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ON VECTOR-VALUED OPERATOR RIESZ SEQUENCE SPACES

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Abstract. In this paper we introduce vector-valued Riesz sequence spaces $R_0^q(X)$, $R_c^q(X)$, $R_\infty^q(X)$ and $R_1^q(X)$ and determine their Köthe-Toeplitz duals. Also, we characterize some matrix classes.

1. Definitions, Notations and Preliminary Results

Let $(X, \|\cdot\|)$ be a Banach space over the complex field and let $w(X)$ and $\Phi(X)$ denote the set of all $X-$valued sequences and the set of all finite sequence in X, respectively. We denote $\ell_\infty(X)$, $c(X)$, and $c_0(X)$, respectively, for the bounded, the convergent and the null sequence space in X, when $X = \mathbb{R}$ or $\mathbb{C}$, the real or complex numbers we shall use the familiar notation ℓ_∞, c and c_0. It is familiar that they are Banach spaces with the norm $\|\mathbf{x}\|_\infty = \sup_k \|x_k\|$ where $x_k \in X$ for $k = 1, 2, ...$. Also by $cs(X)$ and $\ell_1(X)$, we denote convergent and absolutely convergent series in X, respectively, and the space $\ell_1(X)$ is a Banach space under $\|\mathbf{x}\|_1 = \sum_{k=1}^\infty \|x_k\|$. By S we denote the set of all $x \in X$ such that $||x|| \leq 1$. If Y is a Banach space, $B(X,Y)$ is the set of all bounded operators from X into Y and if $T \in B(X,Y)$ the operator norm of T is $\|T\| = \sup_{x\in S} \|Tx\|$. We use the notation X^* to indicate the continuous dual of X, i.e $B(X, \mathbb{C})$. For a Banach space X we use θ for the zero element.

Generalized Köthe-Toeplitz duals of a $X-$valued sequence space E was defined by Maddox [3]. Let X and Y be Banach spaces and (A_k) a sequence in $B(X,Y)$. Then the β-dual and $\alpha-$dual of E are defined as

$$\begin{aligned} E^\beta &= \left\{A = (A_k) : \sum_{k=0}^\infty A_k x_k \quad \text{converges in the } Y \text{ norm for all } x \in E\right\} \\ E^\alpha &= \left\{A = (A_k) : \sum_{k=0}^\infty \|A_k x_k\| < \infty \quad \text{for all } x \in E\right\}. \end{aligned}$$

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Let $A_{nk} \in B(X,Y)$ for all $k,n \in \mathbb{N}$ and $\mathbf{A} = (A_{nk})$ be an infinite matrix. Suppose that E and F are nonempty subsets of $w(X)$. We define the matrix classes (E,F) by saying that $\mathbf{A} \in (E,F)$ if and only if, for every $\mathbf{x} = (x_k) \in E$,

$$\mathbf{A}_n(\mathbf{x}) = \sum_{k=0}^{\infty} A_{nk} x_k$$

converges for each n and the sequence

$$\mathbf{A}\mathbf{x} = \left(\sum A_{nk} x_k \right)_n \text{ belongs to } F.$$

The notion of the group norm of a sequence of bounded and linear operators was came up with by Robinson [2] and was named by Lorentz and Macphil [1].

Let $(B_k) = (B_0, B_1, ...)$ be a sequence in $B(X,Y)$, the group norm of (B_k) be

$$\|(B_k)\| = \sup \left\| \sum_{k=0}^{n} B_k x_k \right\|, \tag{1.1}$$

where the supremum is taken over all $n \in \mathbb{N}$ and all choices of $x_k \in S$.

Now we introduce a generalized Riesz mean or operator Riesz mean, which have been studied different mathematicians, see for example [3],[5] and [6], as follows.

Let Q_n and q_k be bijective elements of $B(X,X)$ and let (q_k) be a sequence. Define the matrix $\mathbf{R}^q = (\mathcal{R}_{nk})$ of the Riesz mean, which is a triangular matrix, by

$$\mathcal{R}_{nk} = \begin{cases} Q_n^{-1} q_k & , \quad 0 \le k \le n, \\ O & , \quad k > n \end{cases} \tag{1.2}$$

where O is the zero element of $B(X,X)$, while Q_n is determined by the relation

$$Q_n^{-1} \sum_{k=0}^{n} q_k x = x, \ x \in X. \tag{1.3}$$

Also, if $\mathbf{x} = (x_k)$ is a sequence in X we write

$$R_n^q(\mathbf{x}) = \sum_{k=0}^{n} Q_n^{-1} q_k x_k, \tag{1.4}$$

which is called $\mathbf{R}^q-$transform of $\mathbf{x}$ where $R^q\mathbf{x} = (R_n^q(\mathbf{x}))$.

