}$,${}^cD^{\beta}$ are the Caputo fractional derivatives of order $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ respectively with the lower limit zero, $A : D(A) \subset X \longrightarrow X$ is the generator of a C_0-semigroup of bounded operators $T(t)_{t\geq 0}$ on a Banach space $(X, \|\ .\ \|)$, $x_0 \in X$, $0 = t_0 = s_0 < t_1 < s_1 < t_2 < s_2 < ... < t_m \leq s_m < t_{m+1} = T$ are fixed numbers, $g_i \in C(J \times X; X)$, h,f, , k are given functions, $F, B, q : C(J; X) \to C(J; X)$ are given by $Bx(t) = \int_0^t B(t, s)x(s)ds$, $Fx(t) = \int_0^t F(t, s)x(s)ds$ and $\{F(t, s); t, s \in J\}$ $\{B(t, s); t, s \in J\}$ are a set of bounded linear operators on X such that:
$F(t, .)x \in C([0, t]; X)$, $F(., s) \in C([s, T]; X)$ for all $t, s \in J$, $B(t, .)x \in C([0, t]; X)$, $B(., s) \in C([s, T]; X)$ for all $t, s \in J$ and $F^* = \sup_{t\in J} \int_0^t \|\ F(t, s)\ \|_{L(X)}\ ds$,
$B^* = \sup_{t\in J} \int_0^t \|\ B(t, s)\ \|_{L(X)}\ ds$, $q^* = \sup_{s\in J} \int_0^s \|q(s - t)\|dt$.

In our work we will need the continuity of $x(t)$ at both points $t = s_i$ and $t = t_i$. The rest of the paper is organized as follows. In section 2 we present the notations, definitions and preliminaries results needed in the following sections. In section 3, a suitable concept of mild solutions for our problems is introduced. Section 4 is concerned with the existence results of our problems.

2. Preliminaries

Let us set $J = [0, T]$, $J_0 = [0, t_1]$, $J_1 = (t_1, t_2]$, ..., $J_{m-1} = (t_{m-1}, t_m]$, $J_m = (t_m, t_{m+1}]$ and introduce the space $PC(J, X) := \{u : J \to X | u \in C(J_k, X),\ k = 0, 1, 2, ..., m,$ and there exists $u(t_k^+)$ and $u(t_k^-)$, $k = 1, 2, ..., m$, with $u(t_k^-) = u(t_k^+)\}$. It is clear that $PC(J, X)$ is a Banach space with the norm $\|u\|_{PC} = sup\{\|u(t)\| : t \in J\}$.
Let Y be a separable reflexive Banach space where controls u takes values, and $P_f(Y)$ is a class of nonempty closed and convex subsets of Y. We suppose that the multivalued map $w : [0, a] \longrightarrow P_f(Y)$ is measurable, $w(.) \subset E$, where E is bounded set of Y, and the admissible control set
$U_{ad} = \Big\{c \in L^p(E) : c(t) \in w(t), a.e\Big\}$, $p > \frac{1}{\tau}$, $(\tau \in (0, \alpha)$), for more detail about admissible control set, we refer the readers to [13].

Lemma 2.1. *(Theorem 2.1 in* [11]*) Suppose* $W \subseteq PC(J, X)$. *If the following conditions are satisfied:*

1. *W is uniformly bounded subset of PC(J,X);*
2. *W is equicontinuous in* $(t_i, t_{i+1}), i = 0, 1, 2, .., m$, *where* $t_0 = 0, t_{m+1} = T$;

(3) $W(t) = \{u(t) : u \in W, t \in J \setminus \{t_1, t_2, ..., t_m\}\}, W(t_i^+) = \{u(t_i^+) : u \in W\}$ *and* $W(t_i^-) = \{u(t_i^-) : u \in W\}$, $i = 1, 2, .., m$, *are relatively compact subsets of X.*
Then W is a relatively compact subset of $PC(J, X)$.

Let us recall the following well-known definitions.

Definition 2.2. ([6]) The Riemann-Liouville fractional integral of order q with lower limit zero for a function f is defined as
$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds, \ q > 0,$

provided the integral exists, where $\Gamma(.)$ is the gamma function.

Definition 2.3. ([6]) The Riemann-Liouville derivative of order q with the lower limit zero for a function $f : [0, \infty) \to \mathbb{R}$ can be written as
${}^L D^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-q-1} f(s) ds, \ n-1 < q < n, t > 0.$

Definition 2.4. ([6]) The Caputo derivative of order q for a function $f : [0, \infty) \longrightarrow \mathbb{R}$ can be written as
${}^c D^q f(t) = {}^L D^q \left(f(t) - \sum_{k=0}^{k} \frac{t^k}{k!} f^k(0) \right), \ n-1 < q < n, \ t > 0.$

Definition 2.5. ([12]-[14]) A function $x \in C(J, X)$ is said to be a mild solution of the following problem:

$$\begin{cases} {}^c D^\alpha x(t) = Ax(t) + y(t), \ t \in (0, T], \\ x(0) = x_0. \end{cases}$$

If it satisfies the integral equation
$x(t) = P_\alpha(t) x_0 + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s) y(s) ds.$

Here

$P_\alpha(t) = \int_0^\infty \xi_\alpha(\theta) T(t^\alpha \theta) d\theta$, $Q_\alpha(t) = \alpha \int_0^\infty \theta \xi_\alpha(\theta) T(t^\alpha \theta) d\theta$,

$\xi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \overline{\omega}_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0,$

$\overline{\omega_\alpha}(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} sin(n\pi\alpha), \theta \in (0, \infty),$

and $\xi_\alpha(\theta) \geq 0$, $\theta \in (0, \infty)$, $\int_0^\infty \xi_\alpha(\theta) d\theta = 1$.

It is easy to verify that

$\int_0^\infty \theta \xi_\alpha(\theta) d\theta = \frac{1}{\Gamma(1+\alpha)}.$

We make the following assumption on A in the whole paper.

H(1): The operator A generates a strongly continuous semigroup $\{T(t) : t \geq 0\}$ in X, and there is a constant $M_A \geq 1$ such that $\sup_{t \in [0,\infty)} \| T(t) \|_{L(X)} \leq M_A$. For any $t > 0$, $T(t)$ is compact.

Lemma 2.6. *([12]-[14]) Let $H(A)$ hold, then the operator P_α and Q_α have the following properties:*

(1) *For any fixed $t \geq 0$, $P_\alpha(t)$ and $Q_\alpha(t)$ are linear and bounded operators, and for any $x \in X$,*
$\| P_\alpha(t)x \| \leq M_A \| x \|$, $\| Q_\alpha(t)x \| \leq \dfrac{\alpha M_A}{\Gamma(1+\alpha)} \| x \|$,
(2) *$\{P_\alpha(t), t \geq 0\}$ and $\{Q_\alpha(t), t \geq 0\}$ are strongly continuous,*
(3) *for every $t > 0$, $P_\alpha(t)$ and $Q_\alpha(t)$ are compact operators.*

We recall a fixed point theorem which will be needed in the sequel.

Theorem 2.7. *(Krasnoselskii fixed point theorem) Let M be a closed, convex, and non-empty subset of a Banach space X. Let A,B be the operators such that:*

(a) *$Ax + By \in M$ for all $x, y \in M$,*
(b) *A is compact and continuous,*
(c) *B is a contraction.*

Then there exists a $x \in M$ such that $x = Ax + Bx$.

