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A Note on Ricci Solitons on Para-Sasakian Manifolds

Bilal Eftal ACET

Adıyaman University, Department of Mathematics, Adıyaman, Turkey

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Abctract
In the present paper, in an almost para-contact metric manifold, especially on para-Sasakian manifolds, Ricci solitons were examined. Also, we consider the *generalized quasi-conformal curvature tensor* on a para-Sasakian manifold admitting Ricci soliton.

Keywords: Para-contact manifolds, Ricci soliton, Semi-symmetric manifolds

Para-Sasakian Manifoldlarda Ricci Solitonlar Üzerine Bir Not

Öz
Bu çalışmada, bir hemen hemen para-kontakt metrik manifoldda, özellikle para-Sasakian manifoldlar üzerinde, Ricci solitonlar incelenmiştir. Ayrıca bir para-Sasakian manifold üzerinde Ricci soliton şartı ile birlikte *genelleştirilmiş quasi-conformal eğrilik tensörü* ele alınmıştır.

Anahtar Kelimeler: Para-kontakt manifoldlar, Ricci soliton, Semi simetrik manifold

1. Introduction

R.S. Hamilton studied the notion of Ricci solitons as which were generalization of an Einstein metric (Hamilton, 1982). On a semi-Riemannian manifold M with semi-Riemannian metric g, a Ricci soliton is a triple (g,V,λ) such that

$$(\pounds_V g)(U,W)+2S(U,W)+2\lambda g(U,W)=0, \quad (1)$$

where W,U vector fields on M, λ is a constant and $\pounds$ denotes the Lie derivative.

The Ricci soliton is called expanding, steady or shrinking if

$$\lambda>0,\ \lambda=0 \ \text{ or } \ \lambda<0.$$

In the recent years, this subject has been introduced in some papers; on Kaehler manifolds Hodes and Fong (2013), on contact manifolds Calin and Crasmereanu (2010); Turan et al., (2012); De et al., (2012) and paracontact manifolds Yüksel Perktaş and Keleş, (2015); Adara et al., (2018).

In 1985, the concept of almost para-contact structure in a semi-Riemannian manifold was introduced in Kaneyuki and Konzai (1985) and then the almost para-complex structure on $M^{2n+1}\times \mathbb{R}$ characterized.

Later, S. Zamkovoy Zamkovoy (2009) investigated almost para-contact structure and some considerable subclasses. Especially, in the recent years many authors have pointed out the importance of para-contact geometry Erdem (2002); Cortes et al. (2006), and in particular of para-Sasakian manifold by several papers (Acet et al. 2014; Acet et al. 2016).

K. Yano and S. Sawaki defined the concept of quasi-conformal curvature tensor (Yano and Sawaki, 1968). Later, generalized quasi-conformal curvature tensor was investigated in (Baishya and Chawdhury, 2016). This tensor is given by

*Corresponding to: eacet@adiyaman.edu.tr

$$\tilde{W}(U,Y)T = \frac{2n-1}{2n+1}\begin{pmatrix}(2nr-s+1) \\ -\{2n(r+s)+1\}t\end{pmatrix}C(U,Y)T$$
$$+(2nr-s+1)E(U,Y)T$$
$$+(2n(s-r))P(U,Y)T$$
$$\frac{2n-1}{2n+1}\begin{pmatrix}2n(t-1)(r+s) \\ +(t-1)\end{pmatrix}K(U,Y)T$$

(2)

for any $Y,U,T \in TM$, where $r,s,t \in \mathbb{R}$ and curvature tensors are given by

$$C(U,Y)T = R(U,Y)T - \frac{1}{2n-1}\begin{pmatrix}S(Y,T)U - S(U,T)Y \\ +g(Y,T)QU - g(U,T)QY\end{pmatrix} + \frac{r}{(2n-1)2n}\left(g(Y,T)U - g(U,T)Y\right)$$

which is known conformal curvature tensor,

$$E(U,Y)T = R(U,Y)T - \frac{\tau}{2n(2n-1)}\left(g(Y,T)U - g(U,T)Y\right)$$

which is known concircular curvature tensor,

$$K(U,Y)T = R(U,Y)T - \frac{1}{2n-1}\begin{pmatrix}S(Y,T)U - S(U,T)Y \\ -g(U,T)QY + g(Y,T)QU\end{pmatrix}$$

which is known conharmonical curvature tensor

$$P(U,Y)T = R(U,Y)T - \frac{1}{2n}\left(S(Y,T)U - S(U,T)Y\right)$$

which is known projective curvature tensor.