We say that $\mathbf{x}$ is (R,q,X) summable to l, written $x_n \to l(R,q,X)$ if and only if there exists $l \in X$ such that $R_n^q(x) \to l$ as $n \to \infty$ in the norm of X.

The generalized matrix domain $\lambda(X)_{\mathbf{A}}$ of an infinite matrix $\mathbf{A} = (A_{nk})$ in a vector valued sequence space $\lambda(X)$ is defined by

$$\lambda(X)_{\mathbf{A}} = \{\mathbf{x} = (x_k) : \mathbf{A}\mathbf{x} \in \lambda(X)\},$$

which is a sequence space.

Köthe-Toeplitz duals of classical sequence spaces and matrix classes of between these spaces have been intensively examined up to now.

Recently the researchers have constructed new complex valued sequence spaces by using matrix domain of some special means such as Cesàro, Riesz and Euler, [8],[9], [10] and [11]. The infinite matrices in the matrix classes examined in these studies have been obtained from complex or real numbers.

The vector-valued sequence spaces were firstly introduced by Robinson in 1950 by using any Banach spaces instead of complex Banach space. Maddox, in his book "Infinite Matrices of Operators" presented this new concept extensively to the interests of the researchers. In this book, the matrix classes between vector-valued sequence spaces were constructed by infinite matrices of linear operators. This is the basic difference with classical sequence spaces.

With these developments, that questions comes to our minds; Can new vector-valued sequence spaces be obtained by using matrix domain of some special mappings? Thus, we have decided to define and introduce new vector-valued sequence spaces by using generalized matrix domain of operator Riesz mappings.

2. The Vector-Valued Sequence Spaces $R^q_\infty(X)$, $R^q_c(X)$, $R^q_0(X)$ and $R^q_1(X)$and Their Köthe-Toeplitz Duals

In this section we define the vector-valued Riesz sequence spaces $R^q_\infty(X)$, $R^q_c(X)$, $R^q_0(X)$ and $R^q_1(X)$ and determine the $\beta-$dual of the spaces $R^q_0(X)$, $R^q_c(X)$ and $R^q_\infty(X)$. Also we determine the $\alpha-$dual of the space $R^q_1(X)$.

If we consider the operator Riesz matrix, then we can define the new vector-valued sequence spaces $R^q_\infty(X)$, $R^q_c(X)$, $R^q_0(X)$ and $R^q_1(X)$ by using the generalized matrix domain as follows:

$$
\begin{aligned}
R^q_\infty(X) &= \left\{\mathbf{x}=(x_k)\in w(X) : \sup_n \left\| \sum_{k=0}^{n} Q_n^{-1} q_k x_k \right\| < \infty \right\} \\
R^q_c(X) &= \left\{\mathbf{x}=(x_k)\in w(X) : \lim_n \left\| \sum_{k=0}^{n} Q_n^{-1} q_k x_k - l \right\| = 0 \text{ for some } l\in X \right\} \\
R^q_0(X) &= \left\{\mathbf{x}=(x_k)\in w(X) : \lim_n \left\| \sum_{k=0}^{n} Q_n^{-1} q_k x_k \right\| = 0 \right\}, \\
R^q_1(X) &= \left\{\mathbf{x}=(x_k)\in w(X) : \sum_{n=1}^{\infty} \left\| \sum_{k=0}^{n} Q_n^{-1} q_k x_k \right\| < \infty \right\},
\end{aligned}
$$

that is, $\{\ell_\infty(X)\}_{\mathbf{R}^q} = R^q_\infty(X)$, $\{c(X)\}_{\mathbf{R}^q} = R^q_c(X)$, $\{c_0(X)\}_{\mathbf{R}^q} = R^q_0(X)$ and $\{\ell_1(X)\}_{\mathbf{R}^q} = R^q_1(X)$.

In the case $X=\mathbb{R}$ or $\mathbb{C}$ and $q_n(x)=q_n$ for all $n\in\mathbb{N}$ and for all $x\in X$, the vector valued sequence spaces $R^q_\infty(X)$, $R^q_c(X)$, $R^q_0(X)$ are, respectively, reduced to the real valued sequence spaces $(\overline{N},q)_\infty$, $(\overline{N},q)$ and $(\overline{N},q)_0$ which are introduced by

Malkowsky and Rakočević [7]. We should record here that the matrix $\mathbf{R}^q = (\mathcal{R}_{nk})$ can be reduced to the Riesz matrix $\overline{N}_q$. So, the results related to the matrix domain of the matrix $R^q_\infty(X)$ and $R^q_c(X)$, $R^q_0(X)$ are more general and comprehensive than the corresponding consequences of the matrix domain of $\overline{N}_q$, and include them.