3. The construction of mild solutions

Let $x \in PC(J, X)$. We first consider the following fractional impulsive problem:

$$\begin{cases} {}^cD^\alpha x(t) = Ax(t) + f(t, x(t), Fx(t), Bx(t)) + \int_0^t q(t-s)k(s, x(s))ds + C(t)u(t),\ t \in (s_i, t_{i+1}], \\ i = 0, 1, 2, ..., m,\ u \in U_{ad}, \\ {}^cD^\beta x(t) = g_i(t, x(t)),\ t \in (t_i, s_i],\ i = 1, 2, ..., m, \\ x(0) = x_0 + h(x). \end{cases}$$

From the property of the Caputo derivative, a general solution can be written as

$$x(t) = \begin{cases} x_0 + h(x) + \dfrac{1}{\Gamma(\alpha} \int_0^t (t-s)^{\alpha-1}[Ax(s) + f(s, x(s), Fx(s), Bx(s)) + \int_0^s q(s-\tau)k(\tau, x(\tau))d\tau + \\ C(s)u(s)]ds,\ t \in (0, t_1], \\ d_{1x} + \dfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g_1(s, x(s))ds,\ t \in (t_1, s_1], \\ K_{1x} + \dfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[Ax(s) + f(s, x(s), Fx(s), Bx(s)) + \int_0^s q(s-\tau)k(\tau, x(\tau))d\tau + \\ C(s)u(s)]ds,\ t \in (s_1, t_2], \\ . \\ . \\ d_{ix} + \dfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g_i(s, x(s))ds,\ t \in (t_i, s_i], \\ K_{ix} + \dfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[Ax(s) + f(s, x(s), Fx(s), Bx(s)) + \int_0^s q(s-\tau)k(\tau, x(\tau))d\tau + \\ C(s)u(s)]ds,\ t \in (s_i, t_{i+1}], \end{cases}$$

where d_{ix} and K_{ix}, $i = 1, 2, ..., m$, are elements of X.
Using [3] pages 5 and 6 we obtain:

$$x(t) = \begin{cases} d_{ix} + \dfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g_i(s, x(s))ds, \ t \in (t_i, s_i], 1 \le i \le m, \\ P_\alpha(t - s_i)K_{ix} + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)[f(s, x(s), Fx(s), Bx(s)) + \int_0^s q(s-\tau)k(\tau, x(\tau))d\tau + \\ +C(s)u(s)]ds, \ t \in (s_i, t_{i+1}], 0 \le i \le m, \\ K_{0x} = x(0). \end{cases}$$

And using the fact that x is continuous at the points t_i, we get :

$$\begin{aligned} x(t_i) &= P_\alpha(t_i - s_{i-1})K_{(i-1)x} + \int_0^{t_i} (t_i - s)^{\alpha-1} Q_\alpha(t_i - s)[f(s, x(s), Fx(s), Bx(s)) + \\ &\quad + \int_0^s q(s-\tau)k(\tau, x(\tau))d\tau + C(s)u(s)]ds \\ &= d_{ix} + \frac{1}{\Gamma(\beta)} \int_0^{t_i} (t_i - s)^{\beta-1} g_i(s, x(s))ds. \end{aligned}$$

Which implies that:

$$\begin{aligned} d_{ix} &= P_\alpha(t_i - s_{i-1})K_{(i-1)x} + \int_0^{t_i} (t_i - s)^{\alpha-1} Q_\alpha(t_i - s)[f(s, x(s), Fx(s), Bx(s)) + \\ &\quad + \int_0^s q(s-\tau)k(\tau, x(\tau))d\tau + C(s)u(s)]ds - \frac{1}{\Gamma(\beta)} \int_0^{t_i} (t_i - s)^{\beta-1} g_i(s, x(s))ds. \end{aligned}$$

Using the fact that x is continuous at the points s_i, we get :

$$\begin{aligned} x(s_i) &= d_{ix} + \frac{1}{\Gamma(\beta)} \int_0^{s_i} (s_i - s)^{\beta-1} g_i(s, x(s))ds \\ &= K_{ix} + \int_0^{s_i} (s_i - s)^{\alpha-1} Q_\alpha(s_i - s) \left[f(s, x(s), Fx(s), Bx(s)) + \int_0^s q(s-\tau)k(\tau, x(\tau))d\tau + C(s)u(s)\right] ds. \end{aligned}$$

Which implies that:

$$\begin{aligned} K_{ix} &= d_{ix} + \frac{1}{\Gamma(\beta)} \int_0^{s_i} (s_i - s)^{\beta-1} g_i(s, x(s))ds - \\ &\quad - \int_0^{s_i} (s_i - s)^{\alpha-1} Q_\alpha(s_i - s) \left[f(s, x(s), Fx(s), Bx(s)) + \int_0^s q(s-\tau)k(\tau, x(\tau))d\tau + C(s)u(s)\right] ds \end{aligned}.$$

Therefore, a mild solution of problem (1.2) is given by

$$x(t) = \begin{cases} P_\alpha(t)(x_0 + h(x)) + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)[f(s, x(s), Fx(s), Bx(s)) + \int_0^s q(s-\tau)k(\tau, x(\tau))d\tau + \\ C(s)u(s)]ds, \ t \in (0, t_1], \\ d_{1x} + \dfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g_1(s, x(s))ds, \ t \in (t_1, s_1], \\ P_\alpha(t - s_1)K_{1x} + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)[f(s, x(s), Fx(s), Bx(s)) + \int_0^s q(s-\tau)k(\tau, x(\tau))d\tau + \\ C(s)u(s)]ds, \ t \in (s_1, t_2], \\ . \\ . \\ d_{ix} + \dfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g_i(s, x(s))ds, \ t \in (t_i, s_i], 1 \le i \le m, \\ P_\alpha(t - s_i)K_{ix} + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)[f(s, x(s), Fx(s), Bx(s)) + \int_0^s q(s-\tau)k(\tau, x(\tau))d\tau + \\ C(s)u(s)]ds, \\ \ t \in (s_i, t_{i+1}], 1 \le i \le m, \end{cases}$$

where

$K_{0x} = x_0 + h(x)$

$$\begin{aligned} d_{ix} &= P_\alpha(t_i - s_{i-1})K_{(i-1)x} + \int_0^{t_i} (t_i - s)^{\alpha-1} Q_\alpha(t_i - s)[f(s, x(s), Fx(s), Bx(s)) + \int_0^s q(s-\tau)k(\tau, x(\tau))d\tau + \\ &\quad +C(s)u(s)]ds - \frac{1}{\Gamma(\beta)} \int_0^{t_i} (t_i - s)^{\beta-1} g_i(s, x(s))ds, \\ K_{ix} &= d_{ix} - \int_0^{s_i} (s_i - s)^{\alpha-1} Q_\alpha(s_i - s) \left[f(s, x(s), Fx(s), Bx(s)) + \int_0^s q(s-\tau)k(\tau, x(\tau))d\tau + C(s)u(s)\right] ds + \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^{s_i} (s_i - s)^{\beta-1} g_i(s, x(s))ds. \end{aligned}$$

Definition 3.1. A function $x \in PC(J;X)$ is said to be a mild solution of problem (1.2) if it satisfies the following relation:

$$x(t) = \begin{cases} P_\alpha(t)K_{0x} + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)[f(s,x(s),Fx(s),Bx(s)) + \int_0^s q(s-\tau)k(\tau,x(\tau))d\tau + \\ C(s)u(s)]ds, \ t \in (0,t_1], \\ d_{ix} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g_i(s,x(s))ds, \ t \in (t_i,s_i], 1 \le i \le m, \\ P_\alpha(t-s_i)K_{ix} + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)[f(s,x(s),Fx(s),Bx(s)) + \int_0^s q(s-\tau)k(\tau,x(\tau))d\tau + \\ C(s)u(s)]ds, \ t \in (s_i,t_{i+1}], 1 \le i \le m. \end{cases}$$

Where
$K_{0x} = x_0 + h(x)$,

$$\begin{aligned} d_{ix} &= P_\alpha(t_i - s_{i-1})K_{(i-1)x} + \int_0^{t_i} (t_i-s)^{\alpha-1} Q_\alpha(t_i-s)[f(s,x(s),Fx(s),Bx(s)) + \int_0^s q(s-\tau)k(\tau,x(\tau))d\tau + \\ &\quad C(s)u(s)]ds - \frac{1}{\Gamma(\beta)} \int_0^{t_i} (t_i-s)^{\beta-1} g_i(s,x(s))ds, \\ K_{ix} &= d_{ix} - \int_0^{s_i} (s_i-s)^{\alpha-1} Q_\alpha(s_i-s)\left[f(s,x(s),Fx(s),Bx(s)) + \int_0^s q(s-\tau)k(\tau,x(\tau))d\tau + C(s)u(s)\right] ds + \\ &\quad \frac{1}{\Gamma(\beta)} \int_0^{s_i} (s_i-s)^{\beta-1} g_i(s,x(s))ds. \end{aligned}$$

Here

$$P_\alpha(t) = \int_0^\infty \xi_\alpha(\theta)T(t^\alpha\theta)d\theta, \ Q_\alpha(t) = \alpha \int_0^\infty \theta\xi_\alpha(\theta)T(t^\alpha\theta)d\theta,$$

$$\xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\overline{\omega}_\alpha(\theta^{-\frac{1}{\alpha}}) \ge 0,$$

$$\overline{\omega_\alpha}(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!} sin(n\pi\alpha), \theta \in (0,\infty),$$

and $\xi_\alpha(\theta) \ge 0$, $\theta \in (0,\infty)$, $\int_0^\infty \xi_\alpha(\theta)d\theta = 1$.