This generalized quasi-conformal curvature tensor is reduced to be

i) *Riemannian curvature tensor* if

$$r = s = 0, \quad t = 0,$$

ii) *Conformal curvature tensor* (Eisenhart, 1949) if

$$r = -\frac{1}{2n-1} = s, t = 1,$$

iii) *Concircular curvature tensor* (Ishii, 1957) if

$$r = s = 0, t = 1,$$

iv) *Conharmonic curvature tensor* (Yano and Bochner, 1953) if

$$r = -\frac{1}{2n-1} = s, t = 0,$$

v) *Projective curvature tensor* (Ishii, 1957) if

$$r = -\frac{1}{2n}, s = 0 = t,$$

vi) m-*Projective curvature tensor* (Pokhariyal and Mishra, 1970) if

$$r = -\frac{1}{4n} = s, t = 0.$$

Preliminaries

A para-contact manifold is a differentiable manifold equipped with a vector field ξ, a 1-form η and a tensor field ϕ of type (1,1) satisfying (Kaneyuki and Konzai, 1985):

$$\phi^2 = I - \eta \otimes \xi,$$

(3)

$$\eta(\xi) = 1, \ \phi\xi = 0.$$

(4)

As a consequence of (4), we get

$$\eta \circ \phi = 0, \ rank\phi = 2n.$$

Moreover if the manifold M is equipped with a semi Riemannian metric g which is known compatible metric satisfying (Zamkovoy, 2009)

$$g(\phi W, \phi X) = -g(W,X) + \eta(W)\eta(X), \quad (5)$$

then one can say that M is an almost para-contact metric manifold.

From the definition, we get (Zamkovoy, 2009),

$$\eta(W) = g(\xi, W).$$

(6)

and

$$g(\phi W, X) = -g(W, \phi X),$$

(7)

The fundamental 2-form Φ of an almost para-contact structure (ϕ, ξ, η, g) is defined by

$$\Phi(W,X) = g(W,\phi X),$$

(8)

If

$$g(W,\phi X) = d\eta(W,X)$$

where

$d\eta(W,X) = (1/2)(W\eta(X) - X\eta(W) - \eta([W,X])),$

an almost para-contact metric structure is called a para-contact metric structure.

For (M,ϕ,ξ,η,g) an almost para-contact metric manifold one can find a local orthonormal basis which is known $\phi-$basis $(X_i,\phi X_i,\xi)$ $(i=1,2,...,n)$ (Zamkovoy, 2009).

An almost para-contact metric manifold M is a para-Sasakian manifold iff (Zamkovoy, 2009)

$$(\nabla_X\phi)W=-g(X,W)\xi+\eta(W)X. \quad (9)$$

In view of (9), one arrive at

$$\nabla_W\xi=-\phi W. \quad (10)$$

Moreover, in para-Sasakian manifolds, we get

$$R(\xi,W)U=-g(U,W)\xi+\eta(U)W, \quad (11)$$

$$g(R(W,Y)X,\xi)=g(W,X)\eta(Y)-g(Y,X)\eta(W), \quad (12)$$

$$R(\xi,W)\xi=W-\eta(W)\xi, \quad (13)$$

$$R(W,X)\xi=\eta(W)X-\eta(X)W, \quad (14)$$

$$S(W,\xi)=-2n\eta(W). \quad (15)$$

Assume that M^{2n+1} is a para-Sasakian manifold and (g,V,λ) is a Ricci soliton on M. From (1), we get

$$2S(W,X)=-(\pounds_V g)(W,X)-2\lambda g(W,X).$$

Using (6) with (10), we arrive at

$$S(W,X)=-\lambda g(W,X). \quad (16)$$

So, we can give:

Theorem 2.1 A para-Sasakian manifold admitting a Ricci soliton is an Einstein manifold.

Moreover, putting $X=\xi$ in (16) and using (15), we get

$$\lambda=2n. \quad (17)$$

So, we get the following.

Theorem 2.2 A Ricci soliton in para-Sasakian manifold M^{2n+1} is expanding.

Main Theorems

Definition 3.1 A para-contact manifold is called semi-symmetric type if the curvature tensor $\overset{\square}{W}$ fulfills the equation

$$R(U,V)\cdot\overset{\square}{W}=0 \quad (18)$$

for all $U,V\in TM$.