It is easy to see that the vector-valued sequence spaces $R^q_\infty(X)$, $R^q_c(X)$ and $R^q_0(X)$ are BK spaces with the norm

$$\|\mathbf{x}\|_\infty = \sup_n \|R^q_n(\mathbf{x})\| = \sup_n \left\| \sum_{k=0}^{n} Q_n^{-1} q_k x_k \right\|,$$

and the space $R^q_1(X)$ is $BK-$space with the norm

$$\|\mathbf{x}\|_1 = \sum_{n=1}^{\infty} \|R^q_n(\mathbf{x})\| = \sum_{n=1}^{\infty} \left\| \sum_{k=0}^{n} Q_n^{-1} q_k x_k \right\|.$$

Also, one can easily show that the spaces $R^q_\infty(X)$, $R^q_c(X)$, $R^q_0(X)$ and $R^q_1(X)$ are linearly isomorphic to the spaces $\ell_\infty(X)$, $c(X)$, $c_0(X)$ and $\ell_1(X)$, respectively.

The following two lemmas, due to Maddox [3]. The first one is characterized the matrix classes $(c(X), c(Y))$, i.e., the operator version of conservative matrices. The second theorem extends the classical theorem of Schur on matrices belonging to the class (ℓ_∞, c). One can also see [3] for their proofs.

Lemma 1. [4, *Theorem 4.2*] *Let $A_{nk} \in B(X, Y)$ for all non-negative integers k, n. Then $(A_{nk}) \in (c(X), c(Y))$ if and only if*

$$\text{there exists } \lim_n A_{nk} \text{ for each } k, \tag{2.1}$$

$$\sup_n \|(A_{nk})\| < \infty, \tag{2.2}$$

$$\sum A_{nk} \text{ converges for each } n, \tag{2.3}$$

$$\text{there exists } \lim_n \sum A_{nk}. \tag{2.4}$$

Lemma 2. [4, *Theorem 4.6*] *Let X and Y be Banach spaces and $(A_{nk}) \in B(X, Y)$. Write for each n, m,*

$$R_{nm} = (A_{nm}, A_{n,m+1}, ...)$$

so that R_{nm} is the m^{th} tail of the n^{th} row of the matrix $\mathrm{A} = (A_{nk})$. Then $(A_{nk}) \in (\ell_\infty(X), c(Y))$ if and only if

$$\text{there exists } lim_n A_{nk} = A_k \text{ for each } k, \tag{2.5}$$

$$\lim_m \|R_{nm}\| = 0 \text{ for each } n, \tag{2.6}$$

$$\sup_n \|R_{nm} - R_m\| \to 0 \ (m \to \infty), \tag{2.7}$$

where $R_m = (A_m, A_{m+1}, ...)$. When (2.5)-(2.7) hold we have

$$\lim_n \sum A_{nk} x_k = \sum A_k x_k$$

for each $\mathrm{x} \in \ell_\infty(X)$.

The next lemma can be deduced from Lemma 1.

Lemma 3. *Let* $A_{nk} \in B(X,Y)$ *for all non-negative integers* k, n. *Then* $(A_{nk}) \in (c_0(X), c(Y))$ *if and only if*

$$\text{there exists } \lim_n A_{nk} \text{ for each } k, \tag{2.8}$$

$$\sup_n \|(A_{nk})\| < \infty, \tag{2.9}$$

$$\sum A_{nk} \text{ converges for each } n. \tag{2.10}$$

Theorem 1. *Let* (B_k) *be a sequence in* $B(X,Y)$. *Then* $(B_k) \in \{R_c^q(X)\}^\beta$ *if and only if*

$$\sup_n \|(D_{nk})\| < \infty, \tag{2.11}$$

$$\sum_{k=0}^{\infty} (B_k q_k^{-1} - B_{k+1} q_{k+1}^{-1}) Q_k \text{ converges}, \tag{2.12}$$

$$\lim_n B_n q_n^{-1} Q_n \text{ exists}. \tag{2.13}$$

Proof. Let $\mathbf{x} = (x_k) \in R_c^q(X)$ and its $\mathbf{R}^q-$transform be $\mathbf{y} = (y_k) = (R_k^q(\mathbf{x}))$ and let (B_k) be a sequence in $B(X,Y)$. Consider the equation

$$\begin{aligned} \sum_{k=0}^{n} B_k x_k &= \sum_{k=0}^{n} B_k q_k^{-1} (Q_k y_k - Q_{k-1} y_{k-1}) \\ &= \sum_{k=0}^{n-1} (B_k q_k^{-1} - B_{k+1} q_{k+1}^{-1}) Q_k y_k + B_n q_n^{-1} Q_n y_n \\ &= (Dy)_n, \end{aligned} \tag{2.14}$$

where $\mathbf{D} = (D_{nk})$ is defined by

$$D_{nk} = \begin{cases} (B_k q_k^{-1} - B_{k+1} q_{k+1}^{-1}) Q_k & , \quad 0 \le k < n \\ B_n q_n^{-1} Q_n & , \quad k = n \\ O & , \quad k > n \end{cases} \tag{2.15}$$

for each non-negative integer n, k. Hence we deduce from (2.14) that

$$\begin{aligned} (B_k x_k) \in cs(Y) \text{ whenever } \mathbf{x} = (x_k) \in R_c^q(X) \quad &\Leftrightarrow \quad \mathbf{Dy} \in c(Y) \text{ whenever } \mathbf{y} = (y_k) \in c(X) \\ &\Leftrightarrow \quad \mathbf{D} = (D_{nk}) \in (c(X), c(Y)). \end{aligned}$$