It is easy to verify that

$$\int_0^\infty \theta\xi_\alpha(\theta)d\theta = \frac{1}{\Gamma(1+\alpha)}.$$

We make the following assumption on A in the whole paper.

H(1): The operator A generates a strongly continuous semigroup $\{T(t) : t \ge 0\}$ in X, and there is a constant $M_A \ge 1$ such that $\sup_{t\in[0,\infty)} \| T(t) \|_{L(X)} \le M_A$. For any $t > 0$, $T(t)$ is compact.

Lemma 3.2. *(*[12]-[14]*) Let $H(A)$ hold, then the operator P_α and Q_α have the following properties:*

(1) *For any fixed $t \ge 0$, $P_\alpha(t)$ and $Q_\alpha(t)$ are linear and bounded operators, and for any $x \in X$,*
$$\| P_\alpha(t)x \| \le M_A \| x \|, \ \| Q_\alpha(t)x \| \le \frac{\alpha M_A}{\Gamma(1+\alpha)} \| x \|,$$
(2) *$\{P_\alpha(t), t \ge 0\}$ and $\{Q_\alpha(t), t \ge 0\}$ are strongly continuous,*
(3) *for every $t > 0$, $P_\alpha(t)$ and $Q_\alpha(t)$ are compact operators.*

We recall a fixed point theorem which will be needed in the sequel.

Theorem 3.3. *(Krasnoselskii fixed point theorem) Let M be a closed, convex, and non-empty subset of a Banach space X. Let A,B be the operators such that:*

(a) $Ax + By \in M$ *for all* $x, y \in M$,
(b) A *is compact and continuous,*
(c) B *is a contraction.*

Then there exists a $x \in M$ *such that* $x = Ax + Bx$.

4. Existence and Uniqueness of Mild Solution

This section deals with the existence results for the problem (1.2).

From Definition (3.1), we define an operator $P : PC(J, X) \to PC(J, X)$ as

$$Px(t) = \begin{cases} P_\alpha(t)(x_0 + h(x)) + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)[f(s, x(s), Fx(s), Bx(s)) + \int_0^s q(s-\tau)k(\tau, x(\tau))d\tau + \\ C(s)u(s)]ds, \ t \in [0, t_1], \\ d_{ix} + \dfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g_i(s, x(s))ds, \ t \in (t_i, s_i], \\ P_\alpha(t-s_i)K_{ix} + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)[f(s, x(s), Fx(s), Bx(s)) + \int_0^s q(s-\tau)k(\tau, x(\tau))d\tau + \\ C(s)u(s)]ds, \ t \in (s_i, t_{i+1}]. \end{cases}$$

To prove our first existence result we introduce the following assumptions.
(H_2) $C : [0, T] \longrightarrow L(Y, X)$ is essentially bounded, ie $C \in L^\infty([0, T], L(Y, X))$,
(H_3) The functions $f \in C(J \times X \times X \times X; X)$,and $k \in C(J \times X; X)$,
(H_4)there exists $C_f, L_f, M_f, L_k > 0$ such that

- $\| f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2) \| \leq L_f \| x_1 - x_2 \| + C_f \| y_1 - y_2 \| + M_f \| z_1 - z_2 \|$, for all $x_1, x_2, y_1, y_2, z_1, z_2 \in X$, and $t \in J$,
- $\| k(t, x_1) - k(t, x_2) \| \leq L_k \| x_1 - x_2 \|$, for all $x_1, x_2 \in X$, and $t \in J$,

(H_5) There exists a constant $D, D' > 0$ such that

- $\| f(t, x, y, z) \| \leq D\left(1 + \| x \|^\mu + \| y \|^\nu + \| z \|^\phi\right)$ for every $t \in J$ and $x, y, z \in X$, $\mu, \nu, \phi \in [0, 1]$,
- $\| k(t, x) \| \leq D' (1 + \| x \|^\mu)$ for every $t \in J$ and $x \in X$, $\mu \in [0, 1]$,

(H_6)

- For $i = 1, 2, ..., m$, $g_i \in C(J \times X; X)$, and there exists $L_g > 0$ such that $\| g_i(t, x) - g_i(t, y) \| \leq L_g \| x - y \|$, for all $x, y \in X$,
- There exists a function $t \longrightarrow \varphi_i(t), i = 1, 2, ..., m$ such that $\| g_i(t, x(t)) \| \leq \varphi_i(t)$ for all $t \in J$, $x \in X$ and $L_{g_i} = \sup_{t \in J} \varphi_i(t)$ and $L'_g = \underset{1 \leq i \leq m}{max} L_{g_i}$,
- $h : PC(J; X) \to X$ and there exists a constant $L_h > 0$ and $\varphi_h \in C([0, \infty); \mathbb{R}^+)$ such that for $x, y \in PC(J; X)$, $\| h(x) - h(y) \| \leq L_h \| x - y \|_{PC}$; $\| h(x) \| \leq \varphi_h(t)$ and $L'_h = \sup_{t \in J} \varphi_h(t)$.

Theorem 4.1. *Assume that* $(H_1) - (H_4)$,(H_6) *are satisfied and*

$$[M^{m+1}L_h + \frac{MT^\alpha + M^2(t_m^\alpha + s_m^\alpha) + .. + M^{m+1}(t_1^\alpha + s_1^\alpha)}{\Gamma(\alpha+1)}(L_f + q^* L_k + M_f B^* + C_f F^*) + \frac{ML_g(t_m^\beta + s_m^\beta) + .. + M^m L_g(t_1^\beta + s_1^\beta)}{\Gamma(\beta+1)}] < 1.$$

Then there exists a unique mild solution of problem (1.2).

Proof. From the assumptions it is easy to show that the operator P is well defined on $PC(J,X)$.
Let $x,y\in PC(J,X)$.
Case 1. For $t\in[0,t_1]$, we have

$$\begin{aligned}
\|(Px)(t)-(Py)(t)\| &\leq \|P_\alpha(t)\,(x_0+h(x)-x_0-h(y))\|+\\
&+\|\int_0^t(t-s)^{\alpha-1}Q_\alpha(t-s)[f(s,x(s),Fx(s),Bx(s))+\int_0^s q(s-\tau)k(\tau,x(\tau))d\tau+\\
&C(s)u(s)-f(s,y(s),Fy(s),By(s))-\int_0^s q(s-\tau)k(\tau,y(\tau))d\tau-C(s)u(s)]ds\|\\
&\leq M_A\|h(x)-h(y)\|+\frac{\alpha M_A}{\Gamma(\alpha+1)}\int_0^t(t-s)^{\alpha-1}\parallel f(s,x(s),Fx(s),Bx(s))-\\
&f(s,y(s),Fy(s),By(s))+\int_0^s q(s-\tau)k(\tau,x(\tau))d\tau-\int_0^s q(s-\tau)k(\tau,y(\tau))d\tau\parallel ds\\
&\leq M_AL_h\parallel x-y\parallel_{PC}+\frac{\alpha M_A}{\Gamma(\alpha+1)}\int_0^t(t-s)^{\alpha-1}(L_f\parallel x(s)-y(s)\parallel+\\
&C_f\parallel Fx(s)-Fy(s)\parallel+M_f\parallel Bx(s)-By(s)\parallel+q^*L_k\parallel x(s)-y(s)\parallel)ds\\
&\leq M_AL_h\parallel x-y\parallel_{PC}+\frac{M_At_1^\alpha}{\Gamma(\alpha+1)}\,(L_f+q^*L_k+M_fB^*+C_fF^*)\parallel x-y\parallel_{PC}\\
&\leq [M_AL_h+\frac{M_At_1^\alpha}{\Gamma(\alpha+1)}\,(L_f+q^*L_k+M_fB^*+C_fF^*)]\parallel x-y\parallel_{PC}\,.
\end{aligned}$$

