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ACTIONS OF INTERNAL GROUPOIDS IN THE CATEGORY OF LEIBNIZ ALGEBRAS

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Abstract. The aim of this paper is to characterize the notion of internal category (groupoid) in the category of Leibniz algebras and investigate some properties of well-known notions such as covering groupoids and groupoid operations (actions) in this category. Further, for a fixed internal groupoid G in the category of Leibniz algebras, we prove that the category of covering groupoids of G and the category of internal groupoid actions of G on Leibniz algebras are equivalent. Finally, we interpret the corresponding notion of covering groupoids in the category of crossed modules of Leibniz algebras.

1. Introduction

Covering groupoids have an important role in the applications of groupoids (see for example [3] and [14]). It is well known that for a groupoid G, the category **GpdAct**(G) of groupoid actions of G on sets, these are also called operations or G-sets, are equivalent to the category **GpdCov**$/G$ of covering groupoids of G. For the topological version of this equivalence, see [6, Theorem 2].

If G is a group-groupoid, which is an internal groupoid in the category of groups, then the category **GpGpdCov**$/G$ of group-groupoid coverings of G is equivalent to the category **GpGpdAct**(G) of group-groupoid actions of G on groups [8, Proposition 3.1]. In [2] this result has been generalized to the case where G is an internal groupoid in an algebraic category **C** which is called a category of groups with operations, acting on a group with operations. Covering groupoids of a categorical group have been studied in [25] and of a categorical ring have been studied in [22].

In [9] it was proved that the categories of crossed modules and group-groupoids, under the name of $\mathcal{G}$-groupoids, are equivalent (see also [18] for an alternative equivalence in terms of an algebraic object called *cat*^{n}*-groups*). By applying this

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equivalence of the categories, normal and quotient objects in the category of group-groupoids have been recently obtained in [27]. The study of internal category theory was continued in the works of Datuashvili [12] and [13]. Moreover, she developed cohomology theory of internal categories in categories of groups with operations [10] and [11] (see also [28] for more information on internal categories in categories of groups with operations). The equivalences of the categories in [9] enable us to generalize some results on group-groupoids to the more general internal groupoids for a certain category of groups with operations **C** (see for example [2], [23], [24] and [21]).

In the mid-twentieth century, Whitehead introduced the notion of crossed module, in a series of papers [30, 31, 32], as algebraic models for (connected) homotopy 2-types (i.e. connected spaces with no homotopy group in degrees above 2), in much the same way that groups are algebraic models for homotopy 1-types. A crossed module consists of groups A and B, where B acts on A by automorphisms, and a homomorphism of groups $\alpha\colon A \to B$ satisfying ***(i)*** $\alpha({}^{b}a) = b + \alpha(a) - b$ and ***(ii)*** ${}^{\alpha(a)}a_1 = a + a_1 - a$ for all $a, a_1 \in A$ and $b \in B$. Crossed modules can be viewed as 2-dimensional groups [4] and have been widely used in: homotopy theory [5]; the theory of identities among relations for group presentations [7]; algebraic K-theory [17]; and homological algebra [15, 19]. See [5, pp.49] for some discussion of the relation of crossed modules to crossed squares and so to homotopy 3-types. In [9] it has been proven that the categories of crossed modules and group groupoids are equivalent and this equivalence has been found important in applications. This equivalence is generalized in [28].

Recently, in [26] authors have interpreted in the category of crossed modules the notion of action of a group-groupoid on a group via a group homomorphism and hence introduced the notion of lifting of a crossed module over groups. Further they showed some results on liftings of crossed modules and proved that the category of liftings of crossed modules, the category of covering crossed modules and the category of group-groupoid actions are equivalent. In order to interpret the notion of liftings in the category of crossed modules over Leibniz algebras one needs the detailed definitions and properties of internal action groupoid, covering groupoid, covering crossed module in this category.

In this paper, first we define and investigate some properties of internal categories (and hence internal groupoids) in the category of Leibniz algebras. Further we define coverings and actions in the category of internal groupoids in the category of Leibniz algebras and prove that the category of internal groupoid actions and the category of covering groupoids of a fixed internal groupoid G in the category of Leibniz algebras are equivalent. Finally, using the equivalence between the categories of internal groupoids in the category of Leibniz algebras and crossed modules in the category of Leibniz algebras, we interpret the notion of covering in the category of crossed modules in the category of Leibniz algebras.

2. Preliminaries

A Leibniz algebra L is a $\mathbb{k}$-vector space equipped with a bilinear map $[-,-] : L \times L \to L$, satisfying the Leibniz identity $[x,[y,z]] = [[x,y],z] - [[x,z],y]$ for all $x,y,z \in L$. Leibniz algebras are the generalization of Lie algebras. Indeed, for a Leibniz algebra L, if $[x,x] = 0$ for all $x \in L$, then L becomes a Lie algebra. On the other hand, every Lie algebra is a Leibniz algebra.

Definition 1. *A Leibniz algebra morphism is a $\mathbb{k}-$linear map $f\colon L \to L'$ which is compatible with the bracket map, i.e.*

$$f[x,y] = [f(x), f(y)]$$

for all $x,y \in L$.

The category of Leibniz algebras consist of Leibniz algebras as objects and Leibniz algebra morphisms as morphisms. This category is denoted by **Lbnz**.