Therefore, if we consider the Lemma 1 then it is obvious that the columns of the matrix $\mathbf{D}$ are in the space $c(Y)$, in the sense of strong operator topology. Moreover, we derive from (2.2) and (2.4) that

$$\sup_n \|(D_{nk})\| < \infty,$$

$$\sum_{k=0}^{\infty}(B_k q_k^{-1} - B_{k+1}q_{k+1}^{-1})Q_k \text{ converges},$$

$$\lim_n B_n q_n^{-1} Q_n \text{ exists}.$$

□

Theorem 2. *Let (B_k) be a sequence in $B(X,Y)$. Then $(B_k) \in \{R_0^q(X)\}^\beta$ if and only if*

$$\sup_n \|(D_{nk})\| < \infty. \tag{2.16}$$

Proof. It can easily be proved according to the proof of Theorem 1 and Lemma 3. □

Theorem 3. *Let (B_k) be a sequence in $B(X,Y)$. Then $(B_k) \in \{R_\infty^q(X)\}^\beta$ if and only if*

$$\sup_n \left(\sup_p \left\| \sum_{k=m}^{m+p} (D_{nk} - D_k) y_k \right\| \right) \to \infty \text{ as } m \to \infty,$$

where $\lim_n D_{nk} = D_k$ for each non-negative integer k and $y_k \in S$.

Proof. Let $\mathbf{x} = (x_k) \in R_\infty^q(X)$ and its $\mathbf{R}^q-$transform be $\mathbf{y} = (y_k) = (R_k^q(\mathbf{x}))$ and let (B_k) be a sequence in $B(X,Y)$. If we consider the equation (2.14) and (2.15), then we deduce

$$\begin{aligned}(B_k x_k) \in cs(Y) \text{ whenever } \mathbf{x} = (x_k) \in R_\infty^q(X) \quad &\Leftrightarrow \quad \mathbf{D}\mathbf{y} \in c(Y) \text{ whenever } \mathbf{y} = (y_k) \in \ell_\infty(X) \\ &\Leftrightarrow \quad \mathbf{D} = (D_{nk}) \in (\ell_\infty(X), c(Y)).\end{aligned}$$

Now, we consider the Lemma 2 for the matrix D_{nk}. It is clear that $\lim_n D_{nk} = (B_k q_k^{-1} - B_{k+1}q_{k+1}^{-1})Q_k$ for each k, i.e., the columns of the matrix $\mathbf{D} = (D_{nk})$ are in the space $c(Y)$. Thus, we shall examine (2.6) and (2.7) for the matrix $\mathbf{D} = (D_{nk})$. Clearly

$$\lim_{m\to\infty} \|R_{nm}\| = \lim_{m\to\infty} \left(\sup_p \left\| \sum_{k=m}^{m+p} D_{nk} y_k \right\| \right) = 0$$

for each n and $y_k \in S$. Now we consider the following equation

$$\sup_n \|R_{nm} - R_m\| = \sup_n \left(\sup_p \left\| \sum_{k=m}^{m+p} (D_{nk} - D_k) y_k \right\| \right).$$

Consequently, $\mathbf{D} = (D_{nk}) \in (\ell_\infty(X), c(Y))$ if and only if

$$\sup_n \left(\sup_p \left\| \sum_{k=m}^{m+p} (D_{nk} - D_k) y_k \right\| \right) \to 0 \text{ as } m \to \infty.$$

□

The next lemma were examined by Maddox in [3] Theorem 4.9. In this theorem if we get $p = 1$, then we can easily obtain the necessary and sufficient conditions for $(A_{nk}) \in (\ell_1(X), \ell_1(Y))$):

Lemma 4. *Let $A_{nk} \in B(X, Y)$ for all $k, n \in \mathbb{N}$. Then $(A_{nk}) \in (\ell_1(X), \ell_1(Y))$ if and only if*

$$\sup \sum_{n=0}^{\infty} \|A_{nk} z\| < \infty \tag{2.17}$$

where the supremum is taken over all $z \in U$ and all non-negative integers k.