Case 2. For $t\in(t_i,s_i]\cup(s_i,t_{i+1}]$.
We prove that for $t\in(t_i,s_i],\ i=1,2,...,m$:

$$\begin{aligned}
\parallel(Px)(t)-(Py)(t)\parallel &\leq [M_A^iL_h+\frac{M_At_i^\alpha+M_A^2(t_{i-1}^\alpha+s_{i-1}^\alpha)+M_A^3(t_{i-2}^\alpha+s_{i-2}^\alpha)+...+M_A^i(t_1^\alpha+s_1^\alpha)}{\Gamma(\alpha+1)}\times\\
&(L_f+q^*L_k+M_fB^*+C_fF^*)+\\
&\frac{L_g(t_i^\beta+s_i^\beta)+M_AL_g(t_{i-1}^\beta+s_{i-1}^\beta)+...+M_A^{i-2}L_g(t_2^\beta+s_2^\beta)+M_A^{i-1}L_g(t_1^\beta+s_1^\beta)}{\Gamma(\beta+1)}]\times\\
&\parallel x-y\parallel_{PC},
\end{aligned}$$

and for $t\in[s_i,t_{i+1}],\ i=1,2,...,m$,

$$\begin{aligned}
\parallel(Px)(t)-(Py)(t)\parallel &\leq [M_A^{i+1}L_h+\frac{M_At_{i+1}^\alpha+M_A^2(t_i^\alpha+s_i^\alpha)+...+M_A^i(t_2^\alpha+s_2^\alpha)+M_A^{i+1}(t_1^\alpha+s_1^\alpha)}{\Gamma(\alpha+1)}\times\\
&(L_f+q^*L_k+M_fB^*+C_fF^*)+\\
&\frac{M_AL_g(t_i^\beta+s_i^\beta)+...+M_A^{i-1}L_g(t_2^\beta+s_2^\beta)+M_A^iL_g(t_1^\beta+s_1^\beta)}{\Gamma(\beta+1)}]\parallel x-y\parallel_{PC}\,.
\end{aligned}$$

For $t\in(t_1,s_1]$

$$\begin{aligned}
\parallel(Px)(t)-(Py)(t)\parallel &\leq \parallel d_1x+\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}g_1(s,x(s))ds-d_1y-\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}g_1(s,y(s))ds\parallel\\
&\leq \parallel d_1x-d_1y\parallel+\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\parallel g_1(s,x(s))-g_1(s,y(s))\parallel ds\\
&\leq \parallel P_\alpha(t_1)(h(x)-h(y))\parallel+\parallel\int_0^{t_1}(t_1-s)^{\alpha-1}Q_\alpha(t_1-s)[f(s,x(s),Fx(s),Bx(s))-\\
&-f(s,y(s),Fy(s),By(s))+\int_0^s q(s-\tau)k(\tau,x(\tau))d\tau-\int_0^s q(s-\tau)k(\tau,y(\tau))d\tau]ds\parallel\\
&+\parallel\frac{1}{\Gamma(\beta)}\int_0^{t_1}(t-s)^{\beta-1}(g_1(s,x(s))-g_1(s,y(s)))ds\parallel+\\
&+\parallel\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}(g_1(s,x(s))-g_1(s,y(s)))ds\parallel\\
&\leq M_AL_h\parallel x-y\parallel_{PC}+\frac{M_At_1^\alpha}{\Gamma(\alpha+1)}(L_f+q^*L_k+M_fB^*+C_fF^*)\parallel x-y\parallel_{PC}+\\
&\frac{(s_1^\beta+t_1^\beta)}{\Gamma(\beta+1)}L_g\parallel x-y\parallel_{PC}\\
&\leq \left[M_AL_h+\frac{M_At_1^\alpha}{\Gamma(\alpha+1)}(L_f+q^*L_k+M_fB^*+C_fF^*)+\frac{(s_1^\beta+t_1^\beta)}{\Gamma(\beta+1)}L_g\right]\parallel x-y\parallel_{PC}\,.
\end{aligned}$$

For $t \in (s_1, t_2]$

$$\begin{aligned}
\| Px(t) - Py(t) \| &\leq \| P_\alpha(t-s_1)K_{1x} - P_\alpha(t-s_1)K_{1y} \| \\
&+ \| \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)[\| f(s,x(s),Fx(s),Bx(s)) - f(s,y(s),Fy(s),By(s)) \\
&+ \int_0^s q(s-\tau)k(\tau,x(\tau))d\tau - \int_0^s q(s-\tau)k(\tau,y(\tau))d\tau]ds \| \\
&\leq M_A \| d_{1x} - d_{1y} \| + M_A \| \int_0^{s_1} (s_1-s)^{\alpha-1} Q_\alpha(s_1-s)[f(s,x(s),Fx(s),Bx(s)) - \\
&f(s,y(s),Fy(s),By(s)) + \int_0^s q(s-\tau)k(\tau,x(\tau))d\tau - \int_0^s q(s-\tau)k(\tau,y(\tau))d\tau]ds \| + \\
&+ M_A \| \frac{1}{\Gamma(\beta)} \int_0^{s_1} (s_1-s)^{\beta-1} [g_1(s,x(s)) - g_1(s,y(s))] ds \| \\
&+ \| \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)[f(s,x(s),Fx(s),Bx(s)) - f(s,y(s),Fy(s),By(s)) + \\
&+ \int_0^s q(s-\tau)k(\tau,x(\tau))d\tau - \int_0^s q(s-\tau)k(\tau,y(\tau))d\tau]ds \| \\
&\leq \left[M_A^2 L_h + \frac{M_A^2 t_1^\alpha + M_A^2 s_1^\alpha + M_A t_2^\alpha}{\Gamma(\alpha+1)} (L_f + q^* L_k + M_f B^* + C_f F^*) + \frac{M_A L_g(s_1^\beta + t_1^\beta)}{\Gamma(\beta+1)} \right] \|
\end{aligned}$$

We suppose that for $1 \leq j \leq i$ we have:
for $t \in (t_j, s_j]$

$$\begin{aligned}
\| Px(t) - Py(t) \| &\leq [M_A^j L_h + \frac{M_A t_j^\alpha + M_A^2(t_{j-1}^\alpha + s_{j-1}^\alpha) + M_A^3(t_{j-2}^\alpha + s_{j-2}^\alpha) + .. + M_A^j(t_1^\alpha + s_1^\alpha)}{\Gamma(\alpha+1)} \times \\
&(L_f + q^* L_k + M_f B^* + C_f F^*) \\
+ &\frac{L_g(t_j^\beta + s_j^\beta) + M_A L_g(t_{j-1}^\beta + s_{j-1}^\beta) + .. + M_A^{j-1} L_g(t_1^\beta + s_1^\beta)}{\Gamma(\beta+1)}] \| x - y \|_{PC},
\end{aligned}$$

and for $t \in (s_j, t_{j+1}]$,

$$\begin{aligned}
\| (Px)(t) - (Py)(t) \| &\leq M_A[M_A^j L_h + \frac{t_{j+1}^\alpha + M_A(t_j^\alpha + s_j^\alpha) + .. + M_A^j(t_1^\alpha + s_1^\alpha)}{\Gamma(\alpha+1)} \times \\
&(L_f + q^* L_k + M_f B^* + C_f F^*) + \\
&\frac{L_g(t_j^\beta + s_j^\beta) + .. + M_A^{j-2} L_g(t_2^\beta + s_2^\beta) + M_A^{j-1} L_g(t_1^\beta + s_1^\beta)}{\Gamma(\beta+1)}] \| x - y \|_{PC} .
\end{aligned}$$

We prove the relations for $j = i+1$.
For $t \in (t_{i+1}, s_{i+1}]$

$$\begin{aligned}
\| Px(t) - Py(t) \| &\leq \| d_{i+1}x + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g_{i+1}(s,x(s))ds - d_{i+1}y - \\
&\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g_{i+1}(s,y(s))ds \| \\
&\leq \| d_{i+1}x - d_{i+1}y \| + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \| g_{i+1}(s,x(s)) - g_{i+1}(s,y(s)) \| ds
\end{aligned}$$

.

$$\begin{aligned}
&\leq \| P_\alpha(t_{i+1}-s_i)(K_{ix} - K_{iy}) \| + \| \int_0^{t_{i+1}} (t_{i+1}-s)^{\alpha-1} Q_\alpha(t_{i+1}-s) \times \\
&[f(s,x(s),Fx(s),Bx(s)) - -f(s,y(s),Fy(s),By(s)) + \\
&\int_0^s q(s-\tau)k(\tau,x(\tau))d\tau - \int_0^s q(s-\tau)k(\tau,y(\tau))d\tau]ds \| \\
&+ \| \frac{1}{\Gamma(\beta)} \int_0^{t_{i+1}} (t_{i+1}-s)^{\beta-1} (g_{i+1}(s,x(s)) - g_{i+1}(s,y(s))ds \| + \\
&+ \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \| (g_{i+1}(s,x(s)) - g_{i+1}(s,y(s))) \| ds
\end{aligned}$$