Definition 2. *A Leibniz algebra with trivial bracket is called an Abelian (or singular) Leibniz algebra.*

Definition 3. *For any Leibniz algebras L and L', a Leibniz action of L on L' consist of two bilinear maps $\Lambda\colon L \times L' \to L', (x,a) \mapsto x \cdot m$ and $\rho\colon L' \times L \to L', (a,x) \mapsto m \cdot x$ satisfying*

i. $x \cdot [m,n] = [x \cdot m, n] - [x \cdot n, m]$,
ii. $[m, x \cdot n] = [m \cdot x, n] - [m,n] \cdot x$,
iii. $[m, n \cdot x] = [m,n] \cdot x - [m \cdot x, n]$,
iv. $x \cdot (y \cdot m) = [x,y] \cdot m - (x \cdot m) \cdot y$,
v. $x \cdot (m \cdot y) = (x \cdot m) \cdot y - [x,y] \cdot m$,
vi. $m \cdot [x,y] = (m \cdot x) \cdot y - (m \cdot y) \cdot x$

for all $x,y \in L$ and $m,n \in L'$.

Let L and L' be two Leibniz algebras. A split extension of L by L' is a short exact sequence

$$\mathcal{E}: \qquad 0 \longrightarrow L' \xrightarrow{\ i\ } E \xrightarrow{\ p\ } L \longrightarrow 0$$

in **Lbnz** with a Leibniz algebra morphism $s\colon L \to E$ such that $ps = 1_L$. Here, note that p is surjective and $\ker p = i$. Given a split extension of L by L', we get derived actions of L on L' defined by

$$x \cdot m = [s(x), m]$$
$$m \cdot x = [m, s(x)]$$

for any $x \in L$ and $m \in L'$. Let a split extension

$$\mathcal{E}: \qquad 0 \longrightarrow L' \xrightarrow{\ i\ } E \underset{p}{\overset{s}{\leftrightarrows}} L \longrightarrow 0$$

is given. Then by using the bijection

$$\begin{array}{rcccl} \theta & : & L' \times L & \longrightarrow & E \\ & & (m,x) & \longmapsto & m + s(x) \end{array}$$

we can define a Leibniz algebra structure on $L' \times L$ as follows:

$$[(m,x),(n,y)] = ([m,n] + m \cdot y + x \cdot n, [x,y])$$

for all $x, y \in L$ and $m, n \in L'$. The inverse of the function θ is defined by

$$\begin{array}{rcccl} \theta^{-1} & : & E & \longrightarrow & L' \times L \\ & & e & \longmapsto & \theta^{-1}(e) = (e - sp(e), p(e)) \end{array}$$

for all $e \in E$. Thus $L' \times L$ cartesian product set becomes a Leibniz algebra which is called semi-direct product of Leibniz algebras and denoted by $L' \rtimes L$.

For any Leibniz algebra L, the obvious action of L on itself corresponds to the extension

$$\mathcal{L}: \qquad 0 \longrightarrow L \xrightarrow{\ i\ } L \rtimes L \underset{p}{\overset{s}{\rightleftarrows}} L \longrightarrow 0$$

where $i(l) = (l, 0)$, $p(l, l_1) = l_1$ and $s(l) = (0, l)$.

Now, we can give the definition of crossed modules of Leibniz algebras due to Porter [28].

Definition 4. [28] *Let L_0 and L_1 be two Leibniz algebras. Given a split extension*

$$\mathcal{L}: \qquad 0 \longrightarrow L_1 \xrightarrow{\ i\ } L_1 \rtimes L_0 \underset{p}{\overset{s}{\rightleftarrows}} L_0 \longrightarrow 0$$

*of L_0 by L_1 and a Leibniz algebra morphism $\partial\colon L_1 \to L_0$, ∂ is called a **crossed module** if $(1_{L_1}, \partial)$ and $(\partial, 1_{L_0})$ are both split extension morphisms in* **Lbnz**.

$$\begin{array}{lccccccccc}
\mathcal{L}_1: & 0 & \longrightarrow & L_1 & \xrightarrow{\ i\ } & L_1 \rtimes L_1 & \xrightarrow{\ p\ } & L_1 & \longrightarrow & 0 \\
& & & \downarrow{\scriptstyle 1_{L_1}} & & \downarrow{\scriptstyle (1_{L_1},\partial)} & & \downarrow{\scriptstyle \partial} & & \\
\mathcal{L}: & 0 & \longrightarrow & L_1 & \xrightarrow{\ i\ } & L_1 \rtimes L_0 & \xrightarrow{\ p\ } & L_0 & \longrightarrow & 0 \\
& & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle (\partial,1_{L_0})} & & \downarrow{\scriptstyle 1_{L_0}} & & \\
\mathcal{L}_0: & 0 & \longrightarrow & L_0 & \xrightarrow{\ i\ } & L_0 \rtimes L_0 & \xrightarrow{\ p\ } & L_0 & \longrightarrow & 0
\end{array}$$

A crossed module is denoted by (L_1, L_0, ∂). It is more practical to have a description in terms of actions and Leibniz bracket. We recall the definitions from [1] and [29].