Theorem 4. *Let (B_k) be a sequence in $B(X, Y)$. Then $(B_k) \in \{R_1^q(X)\}^\alpha$ if and only if*

$$\sup \left\{ \|B_k q_k^{-1} Q_k z\| + \|B_{k+1} q_{k+1}^{-1} Q_k z\| \right\} < \infty,$$

where the supremum is taken over all $z \in U$ and all non-negative integers k.

Proof. Let $\mathbf{x} = (x_k) \in R_1^q(X)$ and its R^q-transform be $\mathbf{y} = (y_k) = (R_k^q(\mathbf{x}))$. Let (B_k) be a sequence in $B(X, Y)$. Then, we have

$$\begin{aligned} B_n x_n &= B_n q_n^{-1} (Q_n y_n - Q_{n-1} y_{n-1}) \\ &= -B_n q_n^{-1} Q_{n-1} y_{n-1} + B_n q_n^{-1} Q_n y_n \\ &= (Cy)_n, \end{aligned}$$

where $\mathbf{C} = (C_{nk})$ is defined by

$$C_{nk} = \begin{cases} -B_n q_n^{-1} Q_{n-1} & , \quad k = n-1 \\ B_n q_n^{-1} Q_n & , \quad k = n \\ O & , \quad \text{otherwise} \end{cases}$$

for each non-negative integers n, k.

$$\begin{aligned} (B_k x_k) \in \ell_1(Y) \text{ whenever } \mathbf{x} = (x_k) \in R_1^q(X) \quad &\Leftrightarrow \quad \mathbf{Cy} \in \ell_1(Y) \text{ whenever } \mathbf{y} = (y_k) \in \ell_1(X) \\ &\Leftrightarrow \quad \mathbf{C} = (C_{nk}) \in (\ell_1(X), \ell_1(Y)). \end{aligned}$$

Therefore, we have by Lemma 4 that

$$(B_k) \in \{R_1^q(X)\}^\alpha \Leftrightarrow \sup_k \left\{\|B_k q_k^{-1} Q_k z\| + \|B_{k+1} q_{k+1}^{-1} Q_k z\|\right\} < \infty.$$

□

3. MATRIX TRANSFORMATIONS

In these section, we characterize the classes $(\mu(X)_{\mathbf{R}^q}, \lambda(Y))$ and $(\lambda(X), \mu(Y)_{\mathbf{R}^q})$ where X and Y are Banach spaces and μ and λ are certain sequence spaces. Firstly, we define the pair of summability methods (operator version) such that one of them is applied to the sequences in the space $\mu(X)_{\mathbf{R}^q}$ and the other one is applied to the sequences in the space $\mu(X)$.

Let E_{nk} and F_{nk} be bounded operators on X into Y and let $\mathbf{E} = (E_{nk})$ and $\mathbf{F} = (F_{nk})$ be infinite matrices such that $\mathbf{E}$ maps the sequence $\mathbf{x} = (x_k)$ to the sequence $\mathbf{u} = (u_n)$ and $\mathbf{F}$ maps the sequence $\mathbf{y} = (y_k) = (\sum_{j=0}^{k} Q_k^{-1} q_j x_j)$ to the sequence $\mathbf{v} = (v_n)$, i.e.,

$$u_n = (\mathbf{Ex})_n = \sum_{k=0}^{\infty} E_{nk} x_k \tag{3.1}$$

$$v_n = (\mathbf{Fy})_n = \sum_{k=0}^{\infty} F_{nk} y_k \tag{3.2}$$

for $n = 0, 1, 2, ..$

It is clear that the method $\mathbf{E} = (E_{nk})$ and $\mathbf{F} = (F_{nk})$ are originally different since $\mathbf{E}$ is applied to the $\mathbf{x} = (x_k)$ while $\mathbf{F}$ is applied to the $\mathbf{R}^q$ transform of $\mathbf{x} = (x_k)$.

Let us assume that the matrix product $\mathbf{FR}^q$. If u_n becomes v_n (or v_n becomes u_n) then we shall say that the methods $\mathbf{E}$ and $\mathbf{F}$ are the pair of summability methods, shortly PSM. Therefore, $\mathbf{FR}^q$ exists and is equal to $\mathbf{E}$ and $(\mathbf{FR}^q)(\mathbf{x}) = \mathbf{F}(\mathbf{R}^q(\mathbf{x}))$ formally holds, if one side exists. This statement is equivalent to the relation

$$E_{nk} = \sum_{j=k}^{\infty} F_{nj} Q_j^{-1} q_k \text{ or } F_{nk} = (E_{nk} q_k^{-1} - E_{n,k+1} q_{k+1}^{-1}) Q_k. \tag{3.3}$$

Now, we show that v_n can be turn into u_n,

$$\begin{aligned} v_n &= (\mathbf{Fy})_n = \sum_{k=0}^{\infty} F_{nk} y_k = \sum_{k=0}^{\infty} F_{nk} \sum_{j=0}^{k} Q_k^{-1} q_j x_j \qquad (3.4) \\ &= \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} F_{nk} Q_k^{-1} q_j x_j = u_n. \end{aligned}$$

But the inversion in the order of summation may not be justified. Hence, $\mathbf{E}$ and $\mathbf{F}$ are not necessarily equivalent.