For

$$\begin{aligned}
&\leq [M_A^{i+1} L_h + \frac{M_A t_{i+1}^\alpha + M_A^2(t_i^\alpha + s_i^\alpha) + ... + M_A^{i+1}(t_1^\alpha + s_1^\alpha)}{\Gamma(\alpha+1)} \times \\
&(L_f + q^* L_k + M_f B^* + C_f F^*) + \\
&\frac{L_g(t_{i+1}^\beta + s_{i+1}^\beta) + M_A^i L_g(t_1^\beta + s_1^\beta) + ... + M_A L_g(t_i^\beta + s_i^\beta)}{\Gamma(\beta+1)}] \| x - y \|_{PC} .
\end{aligned}$$

$t \in (s_{i+1}, t_{i+2}]$

$$\begin{aligned}\| Px(t)-Py(t) \| &\leq \| P_\alpha(t-s_{i+1})K_{(i+1)x}-P_\alpha(t-s_{i+1})K_{(i+1)y} \| \\ &+\| \int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s)[f(s,x(s),Fx(s),Bx(s))-f(s,y(s),Fy(s),By(s)) \\ &+\int_0^s q(s-\tau)k(\tau,x(\tau))d\tau-\int_0^s q(s-\tau)k(\tau,y(\tau))d\tau]ds \| \\ &\leq \leq M_A[\| d_{(i+1)x}-d_{(i+1)y} \| + \frac{s_{i+1}^\alpha M_A}{\Gamma(\alpha+1)}(L_f+q^*L_k+M_fB^*+C_fF^*) \| x-y \|_{PC} + \\ &+\frac{s_{i+1}^\beta}{\Gamma(\beta+1)}L_g \| x-y \|_{PC}] + \frac{t_{i+2}^\alpha M_A}{\Gamma(\alpha+1)}(L_f+q^*L_k+M_fB^*+C_fF^*) \| x-y \|_{PC} \\ &\leq \Bigg[M_A\Bigg[M_A^{i+1}L_h+\frac{t_{i+2}^\alpha+M_A^{i+1}(t_1^\alpha+s_1^\alpha)+...+M_A^2(t_i^\alpha+s_i^\alpha)+M_A(t_{i+1}^\alpha+s_{i+1}^\alpha)}{\Gamma(\alpha+1)}\times \\ &\times(L_f+q^*L_k+M_fB^*+C_fF^*)\Bigg]+ \\ &+\frac{M_AL_g(s_{i+1}^\beta+t_{i+1}^\beta)+M_A^2L_g(s_i^\beta+t_i^\beta)+..+M_A^{i+1}L_g(t_1^\beta+s_1^\beta)}{\Gamma(\beta+1)}\Bigg] \| x-y \|_{PC}\,.\end{aligned}$$

Then it follows that P is a contraction on the space $PC(J,X)$. Hence by the Banach contraction mapping principale, P has a unique fixed point $x\in PC(J,X)$ which is the unique mild solution of problem (1.2) .

□

The next result is based on Krasnoselskii fixed point theorem.

Theorem 4.2. *Assume that $(H_1)-(H_3)$, and $(H_5)-(H_6)$ hold. In addition, let's suppose that the following condition is verified:*

$$max\Bigg\{M_A^{m+1}L_h+\frac{M_AL_g(s_m^\beta+t_m^\beta)+M_A^2L_g(s_{m-1}^\beta+t_{m-1}^\beta)+...+M_A^mL_g(s_1^\beta+t_1^\beta)}{\Gamma(\beta+1)};$$
$$\frac{[M_AT^\alpha+M_A^2(t_m^\alpha+s_m^\alpha)+...+M_A^{m+1}(t_1^\alpha+s_1^\alpha)]}{\Gamma(\alpha+1)}(D(1+B^*+F^*)+D'q^*)\Bigg\}<1.$$

Then the problem (1.2) *has at least one mild solution.*

Proof. We introduce the composition $Q=Q_1+Q_2$ where :

$$Q_1x(t)=\begin{cases}P_\alpha(t)(x_0+h(x))+\int_0^t(t-s)^{\alpha-1}Q_\alpha(t-s)C(s)u(s)ds,\ t\in[0,t_1],\\ d_{i1x}+\dfrac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}g_i(s,x(s))ds,\ t\in(t_i,s_i],\ i=1,2,..,m,\\ P_\alpha(t-s_i)K_{i1x}+\int_0^t(t-s)^{\alpha-1}Q_\alpha(t-s)C(s)u(s)ds,\ t\in(s_i,t_{i+1}],\ i=1,2,..,m,\end{cases}$$

$$Q_2x(t)=\begin{cases}\int_0^t(t-s)^{\alpha-1}Q_\alpha(t-s)\left[f(s,x(s),Fx(s),Bx(s))+\int_0^s q(s-\tau)k(\tau,x(\tau))d\tau\right]ds,\ t\in[0,t_1],\\ d_{i2x},\ t\in(t_i,s_i],\ i=1,2,..,m,\\ P_\alpha(t-s_i)K_{i2x}+\int_0^t(t-s)^{\alpha-1}Q_\alpha(t-s)\left[f(s,x(s),Fx(s),Bx(s))+\int_0^s q(s-\tau)k(\tau,x(\tau))d\tau\right]ds,\\ \ t\in(s_i,t_{i+1}],\ i=1,2,..,m,\end{cases}$$

with

$$\begin{cases} d_{i1x} = P_\alpha(t_i - s_{i-1})K_{(i-1)1x} + \int_0^{t_i}(t_i - s)^{\alpha-1}Q_\alpha(t_i - s)C(s)u(s)ds - \dfrac{1}{\Gamma(\beta)}\int_0^{t_i}(t_i - s)^{\beta-1}g_i(s, x(s))ds, \\ i = 1, 2, .., m, \\ K_{i1x} = d_{i1x} + \dfrac{1}{\Gamma(\beta)}\int_0^{s_i}(s_i - s)^{\beta-1}g_i(s, x(s))ds - \int_0^{s_i}(s_i - s)^{\alpha-1}Q_\alpha(s_i - s)C(s)u(s)ds, \ i = 1, 2, .., m, \\ K_{01x} = x_0 + h(x), \end{cases}$$

and

$$\begin{cases} d_{i2x} = P_\alpha(t_i - s_{i-1})K_{(i-1)2x} + \int_0^{t_i}(t_i - s)^{\alpha-1}Q_\alpha(t_i - s)[f(s, x(s), Fx(s), Bx(s))+ \\ \int_0^s q(s-\tau)k(\tau, x(\tau))d\tau]ds, \ i = 1, 2, .., m, \\ K_{i2x} = d_{i2x} - \int_0^{s_i}(s_i - s)^{\alpha-1}Q_\alpha(s_i - s)\left[f(s, x(s), Fx(s), Bx(s)) + \int_0^s q(s-\tau)k(\tau, x(\tau))d\tau\right]ds, \\ i = 1, 2, .., m, \\ K_{02x} = 0. \end{cases}$$

Our proof will be divided into several steps.

- We show that $QB_r(J) \subset B_r(J)$
  where $B_r = \{x \in PC(J, X); \| x \| \leq r\}$ the ball with radius $r > 0$,
  and $K_{\alpha,\tau} = \alpha\left(\dfrac{1-\tau}{\alpha-\tau}\right)^{1-\tau} \| Cu \|_{L^{1/\tau}}$, $D'' = D + D'q^*$

$$\gamma_1 = M_A^{m+1}[\| x_0 \| + L'_h] + \frac{M_A[D''T^\alpha + K_{\alpha,\tau}T^{\alpha-\tau}]}{\Gamma(\alpha+1)},$$

$$\gamma_2 = \frac{M_A^2[D''(t_m^\alpha + s_m^\alpha) + K_{\alpha,\tau}(t_m^{\alpha-\tau} + s_m^{\alpha-\tau})] + .. + M_A^{m+1}[D''(t_1^\alpha + s_1^\alpha) + K_{\alpha,\tau}(t_1^{\alpha-\tau} + s_1^{\alpha-\tau})]}{\Gamma(\alpha+1)},$$

$$\gamma_3 = \frac{M_A L'_g(t_i^\beta + s_m^\beta) + M_A^2 L'_g(t_{m-1}^\beta + s_{m-1}^\beta) + ... + M_A^m L'_g(t_1^\beta + s_1^\beta)}{\Gamma(\beta+1)},$$

$$\gamma_4 = \frac{[M_A T^\alpha + M_A^2(t_m^\alpha + s_m^\alpha) + ... + M_A^{m+1}(t_1^\alpha + s_1^\alpha)]}{\Gamma(\alpha+1)}(D(1 + B^* + F^*) + D'q^*).$$

Here $\dfrac{\gamma_1 + \gamma_2 + \gamma_3}{1 - \gamma_4} < r.$

For any $x \in B_r$, we have:

Case1. For $t \in [0, t_1]$,

$$\begin{aligned} \| Qx(t) \| &\leq M_A[\| x_0 \| + L'_h] + \frac{M_A t_1^\alpha}{\Gamma(\alpha+1)}[D(1 + r(1 + B^* + F^*)) + D'q^*(1+r)] + \\ &\quad \frac{K_{\alpha,\tau}t_1^{\alpha-\tau}}{\Gamma(\alpha+1)}M_A \\ &\leq M_A\left[\| x_0 \| + L'_h + \frac{D''t_1^\alpha + K_{\alpha,\tau}t_1^{\alpha-\tau}}{\Gamma(\alpha+1)}\right] + M_A\frac{(D(1 + B^* + F^*) + D'q^*)t_1^\alpha}{\Gamma(\alpha+1)}r \\ &\leq r. \end{aligned}$$

Similar to the proof of Theorem(4.1), we prove that:

Case 2. For $t \in (t_i, s_i], i = 1, 2, ..., m,$

$$\begin{aligned} \| Qx(t) \| &\leq M_A^i [\| x_0 \| + L'_h] + \frac{M_A[D'' t_i^\alpha + K_{\alpha,\tau} t_i^{\alpha-\tau}]}{\Gamma(\alpha+1)} + \\ &\frac{M_A^2[D''(t_{i-1}^\alpha + s_{i-1}^\alpha) + K_{\alpha,\tau}(t_{i-1}^{\alpha-\tau} + s_{i-1}^{\alpha-\tau})] + .. + M_A^i[D''(t_1^\alpha + s_1^\alpha) + K_{\alpha,\tau}(t_1^{\alpha-\tau} + s_1^{\alpha-\tau})]}{\Gamma(\alpha+1)} \\ &+ \frac{L'_g(t_i^\beta + s_i^\beta) + M_A L'_g(t_{i-1}^\beta + s_{i-1}^\beta) + ... + M_A^{i-1} L'_g(t_1^\beta + s_1^\beta)}{\Gamma(\beta+1)} \\ &+ \frac{[M_A t_i^\alpha + M_A^2(t_{i-1}^\alpha + s_{i-1}^\alpha) + M_A^3(t_{i-1}^\alpha + s_{i-1}^\alpha) + ... + M_A^i(t_1^\alpha + s_1^\alpha)]}{\Gamma(\alpha+1)} \times \\ &(D(1 + B^* + F^*) + D'q^*))r \leq r. \end{aligned}$$

Case 3. For $t \in (s_i, t_{i+1}]$,

$$\begin{aligned} \| Qx(t) \| &\leq M_A^{i+1} [\| x_0 \| + L'_h] + \frac{M_A[D'' t_{i+1}^\alpha + K_{\alpha,\tau} t_{i+1}^{\alpha-\tau}]}{\Gamma(\alpha+1)} \\ &+ \frac{M_A^2[D''(t_i^\alpha + s_i^\alpha) + K_{\alpha,\tau}(t_i^{\alpha-\tau} + s_i^{\alpha-\tau})] + ... + M_A^{i+1}[D''(t_1^\alpha + s_1^\alpha) + K_{\alpha,\tau}(t_1^{\alpha-\tau} + s_1^{\alpha-\tau})]}{\Gamma(\alpha+1)} + \\ &+ \frac{M_A L'_g(t_i^\beta + s_i^\beta) + M_A^2 L'_g(t_{i-1}^\beta + s_{i-1}^\beta) + ... + M_A^i L'_g(t_1^\beta + s_1^\beta)}{\Gamma(\beta+1)} + \\ &+ \frac{[M_A t_{i+1}^\alpha + M_A^2(t_i^\alpha + s_i^\alpha) + ... + M_A^{i+1}(t_1^\alpha + s_1^\alpha)]}{\Gamma(\alpha+1)} (D(1 + B^* + F^*) + D'q^*)r \leq r. \end{aligned}$$

- Q_1 is a contraction on B_r. Let $x, y \in B_r$

  Case 1. For $t \in [0, t_1]$, we have:
  $\| Q_1x(t) - Q_1y(t) \| \leq M_A L_h \| x - y \|_{PC} < \| x - y \|_{PC}$.
  Similar to the proof of Theorem(4.1), we prove that:
  Case 2. For $t \in [t_i, s_i]$, $1 \leq i \leq m$,

  $$\| Q_1x(t) - Q_1y(t) \| \leq \left[M_A^i L_h + \frac{L_g(s_i^\beta + t_i^\beta) + M_A L_g(s_{i-1}^\beta + t_{i-1}^\beta) + ... + M_A^{i-1} L_g(s_1^\beta + t_1^\beta)}{\Gamma(\beta+1)} \right] \| x - y \|_{PC}$$

  $< \| x - y \|_{PC}$.
  For $t \in (s_i, t_{i+1}], 1 \leq i \leq m$,

  $$\| Q_1x(t) - Q_1y(t) \| \leq \left[M_A^{i+1} L_h + \frac{M_A L_g(s_i^\beta + t_i^\beta) + M_A^2 L_g(s_{i-1}^\beta + t_{i-1}^\beta) + ... + M_A^i L_g(s_1^\beta + t_1^\beta)}{\Gamma(\beta+1)} \right] \| x - y \|_{PC}$$

  $< \| x - y \|_{PC}$.
  This implies that Q_1 is a contraction.

- Q_2 is continuous.

  Let $(x_n)_{n \geq 0}$ be a sequence such that $\lim_{x \to \infty} \| x_n - x \|_{PC} = 0$, we have :
  Case 1. For $t \in [0, t_1]$

$$\begin{aligned} \| Q_2x_n(t) - Q_2x(t) \| &\leq \| \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)([f(s, x_n(s), Fx_n(s), Bx_n(s)) + \\ &\int_0^s q(s-\tau)k(\tau, x_n(\tau))d\tau - f(s, x(s), Fx(s), Bx(s)) - \\ &\int_0^s q(s-\tau)k(\tau, x(\tau))d\tau]ds \| \\ &\leq \frac{M_A t_1^\alpha}{\Gamma(\alpha+1)} \Bigg(\| f(., x_n(.), Fx_n(.), , Bx_n(.)) - f(., x(.), Fx(.), Bx(.)) \|_{PC} + \\ &+ \| q^*[k(., x_n(.)) - k(., x(.))] \|_{PC} \Bigg). \end{aligned}$$

Similar to the proof of Theorem(4.1), we prove that:
Case 2. For $t \in (t_i, s_i],\ i = 1, 2, ..., m,$

$$\| Q_2x_n(t) - Q_2x(t) \| \leq \left[\frac{M_A t_i^\alpha + M_A^2(t_{i-1}^\alpha + s_{i-1}^\alpha) + ... + M_A^i(t_1^\alpha + s_1^\alpha)}{\Gamma(\alpha+1)}\right] \times \Bigg(\| f(., x_n(.), Fx_n(.), , Bx_n(.)) - f(., x(.), Fx(.), Bx(.)) \|_{PC} + \| q^*[k(., x_n(.)) - k(., x(.))] \|_{PC} \Bigg).$$

Case 3. For $t \in (s_i, t_{i+1}]$,

$$\| Q_2x_n(t) - Q_2x(t) \| \leq \left[\frac{M_A t_{i+1}^\alpha + M_A^2(t_i^\alpha + s_i^\alpha) + ... + M_A^{i+1}(t_1^\alpha + s_1^\alpha)}{\Gamma(\alpha+1)}\right] \times \Bigg(\| f(., x_n(.), Fx_n(.), , Bx_n(.)) - f(., x(.), Fx(.), Bx(.)) \|_{PC} + \| q^*[k(., x_n(.)) - k(., x(.))] \|_{PC} \Bigg).$$

- Q_2 is compact.