Proposition 5. *A crossed module of Leibniz algebras is a Leibniz algebra morphism $\partial\colon L_1 \to L_0$ with actions of L_0 on L_1 satisfying the following conditions for all $l_0 \in L_0$ and $l_1, l_1' \in L_1$*

(LXM1) $\partial(l_0 \cdot l_1) = [l_0, \partial(l_1)], \partial(l_1 \cdot l_0) = [\partial(l_1), l_0],$

(LXM2) $l_1 \cdot \partial(l_1{}') = [l_1, l_1{}'], \partial(l_1{}') \cdot l_1 = [l_1{}', l_1].$

Proposition 6. *If* (L_1, L_0, ∂) *is a crossed module, then* $\ker \partial$ *is an Abelian Leibniz algebra.*

Proof. It can easily be shown by using the crossed module condition **(LXM2)**. □

For any two crossed module (L_1, L_0, ∂) and (M_1, M_0, δ) let $f_1 \colon L_1 \to M_1$ and $f_0 \colon L_0 \to M_0$ be two Leibniz algebra morphisms. Then (f_1, f_0) is called a crossed module morphism if the following conditions hold for all $l_0 \in L_0$ and $l_1 \in L_1$:

i. $f_0 \circ \partial = \delta \circ f_1$,
ii. $f_1(l_0 \cdot l_1) = f_0(l_0) \cdot f_1(l_1)$,
iii. $f_1(l_1 \cdot l_0) = f_1(l_1) \cdot f_0(l_0)$

Thus the category **XMod(Lbnz)** of Leibniz crossed modules can be constructed. The objects of this category are Leibniz crossed modules and morphisms are crossed module morphisms.

A groupoid is a category in which every morphism is an isomorphism. Let G be a groupoid. We write $\mathrm{Ob}(G)$ for the set of objects of G and write G for the set of morphisms. We also identify $\mathrm{Ob}(G)$ with the set of identities of G and so an element of $\mathrm{Ob}(G)$ may be written as x or 1_x as convenient. We write $d_0, d_1 \colon G \to \mathrm{Ob}(G)$ for the source and target maps, and, as usual, write $G(x, y)$ for $d_0^{-1}(x) \cap d_1^{-1}(y)$, for $x, y \in \mathrm{Ob}(G)$. The composition $h \circ g$ of two elements of G is defined if and only if $d_0(h) = d_1(g)$, and so the map $(h, g) \mapsto h \circ g$ is defined on the pullback $G_{d_0} \times_{d_1} G$ of d_0 and d_1. The *inverse* of $g \in G(x, y)$ is denoted by $g^{-1} \in G(y, x)$. If $x \in \mathrm{Ob}(G)$, we write $\mathsf{St}_G x$ for $d_0^{-1}(x)$ and call the *star* of G at x.

A groupoid G is *transitive (resp. simply transitive, 1-transitive and totally intransitive)* if $G(x, y) \neq \emptyset$ (resp. $G(x, y)$ has no more than one element, $G(x, y)$ has exactly one element and $G(x, y) = \emptyset$) for all $x, y \in \mathrm{Ob}(G)$ such that $x \neq y$.

3. Internal categories in **Lbnz**

Definition 7. *Let* $\mathbf{C}$ *be an arbitrary category with pullbacks. An internal category* C *in* $\mathbf{C}$ *is a category in which the initial and final point maps* $d_0, d_1 \colon C \to \mathrm{Ob}(C)$, *the object inclusion map* $\varepsilon \colon \mathrm{Ob}(C) \to C$ *and the partial composition* $\circ \colon C_{d_0} \times_{d_1} C \to C, (a, b) \mapsto a \circ b$ *are the morphisms in the category* $\mathbf{C}$.

Let G be an internal category in $\mathbf{C}$. If there exist a morphism $g' \in G$ such that $g \circ g' = \varepsilon d_1(c)$ and $g' \circ g = \varepsilon d_0(c)$ for all morphisms $g \in G$, then G is called an internal groupoid and g' is called the inverse of g which is denoted by g^{-1}.

Let G be an internal category in the category **Lbnz** of Leibniz algebras. Then G and $\mathrm{Ob}(G)$ are Leibniz algebras and the structural maps $(d_0, d_1, \varepsilon, \circ)$ are Leibniz algebra morphisms. Note that the operation $\circ$ being a Leibniz algebra morphism implies that

$$(h \circ g) + (h' \circ g') = (h + h') \circ (g, g')$$
$$[h \circ g, h' \circ g'] = [h, h'] \circ [g, g']$$

for all $g, g', h, h' \in G$ such that $d_1(g) = d_0(h)$ and $d_1(g') = d_0(h')$. These identities are called interchange laws. An application of the interchange laws is that the composition can be expressed by the addition as follows: for $g, h \in G$ such that $d_1(g) = d_0(h)$

$$\begin{aligned} h \circ g &= (h + 0) \circ (\varepsilon d_1(g) + (-\varepsilon d_1(g) + g)) \\ &= (h \circ \varepsilon d_1(g)) + (0 \circ (-\varepsilon d_1(g) + g)) \\ &= h - \varepsilon d_1(g) + g \end{aligned}$$

and similarly $h \circ g = g - \varepsilon d_1(g) + h$.

Clearly, one can see that any internal category in **Lbnz** is an internal groupoid. Indeed, for any $g \in G$,

$$\begin{aligned} \varepsilon d_0(g) &= h \circ g \\ \varepsilon d_0(g) &= h - \varepsilon d_1(g) + g \\ h &= \varepsilon d_0(g) - g + \varepsilon d_1(g) \end{aligned}$$

so $h = g^{-1} = \varepsilon d_0(g) - g + \varepsilon d_1(g)$ is the inverse morphism of g. Hence, we will use internal groupoid instead of internal category.

For other maps, we can give the following lemma.

Lemma 8. *Let G be an internal groupoid in* **Lbnz**. *Then for all $x, y \in \mathrm{Ob}(G)$ and $g, g' \in G$*

i. $d_0([g, g']) = [d_0(g), d_0(g')]$,
ii. $d_1([g, g']) = [d_1(g), d_1(g')]$,
iii. $\varepsilon([x, y]) = [\varepsilon(x), \varepsilon(y)]$, *i.e.* $1_{[x,y]} = \left[1_x, 1_y\right]$,
iv. $[g, g']^{-1} = \left[g^{-1}, (g')^{-1}\right]$.