We consider the partial sums of the (3.1) and (3.2)

$$\sum_{k=0}^{m} E_{nk}x_k = \sum_{k=0}^{m-1}(E_{nk}q_k^{-1} - E_{n,k+1}q_{k+1}^{-1})Q_k y_k + E_{nm}q_m^{-1}Q_m y_m. \tag{3.5}$$

Thus, one of the series (3.1) and (3.2) converges then the other converges if and only if

$$\lim_{m\to\infty} E_{nm}q_m^{-1}Q_m y_m = \alpha_n \quad \text{(for each n)}. \tag{3.6}$$

If (3.6) holds, the sums are connected by the equation

$$u_n = v_n + \alpha_n. \tag{3.7}$$

Hence, if (y_n) is summable by one of the methods **E**, **F**, then it is summable by the other one if and only if (3.7) holds and

$$\lim_{n\to\infty} \alpha_n = \rho. \tag{3.8}$$

Therefore, the limits of **F** and **E** differ by ρ. Thus, a necessary and sufficient condition for **F** and **E** methods summable any sequence to the same limit, if **F** summability implies that (3.7) holds with $\rho = 0$. Even if we take **E** for **F** the result does not changed. In the case that when (3.7) holds with $\rho \neq 0$ for some sequence, we say that the methods **E** and **F** are inconsistent.

Now we give two basic theorems. The first one characterizes the matrix classes of $(\mu(X)_{\mathbf{R}^q}, \lambda(Y))$ and the other one characterizes the matrix classes of $(\lambda(X), \mu(Y)_{\mathbf{R}^q})$.

Theorem 5. *Let X, Y be Banach spaces and E_{nk}, F_{nk} be bounded operators on X into Y. Then $\boldsymbol{E} = (E_{nk}) \in (\lambda(X)_{\boldsymbol{R}^q}, \mu(Y))$ if and only if $\boldsymbol{F} = (F_{nk}) \in (\lambda(X), \mu(Y))$ and*

$$\boldsymbol{F}^{(n)} \in (\lambda(X), c(Y)) \tag{3.9}$$

for every fixed $n = 0, 1, 2...$ where $\boldsymbol{F}^{(n)} = (F_{mk}^{(n)})$ by

$$F_{mk}^{(n)} = \begin{cases} (E_{nk}q_k^{-1} - E_{n,k+1}q_{k+1}^{-1})Q_k & , \quad k < m \\ E_{nm}q_m^{-1}Q_m & , \quad k = m \\ O & , \quad otherwise \end{cases}$$

for each non-negative integers m, k.

Proof. Let $\mathbf{E} = (E_{nk}) \in (\lambda(X)_{\mathbf{R}^q}, \mu(Y))$ and take $\mathbf{x} \in \lambda(X)_{\mathbf{R}^q}$. Then $\mathbf{FR}^q$ exists and $(E_{nk})_k \in \{\lambda(X)_{\mathbf{R}^q}\}^\beta$. So we have that (3.9) is necessary and $(F_{nk})_k \in \{\lambda(X)\}^\beta$ for $n = 0, 1, 2....$ Hence $\mathbf{Fy}$ exists for each $\mathbf{y} \in \lambda(X)$. Letting $m \to \infty$ in the equality (3.5), we have by (3.3) $\mathbf{Ex} = \mathbf{Fy}$. Consequently we get $\mathbf{F} \in (\lambda(X), \mu(Y))$.

Conversely, let $\mathbf{F} \in (\lambda(X), \mu(Y))$ and (3.9) hold and take any $\mathbf{y} \in \lambda(X)$. Then we have $(F_{nk})_k \in \{\lambda(X)\}^\beta$ which implies by (3.9) that $(E_{nk})_k \in \{\lambda(X)_{\mathbf{R}^q}\}^\beta$ for each non-negative n. Hence $\mathbf{Ex}$ exists. Therefore, letting $m \to \infty$ in the equality 3.5), then we have $\mathbf{Fy} = \mathbf{Ex}$ and this show that $\mathbf{E} = (E_{nk}) \in (\lambda(X)_{\mathbf{R}^q}, \mu(Y))$. □