1. We have $Q_2B_r \subseteq B_r$, then Q_2 is uniformly bounded on B_r,
2. For $x \in B_r$, we have the following:

for $0 \leq t' < t'' \leq t_1$, we have:

$$\begin{aligned} \| Q_2x(t'') - Q_2x(t') \| &\leq \| \textstyle\int_0^{t''} (t''-s)^{\alpha-1} Q_\alpha(t''-s) \Bigg(f(s, x(s), Fx(s), Bx(s)) + \\ &\quad \textstyle\int_0^s q(s-\tau)k(\tau, x(\tau))d\tau \Bigg) ds - \int_0^{t'} (t'-s)^{\alpha-1} Q_\alpha(t'-s) \times \\ &\quad \Bigg(f(s, x(s), Fx(s), Bx(s)) + \textstyle\int_0^s q(s-\tau)k(\tau, x(\tau))d\tau \Bigg) ds \| \\ &\leq I_1 + I_2 + I_3, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \| \textstyle\int_{t'}^{t''} (t''-s)^{\alpha-1} Q_\alpha(t''-s) \Bigg(f(s, x(s), Fx(s), Bx(s)) + \int_0^s q(s-\tau)k(\tau, x(\tau))d\tau \Bigg) ds \| \\ I_2 &= \| \textstyle\int_0^{t'} (t'-s)^{\alpha-1} \left[Q_\alpha(t''-s) - Q_\alpha(t'-s)\right] \Bigg(f(s, x(s), Fx(s), Bx(s)) + \\ &\quad \textstyle\int_0^s q(s-\tau)k(\tau, x(\tau))d\tau \Bigg) ds \| \\ I_3 &= \| \textstyle\int_0^{t'} \left[(t''-s)^{\alpha-1} - (t'-s)^{\alpha-1}\right] Q_\alpha(t''-s) \Bigg(f(s, x(s), Fx(s), Bx(s)) + \\ &\quad \textstyle\int_0^s q(s-\tau)k(\tau, x(\tau))d\tau \Bigg) ds \| . \end{aligned}$$

$$I_1 \leq \frac{\alpha M_A[D\,(1 + r(1 + B^* + F^*)) + D'q^*(1+r)]}{\Gamma(\alpha+1)} \int_{t'}^{t''} (t''-s)^{\alpha-1} ds$$
$$\leq \frac{M_A[D\,(1 + r(1 + B^* + F^*)) + D'q^*(1+r)]}{\Gamma(\alpha+1)} (t''-t')^\alpha \longrightarrow 0,\ t''-t' \longrightarrow 0.$$

$$I_2 \leq [D\,(1 + r(1 + B^* + F^*)) + D'q^*(1+r)] \int_0^{t'} (t'-s)^{\alpha-1} \| Q_\alpha(t''-s)^{\alpha-1} - Q_\alpha(t'-s)^{\alpha-1} \| ds \longrightarrow 0,\ t''-t' \longrightarrow 0.$$

$$I_3 \leq \frac{M_A[D\,(1+r(1+B^*+F^*))+D'q^*(1+r)]}{\Gamma(\alpha)}\int_0^{t'}\left((t''-s)^{\alpha-1}-(t'-s)^{\alpha-1}\right)ds$$
$$\leq \frac{M_A[D\,(1+r(1+B^*+F^*))+D'q^*(1+r)]}{\Gamma(\alpha+1)}(t''-t')^{\alpha} \longrightarrow 0; t''-t' \longrightarrow 0.$$

Case 1. For $t_i \leq t' < t'' \leq s_i$,
$\| Q_2x(t'') - Q_2x(t') \| = 0$.
Case 2. For $s_i \leq t' < t'' \leq t_{i+1}$,

$$\| Q_2x(t'') - Q_2x(t') \| \leq I_1 + I_2 + I_3 + \| \left(P_\alpha(t''-s_i) - P_\alpha(t'-s_i)\right) K_{i2x} \| . \quad (4.1)$$

From (H_1) and the proof of lemma 3.4 in [14] we have that the continuity of $P_\alpha(t)$ and $Q_\alpha(t)$ $(t > 0)$ in t is in the uniform operator topology, we deduce that the right-hand side of (4.1) tends to 0 independently of $x \in B_r$ as $t'' \to t'$.

Case 3. For $t_i \leq t' < s_i < t'' \leq t_{i+1}$,

$$\| Q_2x(t'') - Q_2x(t') \| \leq \| P_\alpha(t''-s_i)K_{i2x} + \int_0^{t''}(t''-s)^{\alpha-1}Q_\alpha(t''-s)\Bigg(f(s,x(s),Fx(s),Bx(s))+ + \int_0^s q(s-\tau)k(\tau,x(\tau))d\tau\Bigg)ds - d_{i2x} \| \longrightarrow 0$$

indenpendently of $x \in B_r$, as $t'' \longrightarrow t'$ we have $(t'' \longrightarrow s_i)$.

In conclusion, $\| Q_2x(t'') - Q_2x(t') \| \longrightarrow 0$, as $t'' - t' \longrightarrow 0$, which implies that $Q_2(B_r(J))$ is equicontinuous .
We have $Q_2B_r \subseteq B_r$, let $Q_2B_r(t) = \{Q_2x(t); x \in B_r\}$ for $t \in J$.

3.$Q_2B_r(t)$ is relatively compact.
$Q_2B_r(0) = \{0\}$ is compact.
For $t \in [0, t_1]$.
For each $\epsilon \in (0, t)$ and $\delta > 0$, we define a set:
$Q_2^{\epsilon,\delta}(B_r)(t) = \left\{Q_2^{\epsilon,\delta}x(t); x \in B_r\right\}$
with

$$\begin{aligned} Q_2^{\epsilon,\delta}x(t) &= \alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)\Bigg(f(s,x(s),Fx(s),Bx(s))+ \\ &\quad \int_0^s q(s-\tau)k(\tau,x(\tau))d\tau\Bigg)d\theta ds \\ &= \alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)\left[T(\epsilon^\alpha\delta)T((t-s)^\alpha\theta-\epsilon^\alpha\delta)\right]\Bigg(f(s,x(s),Fx(s),Bx(s))+ \\ &\quad \int_0^s q(s-\tau)k(\tau,x(\tau))d\tau\Bigg)d\theta ds \\ &= \alpha T(\epsilon^\alpha\delta)\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta-\epsilon^\alpha\delta)\Bigg(f(s,x(s),Fx(s),Bx(s))+ \\ &\quad \int_0^s q(s-\tau)k(\tau,x(\tau))d\tau\Bigg)d\theta ds. \end{aligned}$$

(Observe that $\theta \geq \delta$ and $t - \epsilon \geq s$, hence $(t-s)^\alpha\theta - \epsilon^\alpha\delta \geq 0$) Since the operator $T(\epsilon^\alpha\delta)\,(\epsilon^\alpha\delta > 0)$ is compact, the set $Q_2^{\epsilon,\delta}B_r(t)$ is relatively compact in X. Moreover, for every $x \in B_r$ we have

$$\begin{aligned} \| Q_2x(t)-Q_2^{\epsilon,\delta}x(t) \| &\leq \alpha \| \int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)\Big(f(s,x(s),Fx(s),Bx(s))+ \\ &+\int_0^s q(s-\tau)k(\tau,x(\tau))d\tau\Big)d\theta ds+\int_0^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)\times \\ &\times\Big(f(s,x(s),Fx(s),Bx(s))+\int_0^s q(s-\tau)k(\tau,x(\tau))d\tau\Big)d\theta ds \\ &-\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)\Big(f(s,x(s),Fx(s),Bx(s))+ \\ &\int_0^s q(s-\tau)k(\tau,x(\tau))d\tau\Big)d\theta ds \| \\ &\leq G_1+G_2. \end{aligned}$$

With

$$G_1=\alpha \| \int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)\Big(f(s,x(s),Fx(s),Bx(s))+\int_0^s q(s-\tau)k(\tau,x(\tau))d\tau\Big)d\theta ds \|$$

and

$$G_2=\alpha \| \int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)\Big(f(s,x(s),Fx(s),Bx(s))+\int_0^s q(s-\tau)k(\tau,x(\tau))d\tau\Big)d\theta ds \| .$$

We have

$$\begin{aligned} G_1 &\leq \alpha M_A\int_0^t(t-s)^{\alpha-1} \| \Big(f(s,x(s),Fx(s),Bx(s))+\int_0^s q(s-\tau)k(\tau,x(\tau))d\tau\Big) \| ds\int_0^\delta \theta\xi_\alpha(\theta)d\theta \\ &\leq M_At_1^\alpha[D\,(1+r(1+B^*+F^*))+D'q^*(1+r)]\int_0^\delta \theta\xi_\alpha(\theta)d\theta, \end{aligned}$$

and

$$\begin{aligned} G_2 &\leq \alpha M_A\int_{t-\epsilon}^t(t-s)^{\alpha-1} \| \Big(f(s,x(s),Fx(s),Bx(s))+\int_0^s q(s-\tau)k(\tau,x(\tau))d\tau\Big) \| ds\int_\delta^\infty \theta\xi_\alpha(\theta)d\theta \\ &\leq M_A\epsilon^\alpha[D\,(1+r(1+B^*+F^*))+D'q^*(1+r)]\int_0^\infty \theta\xi_\alpha(\theta)d\theta \\ &\leq \frac{M_A[D\,(1+r(1+B^*+F^*))+D'q^*(1+r)]}{\Gamma(\alpha+1)}\epsilon^\alpha. \end{aligned}$$

Then $\| (Q_2x(t)-Q_2^{\epsilon,\delta}x(t) \| \longrightarrow 0$, as $\epsilon\longrightarrow 0;\delta\longrightarrow 0$

This means that there are relatively compact sets arbitrarily close to the set $Q_2B_r(t)$. Hence the set $Q_2B_r(t)$ is also relatively compact in X.