Example 9. *Every Abelian Leibniz algebra L is an internal groupoid in* **Lbnz** *where algebra of objects $\mathrm{Ob}(L)$ is trivial, i.e. singleton.*

Example 10. *Let L be a Leibniz algebra. Then $L \times L$ becomes an internal groupoid in* **Lbnz** *where algebra of object is L. Here $d_0(l, l') = l$, $d_1(l, l') = l'$, $\varepsilon(l) = (l, l)$ and the composition $(l', l'') \circ (l, l') = (l, l'')$ for all $l, l', l'' \in L$.*

Proposition 11. *Let G be an internal groupoid in* **Lbnz**. *Then $\mathsf{St}_G 0 = \ker d_0$ is an ideal of G.*

Proof. It can be shown by an easy calculation. □

Following lemmas are special cases given in [28] where the category of groups with operations is the category **Lbnz** of Leibniz algebras.

Lemma 12. *Let G be an internal groupoid in* **Lbnz**. *If $g_1 \in \ker d_0$ and $g_2 \in \ker d_1$, then*

$$[g_1, g_2] = [g_2, g_1] = 0.$$

Proof. Assume that $g_1 \in \ker d_0$ and $g_2 \in \ker d_1$. So compositions $g_1 \circ \varepsilon(0)$ and $\varepsilon(0) \circ g_2$ are defined, where $\varepsilon(0) = 0$ the identity element of addition operation and hence, of bracket operation. Then,

$$\begin{aligned} [g_1, g_2] &= [g_1 \circ 0, 0 \circ g_2] \\ &= [g_1, 0] \circ [0, g_2] && \text{(by interchange law)} \\ &= 0 \circ 0 \\ &= 0 \end{aligned}$$

□

Lemma 13. *Let G be an internal groupoid in* **Lbnz**. *If $g_1 \in \ker d_0$, then we have*

$$[g_1, \varepsilon d_1(g)] = [g_1, g]$$

and

$$[\varepsilon d_1(g), g_1] = [g, g_1] \,.$$

Proof. Since $g_1 \in \ker d_0$ and $g - \varepsilon d_1(g) \in \ker d_1$, one can prove the assertion of the Lemma by using Lemma 12. □

Let G and H be two internal groupoids in **Lbnz**. An internal groupoid morphism (internal functor) $f\colon G \to H$ is a morphism of underlying groupoids and Leibniz algebra morphism on both the algebra of morphisms and the algebra of objects. So, we can construct the category of internal groupoids in **Lbnz**. This category may be denoted by **Cat**(**Lbnz**) or **Gpd**(**Lbnz**).

In [28] it has been proven that for any category **C** of groups with operations the category of internal categories within **C** and the category of crossed modules in **C** are equivalent. Following theorem is the special case for **C** = **Lbnz**.

Theorem 14. *The category* **XMod(Lbnz)** *of crossed modules in the category of Leibniz algebras and the category* **Cat**(**Lbnz**) *of internal categories (groupoids) in the category of Leibniz algebras are naturally equivalent.*

Proof. We give a sketch of proof and leave the details to the reader. Let G be an internal groupoid in **Lbnz**. Then $\ker d_0$ and $\mathrm{Ob}(G)$ are both Leibniz algebras and the restriction of the final point map

$$d_1\colon \ \ker d_0 \to \mathrm{Ob}(G)$$

is a Leibniz algebra morphism. Moreover $\mathrm{Ob}(G)$ acts on $\ker d_0$ by the maps

$$\begin{array}{ccc} \mathrm{Ob}(G) \times \ker d_0 & \longrightarrow & \ker d_0 \\ (x, g) & \longmapsto & x \cdot g = [\varepsilon(x), g] \end{array}$$

and

$$\begin{array}{ccc} \ker d_0 \times \mathrm{Ob}(G) & \longrightarrow & \ker d_0 \\ (g, x) & \longmapsto & g \cdot x = [g, \varepsilon(x)] \end{array}$$

These are derived actions, since these are obtained from the split extension

$$0 \longrightarrow \ker d_0 \xrightarrow{i} G \underset{d_0}{\overset{\varepsilon}{\rightleftarrows}} \mathrm{Ob}(G) \longrightarrow 0$$

Here we note that

$$\ker d_0 \rtimes \mathrm{Ob}(G) \cong G.$$

Also $(\ker d_0, \mathrm{Ob}(G), d_1)$ is a crossed module. Indeed,

i. for all $x \in \mathrm{Ob}(G)$ and $g \in \ker d_0$

$$\begin{aligned} d_1(g \cdot x) &= d_1([g, \varepsilon(x)]) \\ &= [d_1(g), d_1(\varepsilon(x))] \\ &= [d_1(g), x] \end{aligned}$$

and similarly

$$\begin{aligned} d_1(x \cdot g) &= d_1([\varepsilon(x), g]) \\ &= [d_1(\varepsilon(x)), d_1(g)] \\ &= [x, d_1(g)] \end{aligned}$$

ii. for all $g, g_1 \in \ker d_0$

$$\begin{aligned} g \cdot d_1(g_1) &= [g, \varepsilon(d_1(g_1))] \\ &= [g, g_1] \end{aligned}$$

and similarly

$$\begin{aligned} d_1(g_1) \cdot g &= [\varepsilon(d_1(g_1)), g] \\ &= [g_1, g] \end{aligned}$$