We write throughout for brevity that $\tilde{A}_{nk} = (E_{nk}q_k^{-1} - E_{n,k+1}q_{k+1}^{-1})Q_k$ and consider the following conditions

$$\sup_n \|(A_{nm})\| < \infty \tag{3.10}$$

$$\text{there exists } \lim_n A_{nk} = A_k \text{ for each } k \tag{3.11}$$

$$\sum A_{nk} \text{ converges for each } n \tag{3.12}$$

$$\text{there exists} \lim_n \sum A_{nk} \tag{3.13}$$

$$\lim_m \|R_{nm}\| = 0 \text{ for each } n \tag{3.14}$$

$$\sup_n \|R_{nm} - R_m\| \to 0 \text{ (m} \to \infty). \tag{3.15}$$

Corollary 1. *Let X and Y be Banach spaces, and $A_{nk} \in B(X,Y)$.*

(i) $\boldsymbol{A} = (A_{nk}) \in (R_c^q(X), c(Y))$ if and only if (3.10)-(3.13) hold with $\tilde{A}_{nk}$ instead of A_{nk} and (3.9) holds with $\lambda = c$.

(ii) $\boldsymbol{A} = (A_{nk}) \in (R_\infty^q(X), c(Y))$ if and only if (3.14) and (3.15) hold with $\tilde{A}_{nk}$ instead of A_{nk} and (3.9) holds with $\lambda = \ell_\infty$.

(iii) $\boldsymbol{A} = (A_{nk}) \in (R_0^q(X), c(Y))$ if and only if (3.10)-(3.12) hold with $\tilde{A}_{nk}$ instead of A_{nk} and (3.9) holds with $\lambda = c_0$.

Theorem 6. *Let X and Y be Banach spaces, and $A_{nk} \in B(X,Y)$, and $T_{nk} \in B(Y,Y)$ and $\boldsymbol{T} = (T_{nk})$ be a triangle. Then $\boldsymbol{A} = (A_{nk}) \in (\mu(X), \lambda(Y)_{\boldsymbol{T}})$ if and only if $\boldsymbol{TA} \in (\mu(X), \lambda(Y))$.*

Proof. Let we define $\mathbf{C} = (C_{mk})$ with $\mathbf{C} = \mathbf{TA}$ i.e.,

$$C_{mk}(x) = \sum_{n=0}^{\infty} T_{mn}A_{nk}x = \sum_{n=0}^{m} T_{mn}A_{nk}x$$

for all $x \in X$ and $m, k \in \{0, 1, 2, ...\}$.

Let $\mathbf{A} = (A_{nk}) \in (\mu(X), \lambda(Y)_{\mathbf{T}})$. Hence $A_n = (A_{nk})_k \in \{\mu(X)\}^\beta$ for each $n \in \{0, 1, 2, ...\}$.

$$\begin{aligned} C_m(\mathbf{x}) &= \sum_{k=0}^{\infty} C_{mk} x_k = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{m} T_{mn} A_{nk} \right) x_k \\ &= \sum_{k=0}^{\infty} T_{mk} \sum_{j=0}^{\infty} A_{kj} x_j = T_m(\mathbf{A}(\mathbf{x})) \end{aligned} \tag{3.16}$$

for all $\mathbf{x} = (x_k) \in \mu(X)$ and for all $m \in \{0, 1, 2, ...\}$. Since $\mathbf{A}(\mathbf{x}) \in \lambda(Y)_{\mathbf{T}}$ for all $\mathbf{x} \in \mu(X)$ we have that by (3.16)

$$\mathbf{C}(\mathbf{x}) = (\mathbf{TA})(\mathbf{x}) = \mathbf{T}(\mathbf{A}(\mathbf{x})) \in \lambda(Y),$$

which leads us to $\mathbf{C} = \mathbf{TA} \in (\mu(X), \lambda(Y))$.

Conversely, let $\mathbf{C} = (C_{mk}) \in (\mu(X), \lambda(Y))$. We now show that $A_n = (A_{nk})_k \in \{\mu(X)\}^\beta$ for each $n \in \{0, 1, 2, ...\}$. First $C_0 = (C_{0k})_k \in \{\mu(X)\}^\beta$ implies convergence of the series

$$A_0(\mathbf{x}) = \sum_{k=0}^{\infty} A_{0k} x_k = \sum_{k=0}^{\infty} T_{00}^{-1} C_{0k} x_k = T_{00}^{-1} C_0(\mathbf{x})$$

for all $\mathbf{x} = (x_k) \in \mu(X)$, i.e., $A_0 \in \{\mu(X)\}^\beta$. We assume $A_p \in \{\mu(X)\}^\beta$ for $0 \le p \le n$ for some $n \ge 0$. Since $C_{n+1} \in \{\mu(X)\}^\beta$, the series