For $t_i<t\leq s_i$, $i=1,2,...,m$, in such case

$Q_2B_r(t)=\{d_{i2x},x\in B_r\}$ is compact.

For $s_i<t\leq t_{i+1}$, $i=1,2,...,m$,

$$Q_2B_r(t)=\Big\{P_\alpha(t-s_i)K_{i2x}+\int_0^t(t-s)^{\alpha-1}Q_\alpha(t-s)\Big(f(s,x(s),Fx(s),Bx(s))+\int_0^s q(s-\tau)k(\tau,x(\tau))d\tau\Big)ds,x\in B_r\Big\}.$$

By the same argument in case 1 ($t\in[0,t_1]$) and $P_\alpha(t-s_i)$ is a compact operator, we know that $Q_2B_r(t)$ is relatively compact.

□

Example

In this section, we give two examples which illustrate the applicability of our results.

Throughout this section, we let $X = L^2(0,1)$, $J = [0,1]$, $t_0 = s_0 = 0$, $t_1 = \frac{1}{3}$, $s_1 = \frac{2}{3}$, $T = 1$.

Define $Ax = \dfrac{\partial^2}{\partial^2 v}x$ for $x \in D(A) = \left\{x \in X : \dfrac{\partial x}{\partial v}, \dfrac{\partial^2 x}{\partial^2 v} \in X, x(0) = x(1) = 0\right\}$.

Then A is the infinitesimal generator of strongly continuous semigroup $\{T(t), t \geq 0\}$ on X. In addition $T(t)$ is compact and $\| T(t) \| \leq 1 = M_A$, for all $t \geq 0$. See [10]

Example 4.3. Consider

$$\begin{cases} \dfrac{\partial^\alpha}{\partial t^\alpha}y(t,v) = \dfrac{\partial^2}{\partial v^2}y(t,v) + \dfrac{1}{24}sin\left[y(t,v) + \int_0^t \dfrac{e^{-(s-t)}}{80}y(s,v) + \int_0^t \dfrac{e^{-2(s-t)}}{160}y(s,v)ds\right] + \\ \int_0^t e^{-(s-t)}\dfrac{1}{24}cos(y(s,v))ds + C(t,v),\ v \in (0,1),\ t \in [0,\dfrac{1}{3}] \cup (\dfrac{2}{3},1], \\ \dfrac{\partial^\beta}{\partial t^\beta}y(t,v) = \dfrac{1}{8}cos(y(t,v)),\ t \in (\dfrac{1}{3},\dfrac{2}{3}], \\ y(t,v) = y_0 + \dfrac{1}{8e^t}\left(y(s_1,v) + y(t_1,v)\right). \end{cases}$$

Denote $x(t)(v) = y(t,v)$ and $C(t,v) = C(t)u(t)(v)$

This problem can be abstracted into;

$$(P)\begin{cases} {}^cD^\alpha x(t)(v) = Ax(t)(v) + f(t,x(t),Fx(t),Bx(t))(v) + \int_0^t q(t-s)k(s,x(s)(v))ds + C(t)u(t)(v) \\ \ t \in [0,\dfrac{1}{3}] \cup (\dfrac{2}{3},1],\ u \in U_{ad}, \\ {}^cD^\beta x(t) = g_1(t,x(t)),\ t \in (t_i,s_i],\ t \in (\dfrac{1}{3},\dfrac{2}{3}], \\ x(0) = x_0 + h(x). \end{cases}$$

Where: $Bx(t)(v) = \int_0^t \dfrac{e^{-2(s-t)}}{160}y(s,v)ds$
$Fx(t)(v) = \int_0^t \dfrac{e^{-(s-t)}}{80}y(s,v)ds$, $f(t,x(t),Fx(t),Bx(t))(v) = \dfrac{1}{24}sin[y(t,v) + \int_0^t \dfrac{e^{-(s-t)}}{80}y(s,v) + \int_0^t \dfrac{e^{-2(s-t)}}{160}y(s,v)ds]$

$k(t,x(t))(v) = \dfrac{1}{24}cos(x(t)(v))$, $g_1(t,x(t))(v) = \dfrac{1}{8}cos(x(t)(v))$, $h(t,x(t))(v) = \dfrac{1}{8}\left(x(s_1)(v) + x(t_1)(v)\right)$,
$q(t) = \dfrac{e^t}{4}$ and $\alpha = 0.85$, $\beta = 0.95$,

In this case we have: $L_k = L_f = C_f = M_f = \dfrac{1}{24}$, $L_g = L_h = \dfrac{1}{8}$, and $F^* = B^* = \dfrac{1}{40}$, $q^* = \dfrac{e-1}{4}$, $\left[L_h + \dfrac{1 + (t_1^\alpha + s_1^\alpha)}{\Gamma(\alpha+1)}(L_f + q^*L_k + M_fB^* + C_fF^*) + \dfrac{L_g(t_1^\beta + s_1^\beta)}{\Gamma(\beta+1)}\right] \approx 0.393717 < 1$.

Which implies that all the assumptions of theorem 4.1 are satisfied. Therefore, there exists a unique mild solution to our problem.

Example 4.4. Consider

$$\begin{cases} \frac{\partial^{\alpha}}{\partial t^{\alpha}}y(t,v) = \frac{\partial^{2}}{\partial v^{2}}y(t,v) + \frac{1}{24e^{t}}\frac{|y(t,v)+By(t,v)+Fy(t,v)|}{1+|y(t,v)+By(t,v)+Fy(t,v)|} + \int_0^t \frac{1}{24e^{-(t-s)}}\frac{e^{s}|y(s,v)|}{1+|y(s,v)|}ds + C(t,v), \\ v \in (0,1),\ t \in [0,\frac{1}{3}] \cup (\frac{2}{3},1], \\ \frac{\partial^{\beta}}{\partial t^{\beta}}y(t,v) = \frac{1}{8e^{t}}\frac{|y(t,v)|}{1+|y(t,v)|},\ t \in (\frac{1}{3},\frac{2}{3}], \\ y(t,v) = y_0 + \frac{1}{8e^{t}}\left(y(s_1,v)+y(t_1,v)\right). \end{cases}$$

Denote $x(t)(v) = y(t,v)$ and $C(t,v) = C(t)u(t)(v)$.

This problem can be abstracted into (P). Where: $Bx(t)(v) = \int_0^t \frac{e^{-2(s-t)}}{160}y(s,v)ds$

$Fx(t)(v) = \int_0^t \frac{e^{-(s-t)}}{80}y(s,v)ds$, $f(t,x(t),Fx(t),Bx(t))(v) = \frac{1}{24e^{t}}\frac{|x(t)(v)+Bx(t)(v)+Fx(t)(v)|}{1+|x(t)(v)+Bx(t)(v)+Fx(t)(v)|}$

$k(t,x(t))(v) = \frac{1}{24}\frac{e^{t}|y(t,v)|}{1+|y(t,v)|}$, $g_1(t,x(t))(v) = \frac{1}{8e^{t}}\frac{|y(t,v)|}{1+|y(t,v)|}$, $h(t,x(t))(v) = \frac{1}{8e^{t}}\left(x(s_1)(v)+x(t_1)(v)\right)$,

$q(t) = \frac{e^{t}}{4}$ and $\alpha = 0.75$, $\beta = 0.65$.

In this case we have: $L_h = L_g = \frac{1}{8}$, $D = D' = \frac{1}{24}$, and $B^* = F^* = \frac{1}{40}$, $q^* = \frac{e-1}{4}$

$L_h + L_g\frac{(t_1^{\beta}+s_1^{\beta})}{\Gamma(\beta+1)} \approx 0.3$

and $\frac{1+(t_1^{\alpha}+s_1^{\alpha})}{\Gamma(\alpha+1)}(D(1+B^*+F^*)+D'q^*) \approx 0.145993.$

We have $max\,\{0.3; 0.145993\} < 1.$

This implies that all assumptions of theorem 4.2 are satisfied. Then, our problem has at least one mild solution.

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