This construction defines a functor, η, from the category **Cat**(**Lbnz**) of internal categories in the category of Leibniz algebras to the category **XMod(Lbnz)** of crossed modules in the category of Leibniz algebras.

$$\eta \colon \mathbf{Cat}(\mathbf{Lbnz}) \longrightarrow \mathbf{XMod(Lbnz)}$$

Conversely, let (L_1, L_0, ∂) be a crossed module of Leibniz algebras. Then $(L_1 \rtimes L_0, L_0, d_0, d_1, \varepsilon, \circ)$ becomes an internal groupoid in **Lbnz**, where $d_0(l_1, l_0) = l_0$, $d_1(l_1, l_0) = \partial(l_1) + l_0$, $\varepsilon(l_0) = (0, l_0)$, the composition

$$(l_1', l_0') \circ (l_1, l_0) = (l_1' + l_1, l_0)$$

for $l_0' = \partial(l_1) + l_0$ and the inverse $(l_1, l_0)^{-1} = (-l_1, \partial(l_1) + l_0)$. Now we need to show that these structural maps are Leibniz algebra morphisms. For all $(l_1, l_0), (l_1', l_0') \in L_1 \rtimes L_0$

$$\begin{aligned} d_0([(l_1, l_0), (l_1', l_0')]) &= d_0(([l_1, l_1'] + l_1 \cdot l_0' + l_0 \cdot l_1', [l_0, l_0'])) \\ &= [l_0, l_0'] \\ &= [d_0(l_1, l_0), d_0(l_1', l_0')], \end{aligned}$$

$$\begin{aligned}
d_1\left([(l_1,l_0),(l'_1,l'_0)]\right) &= d_1\left(([l_1,l'_1]+l_1\cdot l'_0+l_0\cdot l'_1,[l_0,l'_0])\right)\\
&= \partial\left([l_1,l'_1]+l_1\cdot l'_0+l_0\cdot l'_1\right)+[l_0,l'_0]\\
&= \partial\,[l_1,l'_1]+\partial(l_1\cdot l'_0)+\partial(l_0\cdot l'_1)+[l_0,l'_0]\\
&= [\partial(l_1),\partial(l'_1)]+[\partial(l_1),l'_0]+[l_0,\partial(l'_1)]+[l_0,l'_0]\\
&= [\partial(l_1),\partial(l'_1)+l'_0]+[l_0,\partial(l'_1)+l'_0]\\
&= [\partial(l_1),+l_0,\partial(l'_1)+l'_0]\\
&= [d_1(l_1,l_0),d_1(l'_1,l'_0)]
\end{aligned}$$

$$\begin{aligned}
\varepsilon\left([l_0,l'_0]\right) &= (0,[l_0,l'_0])\\
&= \left([0,0]+[0,l'_0]+[l_0,0]\,,[l_0,l'_0]\right)\\
&= [(0,l_0),(0,l'_0)]\\
&= [\varepsilon(l_0),\varepsilon(l'_0)]
\end{aligned}$$

To see that the composition is a Leibniz algebra morphism, we need to verify the interchange law for bracket operation. Let $(l_1,l_0),(l'_1,l'_0),(l''_1,l''_0),(l'''_1,l'''_0)\in L_1\rtimes L_0$ such that $(l_1,l_0),(l'_1,l'_0)$ and $(l''_1,l''_0),(l'''_1,l'''_0)$ are composable, i.e. $l'_0=\partial(l_1)+l_0$ and $l'''_0=\partial(l''_1)+l''_0$. Then

$$\begin{aligned}
\left[(l'_1,l'_0)\circ(l_1,l_0),(l'''_1,l'''_0)\circ(l''_1,l''_0)\right] &= \left[(l'_1+l_1,l_0),(l'''_1+l''_1,l''_0)\right]\\
&= \left(\left[l'_1+l_1,l'''_1+l''_1\right]+\left(l'_1+l_1\right)\cdot l''_0+l_0\cdot\left(l'''_1+l''_1\right),\left[l_0,l''_0\right]\right)\\
&= \left(\left[l'_1,l'''_1\right]+\left[l_1,l'''_1\right]+\left[l'_1,l''_1\right]+\left[l_1,l''_1\right]+l'_1\cdot l''_0+l_1\cdot l''_0+l_0\cdot l'''_1+l_0\cdot l''_1,\left[l_0,l''_0\right]\right)\\
&= \left(\left[l'_1,l'''_1\right]+\partial(l_1)\cdot l'''_1+l'_1\cdot\partial(l''_1)+\left[l_1,l''_1\right]+l'_1\cdot l''_0+l_1\cdot l''_0+l_0\cdot l'''_1+l_0\cdot l''_1,\left[l_0,l''_0\right]\right)\\
&= \left(\left[l'_1,l'''_1\right]+(\partial(l_1)+l_0)\cdot l'''_1+l'_1\cdot(\partial(l''_1)+l''_0)+\left[l_1,l''_1\right]+l_1\cdot l''_0+l_0\cdot l''_1,\left[l_0,l''_0\right]\right)\\
&= \left(\left[l'_1,l'''_1\right]+l'_0\cdot l'''_1+l'_1\cdot l'''_0+\left[l_1,l''_1\right]+l_1\cdot l''_0+l_0\cdot l''_1,\left[l_0,l''_0\right]\right)\\
&= \left(\left[l'_1,l'''_1\right]+l'_0\cdot l'''_1+l'_1\cdot l'''_0,\left[l'_0,l'''_0\right]\right)\circ\left(\left[l_1,l''_1\right]+l_1\cdot l''_0+l_0\cdot l''_1,\left[l_0,l''_0\right]\right)\\
&= \left[(l'_1,l'_0),(l'''_1,l'''_0)\right]\circ\left[(l_1,l_0),(l''_1,l''_0)\right]
\end{aligned}$$