$$\begin{aligned} C_{n+1}(\mathbf{x}) - \sum_{p=0}^{n} T_{n+1,p} A_p(\mathbf{x}) &= \sum_{k=0}^{\infty} (TA)_{n+1,k} x_k - \sum_{p=0}^{n} T_{n+1,p} A_p(\mathbf{x}) \\ &= \sum_{k=0}^{\infty} \sum_{p=0}^{n+1} T_{n+1,p} A_{pk} x_k - \sum_{k=0}^{\infty} \sum_{p=0}^{n} T_{n+1,p} A_{pk} x_k \\ &= \sum_{k=0}^{\infty} T_{n+1,n+1} A_{n+1,k} x_k \\ &= T_{n+1,n+1} A_{n+1}(\mathbf{x}) \end{aligned}$$

for all $\mathbf{x} = (x_k) \in \mu(X)$ which show that $A_{n+1}(\mathbf{x})$ convergence for all $\mathbf{x} = (x_k) \in \mu(X)$. Hence $A_{n+1} \in \{\mu(X)\}^\beta$. Consequently, $\mathbf{C}(\mathbf{x}) \in \lambda(Y)$ for all $\mathbf{x} = (x_k) \in \mu(X)$ implies $\mathbf{T}(\mathbf{A}(\mathbf{x})) \in \lambda(Y)$ for all $\mathbf{x} = (x_k) \in \mu(X)$. Hence $\mathbf{A}(\mathbf{x}) \in (\mu(X), \lambda(Y)_{\mathbf{T}})$. □

Remark 1. *Let $A_{nk} \in B(X, Y)$ and $\boldsymbol{A} = (A_{nk})$ be any infinite matrix. If we take $\boldsymbol{T} = \boldsymbol{R}_q = (R_{nj}^q)$ in Theorem 6, then we have the matrix $\boldsymbol{C} = \boldsymbol{R}^q \boldsymbol{A} = (C_{nk})$ as*

$$C_{nk}(x) = \sum_{j=0}^{\infty} R_{nj}^q A_{jk} x = Q_n^{-1} \sum_{j=0}^{n} q_j A_{jk} x$$

for all $n, k \in \{0, 1, 2, ...\}$. Hence $\boldsymbol{C} = (C_{nk}) = (Q_n^{-1} \sum_{j=0}^{n} q_j A_{jk})$

Corollary 2. *Let X and Y be Banach spaces, and $A_{nk} \in B(X, Y)$*

(i) $\boldsymbol{A} = (A_{nk}) \in (c(X), R_c^q(Y))$ if and only if (3.10)-(3.13) hold with C_{nk} instead of A_{nk}.

(ii) $\boldsymbol{A} = (A_{nk}) \in (\ell_\infty(X), R_c^q(Y))$ if and only if (3.14) and (3.15) hold with C_{nk} instead of A_{nk}.

(iii) $\boldsymbol{A} = (A_{nk}) \in (c_0(X), R_c^q(Y))$ if and only if (3.10)-(3.12) hold with C_{nk} instead of A_{nk}.

References

[1] Lorentz, G. G. and Machphail, M. S., Unbounded operators and a theorem of A. Robinson, *Trans. Royal Soc. of C* XLVI, (1952), 33-37.

[2] Robinson, A.R., On functional transformations and summability, *Proc. London Math. Soc.*, 52, (1995), 132-160.

[3] Maddox, I. J., *Infinite matrices of operators*, Lecture notes in Mathematics, 786 Springer-Verlag, Berlin.

[4] Maddox, I. J., Spaces of strongly summable sequences, *The Quarterly Journal of Mathematics,* 18(1), (1967), 345-53.

[5] Nappus, A. and Sõrmus, T., Einige verallgemeinerte matrixverfahren, *Proc. Estonian Acad. Sci. Phys.Math.*, **45**(2-3), (1996), 201-210.

[6] Aasma, A., Matrix transformations of summability domains of generalized matrix methods in Banach spaces, *Rendiconti del Circolo Matematico di Palermo*, 58, (2009), 467-476 .

[7] Malkowsky, E. and Rakočević, V., Measure of noncompactness of linear operators between spaces of sequences that are $(\overline{N}, q)$ summable or bounded, *Czechoslovak Mathematical Journal*, 51, (126), (2001), 505-522.

[8] Şengönül, M. and Başar, F., Some new Cesàro sequence spaces of non-absolute type which include the spaces c_0 and c, *Soochow Journal of Mathematics,* 31(1), (2005), 107-119.

[9] Altay, B. and Başar, F., On the paranormed Riesz sequence spaces of non-absolute type, *Southeast Asian Bulletin of Mathematics,* 24(2), (2003), 701-715.

[10] Altay, B. and Başar, F., Some paranormed Riesz sequence spaces of non-absolute type, *Southeast Asian Bulletin of Mathematics*, 30(4), (2006), 591-608.

[11] Altay, B. and Başar, F., Some Euler sequence spaces of non-absolute type, *Ukrainian Mathematical Journal*, 57(2), (2005), 1-17.

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