This shows that the composition $\circ$ is a morphism of Leibniz algebras. Thus $L_1\rtimes L_0$ becomes an internal groupoid on L_0 in **Lbnz**. Above construction also defines a functor, δ, from the category **XMod(Lbnz)** of crossed modules in the category of Leibniz algebras to the category **Cat**(**Lbnz**) of internal categories in the category of Leibniz algebras.

$$\delta\colon \textbf{XMod(Lbnz)}\longrightarrow \textbf{Cat}(\textbf{Lbnz})$$

It is straightforward to show that these functors, η and δ, gives a natural equivalence between the categories **XMod(Lbnz)** and **Cat**(**Lbnz**), i.e. $\eta\delta\simeq 1_{\textbf{XMod(Lbnz)}}$ and $\delta\eta\simeq 1_{\textbf{Cat(Lbnz)}}$. □

Coverings and actions of internal groupoids in Lbnz

First we will recall the definitions of coverings over groupoids from [3].

Definition 15. *(cf. [3]) Let $p\colon \widetilde{G} \to G$ be a morphism of groupoids. Then p is called a* covering morphism *and $\widetilde{G}$ a* covering groupoid *of G if for each $\widetilde{x} \in \mathrm{Ob}(\widetilde{G})$ the restriction $\mathsf{St}_{\widetilde{G}}\widetilde{x} \to \mathsf{St}_{G}p(\widetilde{x})$ is bijective.*

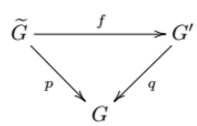
Assume that $p\colon \widetilde{G} \to G$ is a covering morphism. Then we have a lifting function $S_p\colon G_{d_0} \times_{\mathrm{Ob}(p)} \mathrm{Ob}(\widetilde{G}) \to \widetilde{G}$ assigning to the pair (a, x) in the pullback $G_{d_0} \times_{\mathrm{Ob}(p)} \mathrm{Ob}(\widetilde{G})$ the unique element b of $\mathsf{St}_{\widetilde{G}}\widetilde{x}$ such that $p(b) = a$. Clearly S_p is inverse to $(p, d_0)\colon \widetilde{G} \to G_{d_0} \times_{\mathrm{Ob}(p)} \mathrm{Ob}(\widetilde{G})$. So it is stated that $p\colon \widetilde{G} \to G$ is a covering morphism if and only if (p, d_0) is a bijection [6].

Definition 16. *An internal groupoid morphism $p\colon \widetilde{G} \to G$ is a covering morphism if and only if $(p, d_0)\colon \widetilde{G} \to G_{d_0} \times_{\mathrm{Ob}(p)} \mathrm{Ob}(\widetilde{G})$ is an isomorphism in* **Lbnz**.

A covering morphism $p\colon \widetilde{G} \to G$ is called *transitive* if both $\widetilde{G}$ and G are transitive. A transitive covering morphism $p\colon \widetilde{G} \to G$ is called *universal* if for every covering morphism $q\colon \widetilde{H} \to G$ there is a unique morphism of groupoids $\widetilde{p}\colon \widetilde{G} \to \widetilde{H}$ such that $q\widetilde{p} = p$ (and hence $\widetilde{p}$ is also a covering morphism), this is equivalent to that for $\widetilde{x}, \widetilde{y} \in \mathrm{Ob}(\widetilde{G})$ the set $\widetilde{G}(\widetilde{x}, \widetilde{y})$ has not more than one element.

Remark 17. *Since for an internal groupoid G in* **Lbnz**, *the star St_G0 is also a Leibniz algebra, we have that if $p\colon \widetilde{G} \to G$ is a covering morphism of internal groupoids, then the restriction of p to the stars $\mathsf{St}_{\widetilde{G}}\widetilde{0} \to \mathsf{St}_G0$ is an isomorphism in* **Lbnz**.

Let $p\colon \widetilde{G} \to G$ and $q\colon G' \to G$ be two coverings of G. A morphism $f\colon \widetilde{G} \to G'$ of coverings is a morphism of internal groupoids in **Lbnz** such that $qf = p$, i.e. following diagram is commutative.



Hence we can construct the category of covering internal groupoids of an internal groupoid G in **Lbnz** which has covering morphisms of G as objects and has morphisms of coverings as morphisms. This category will be denoted by $\mathbf{Cov}_{\mathbf{Cat(Lbnz)}}/G$.

Recall that an action of a groupoid G on a set S via a function $\omega\colon S \to \mathrm{Ob}(G)$ is a function $G_{d_0} \times_\omega S \to S, (g, s) \mapsto g \bullet s$ satisfying the usual rules for an action: $\omega(g \bullet s) = d_1(g)$, $1_{\omega(s)} \bullet s = s$ and $(h \circ g) \bullet s = h \bullet (g \bullet s)$ whenever $h \circ g$ and $g \bullet s$ are defined. A morphism $f\colon (S, \omega) \to (S', \omega')$ of such actions is a function $f\colon S \to S'$ such that $w'f = w$ and $f(g \bullet s) = g \bullet f(s)$ whenever $g \bullet s$ is defined. This gives a category $\mathbf{GpdAct}(G)$ of actions of G on sets. For such an action, the action groupoid $G \ltimes S$ is defined to have object set S, morphisms the pairs (g, s) such that $d_0(g) = \omega(s)$, source and target maps $d_0(g, s) = s$, $d_1(g, s) = g \bullet s$, and the composition

$$(g', s') \circ (g, s) = (g \circ g', s)$$

whenever $s' = g \bullet s$. The projection $q\colon G \ltimes S \to G, (g,s) \mapsto s$ is a covering morphism of groupoids and the functor assigning this covering morphism to an action gives an equivalence of the categories $\mathsf{GpdAct}(G)$ and GpdCov/G. Following equivalence of the categories was given in [8].

Proposition 18. *(cf. [8]) The categories* $\mathsf{GpGpdCov}/G$ *and* $\mathsf{GpGpdAct}(G)$ *are equivalent.*

Definition 19. *Let G be an internal groupoid in* Lbnz. *An action of the internal groupoid G on a Leibniz algebra L via ω consists of a Leibniz algebra morphism $\omega\colon L \to \mathrm{Ob}(G)$ from L to the algebra of objects $\mathrm{Ob}(G)$ and a Leibniz algebra morphism*

$$\begin{array}{ccc} G_{d_0}\times_{\omega} L & \longrightarrow & L \\ (g,l) & \longmapsto & g \bullet l, \end{array}$$

which is called action, satisfying

i. $\omega(g \bullet l) = d_1(g)$,

ii. $1_{\omega(l)} \bullet l = l$,

iii. $(h \circ g) \bullet l = h \bullet (g \bullet l)$,

whenever $h \circ g$ and $g \bullet l$ are defined.

Note that the action being a Leibniz algebra morphism implies the following so called interchange laws:

$$(g \bullet l) + (g' \bullet l') = (g + g') \bullet (l + l')$$
$$[(g \bullet l), (g' \bullet l')] = [g, g'] \bullet [l, l']$$

for all $g, g' \in G$ and $l, l' \in L$, whenever both sides are defined.

The notion of an internal diagram on an internal category in a category with finite limits introduced in [16] and given in [11] for an arbitrary category of groups with operations. Let G be an internal groupoid within a category C of groups with operations. An internal diagram F on G consists of an object $\gamma_0\colon F_0 \to \mathrm{Ob}(G)$ of the slice category $\mathsf{C}/\mathrm{Ob}(G)$ and a morphism $e\colon G \times_{\mathrm{Ob}(G)} F_0 \to F_0$ such that $\gamma_0 e = d_1\pi_1$, $e(\varepsilon \times 1) = 1_{F_0}$, and $e(1 \times e) = e(m \times 1)\colon G \times_{\mathrm{Ob}(G)} G \times_{\mathrm{Ob}(G)} F_0 \to F_0$. Obviously, internal diagrams and internal groupoid actions are the same.

A morphism $f\colon (L,\omega) \to (L',\omega')$ of such actions is a morphism $f\colon L \to L'$ of Leibniz algebras such that $\omega' f = \omega$. This gives a category $\mathsf{Act}_{\mathsf{Cat(Lbnz)}}(G)$ of actions of G on Leibniz algebras.

For an action of G on a Leibniz algebra L via ω, the action groupoid $G \ltimes L$ has a Leibniz algebra structure defined by

$$(g,l) + (g',l') = (g + g', l + l'),$$
$$[(g,l),(g',l')] = ([g,g'],[l,l'])$$

and with this operations $G \ltimes L$ becomes an internal groupoid in Lbnz.

Proposition 20. *Let G be an internal groupoid in* Lbnz. *The categories* $\mathsf{Cov}_{\mathsf{Cat(Lbnz)}}/G$ *and* $\mathsf{Act}_{\mathsf{Cat(Lbnz)}}(G)$ *are equivalent.*

Proof. Let $p\colon \widetilde{G} \to G$ be a covering morphism in $\mathsf{Cat}(\mathsf{Lbnz})$. Then G acts on $\mathrm{Ob}(\widetilde{G})$ via $p_0\colon \mathrm{Ob}(\widetilde{G}) \to \mathrm{Ob}(G)$ by

$$\begin{array}{ccc} G_{d_0}\times_{p_0}\mathrm{Ob}(\widetilde{G}) & \longrightarrow & \mathrm{Ob}(\widetilde{G}) \\ (g,\widetilde{x}) & \longmapsto & g\bullet\widetilde{x} = d_1(\widetilde{g}), \end{array}$$

where $\widetilde{g}$ is the unique lifting of g with initial point $\widetilde{x}$. It is easy to verify that this map is an action and a Leibniz algebra morphism, since p is a Leibniz algebra morphism.

Conversely, let G acts on a Leibniz algebra L via $\omega\colon L \to \mathrm{Ob}(G)$. Then $q\colon G \ltimes L \to G, (g,l) \mapsto g$ is a covering morphism in $\mathsf{Cat}(\mathsf{Lbnz})$. It is straightforward to confirm that these constructions define the intended natural equivalence. □

Example 21. *Let G be an internal groupoid in* Lbnz*. Then $1_G\colon G \to G$ is a covering morphism in* $\mathsf{Cat}(\mathsf{Lbnz})$*. The corresponding action to 1_G is constructed as follows: G acts on $\mathrm{Ob}(G)$ via $1_{\mathrm{Ob}(G)}\colon \mathrm{Ob}(G) \to \mathrm{Ob}(G)$ where the action is*

$$\begin{array}{ccc} G_{d_0}\times_{1_{\mathrm{Ob}(G)}}\mathrm{Ob}(G) & \longrightarrow & \mathrm{Ob}(G) \\ (g,x) & \longmapsto & g\bullet x = d_1(g). \end{array}$$

In this case the action groupoid

$$G \ltimes \mathrm{Ob}(G) = \{(g,x) \mid d_0(g) = x\}$$

is isomorph to G as an internal groupoid in Lbnz*, i.e., $G \ltimes \mathrm{Ob}(G) \cong G$.*

4. Covering crossed modules in Lbnz

The notion of coverings for crossed modules in the category of groups is introduced in [8] (see also [20]). In a similar way, by using the equivalence of the categories $\mathsf{Cat}(\mathsf{Lbnz})$ and $\mathsf{XMod}(\mathsf{Lbnz})$, we can interpret the notion of coverings in $\mathsf{XMod}(\mathsf{Lbnz})$.

Definition 22. *Let (L_1, L_0, ∂) and $(\widetilde{L_1}, \widetilde{L_0}, \widetilde{\partial})$ be two crossed modules of Leibniz algebras and $p_1\colon \widetilde{L_1} \to L_1$, $p_0\colon \widetilde{L_0} \to L_0$ be Leibniz algebra morphisms such that $(p_1,p_0)\colon (\widetilde{L_1}, \widetilde{L_0}, \widetilde{\partial}) \to (L_1, L_0, \partial)$ is a crossed module morphism. If $p_1\colon \widetilde{L_1} \to L_1$ is an isomorphism of Leibniz algebras, then we say that $(\widetilde{L_1}, \widetilde{L_0}, \widetilde{\partial})$ is a covering crossed module of (L_1, L_0, ∂) and that (p_1, p_0) is a covering morphism of crossed modules.*

Example 23. *Let (L_1, L_0, ∂) be a crossed module of Leibniz algebras. Then $(1_{L_1}, 1_{L_0})\colon (L_1, L_0, \partial) \to (L_1, L_0, \partial)$ is a covering.*

Let $(p_1,p_0)\colon (\widetilde{L_1}, \widetilde{L_0}, \widetilde{\partial}) \to (L_1, L_0, \partial)$ and $(q_1,q_0)\colon (L_1', L_0', \partial') \to (L_1, L_0, \partial)$ be two coverings of (L_1, L_0, ∂). A morphism of coverings is a crossed module morphism $(f_1,f_0)\colon (\widetilde{L_1}, \widetilde{L_0}, \widetilde{\partial}) \to (L_1', L_0', \partial')$ such that $(q_1,q_0)\circ(f_1,f_0) = (p_1,p_0)$, i.e. $q_1 f_1 = p_1$ and $q_0 f_0 = p_0$. Now we can construct the category of coverings of (L_1, L_0, ∂) which will be denoted by $\mathsf{Cov}_{\mathsf{XMod(Lbnz)}}/(L_1, L_0, \partial)$.

Proposition 24. *Let (L_1, L_0, ∂) be a crossed module of Leibniz algebras and G be the corresponding internal groupoid according to Theorem 14. Then the category $\mathsf{Cov}_{\mathsf{XMod(Lbnz)}}/(L_1, L_0, \partial)$ of coverings of (L_1, L_0, ∂) and the category $\mathsf{Cov}_{\mathsf{Cat(Lbnz)}}/G$ covering internal groupoids of G are equivalent.*

Proof. Let $p\colon \widetilde{G} \to G$ be a covering in $\mathsf{Cat}(\mathsf{Lbnz})$ and $(\widetilde{L_1}, \widetilde{L_0}, \widetilde{\partial})$ be the corresponding crossed modules to $\widetilde{G}$. Then by Theorem 14, $\mathsf{St}_{\widetilde{G}}\widetilde{0} = \ker \widetilde{d_0} = \widetilde{L_1}$ and $\mathsf{St}_G 0 = \ker d_0 = L_1$. Since p is a covering then by Remark 17 the restriction of p on $\widetilde{L_1}$ defines an isomorphism $\widetilde{L_1} \cong L_1$. Hence $(\widetilde{L_1}, \widetilde{L_0}, \widetilde{\partial})$ is a covering crossed module of (L_1, L_0, ∂).

Conversely, let $(p_1, p_0)\colon (\widetilde{L_1}, \widetilde{L_0}, \widetilde{\partial}) \to (L_1, L_0, \partial)$ be a covering of (L_1, L_0, ∂) and $\widetilde{G}$ be the corresponding internal groupoid to $(\widetilde{L_1}, \widetilde{L_0}, \widetilde{\partial})$. Here $\widetilde{G} = \widetilde{L_1} \rtimes \widetilde{L_0}$, $\mathrm{Ob}(\widetilde{G}) = \widetilde{L_0}$ and the corresponding internal groupoid morphism is $p = p_1 \times p_0\colon \widetilde{G} \to G$. Let $x \in L_1$. Since $\widetilde{L_1} \cong L_1$ then there exist a unique $\widetilde{x} \in \widetilde{L_1}$ such that $p_1(\widetilde{x}) = x$. Hence

$$\begin{array}{rcccc} S_p & : & (L_1 \rtimes L_0)\,{}_{d_0}\!\times_{\mathrm{Ob}(p)} \widetilde{L_0} & \longrightarrow & \widetilde{L_1} \rtimes \widetilde{L_0} \\ & & ((x, m), \widetilde{m}) & \longmapsto & (\widetilde{x}, \widetilde{m}) \end{array}$$

defines an isomorphism of Leibniz algebras. One can easily see that these constructions are functorial and defines a natural equivalence between the categories $\mathsf{Cov}_{\mathsf{XMod(Lbnz)}}/(L_1, L_0, \partial)$ and $\mathsf{Cov}_{\mathsf{Cat(Lbnz)}}/G$. □

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