We suppose that the equation (18) fulfills on M. So we get

$$R(\xi,U)\overset{\square}{W}(Z,Y)X-\overset{\square}{W}(R(\xi,U)Z,Y)X$$
$$-\overset{\square}{W}(Z,R(\xi,U)Y)X-\overset{\square}{W}(Z,Y)R(\xi,U)X \quad (19)$$
$$=0.$$

Using the equation (13) in (19), we have

$$\eta(\overset{\square}{W}(Z,Y)X)U-g(U,\overset{\square}{W}(Z,Y)X)\xi$$
$$+g(U,Z)\overset{\square}{W}(\xi,Y)X-\eta(Z)\overset{\square}{W}(U,Y)X$$
$$+g(U,Y)\overset{\square}{W}(Z,\xi)X-\eta(Y)\overset{\square}{W}(Z,U)X \quad (20)$$
$$+g(U,X)\overset{\square}{W}(Z,Y)\xi-\eta(X)\overset{\square}{W}(Z,Y)U$$
$$=0.$$

Taking an inner product with ξ, we obtain

$$\eta(\overset{\square}{W}(Z,Y)X)\eta(U)-g(U,\overset{\square}{W}(Z,Y)X)$$
$$+g(U,Z)\eta(\overset{\square}{W}(\xi,Y)X)$$
$$-\eta(Z)\eta(\overset{\square}{W}(U,Y)X)$$
$$+g(U,Y)\eta(\overset{\square}{W}(Z,\xi)X)-\eta(Y)\eta(\overset{\square}{W}(Z,U)X) \quad (21)$$

$$=0.$$

Using (2) in (21) and after some calculations, we get

$$\begin{bmatrix} 1+\lambda(r+s) \\ +\frac{t\tau}{2n+1}\left(\frac{1}{2n}+r+s\right) \end{bmatrix}\begin{pmatrix} g(U,Y)g(Z,X) \\ -g(U,Z)g(Y,X) \end{pmatrix}$$

$$-g(U,\tilde{W}(Z,Y)X) \tag{22}$$

$$=0.$$

In view of (2) with (16) in (22), we find

$$-g(\tilde{W}(Z,Y)X,U)+g(Z,X)g(U,Y) -g(U,Z)g(Y,X)=0. \tag{23}$$

Since $\{e_i, \phi e_i, \xi\}$ is a basis of M . from (23), we have

$$S(Y,X)+2ng(Y,X)=0. \tag{24}$$

Now, one can state:

Theorem 3.1 A para-Sasakian manifold satisfying $R(\xi,X)\cdot\tilde{W}=0$ admitting a Ricci soliton (g,V,λ) is an Einstein manifold.

Taking $X=\xi=Y$ in (24) and in view of (16), we arrive at

$$\lambda=2n,$$

which yields λ is positive. Thus, we have :

Theorem 3.2 Let (g,V,λ) be a Ricci soliton on para-Sasakian manifold. If the semi-symmetric condition $R(\xi,X)\cdot\tilde{W}=0$ satisfies on M, then the Ricci soliton is expanding.

Now, assume that (g,V,λ) is a Ricci soliton on a para-Sasakian manifold M and the equation $S(\xi,X)\cdot\tilde{W}=0$ satisfies on M . By use of the following equations

$$(S(\xi,X)\cdot\tilde{W})(U,Y)Z=((X\wedge_S\xi)\tilde{W})(U,Y)Z$$

$$=(X\wedge_S\xi)\tilde{W}(U,Y)Z+\tilde{W}((X\wedge_S\xi)U,Y)Z$$

$$+\tilde{W}(U,(X\wedge_S\xi)Y)Z+\tilde{W}(U,Y)(X\wedge_S\xi)Z,$$

where the endomorphisim

$$(U\wedge_S Y)Z=S(Y,Z)U-S(U,Z)Y,$$

we obtain

$$(S(X,\xi)\cdot\tilde{W})(U,Y)Z=S(\xi,\tilde{W}(U,Y)Z)X$$

$$-S(X,\tilde{W}(U,Y)Z)\xi+S(\xi,U)\tilde{W}(X,Y)Z$$

$$-S(X,U)\tilde{W}(\xi,Y)Z+S(\xi,Y)\tilde{W}(U,X)Z$$

$$-S(X,Y)\tilde{W}(U,\xi)Z \tag{25}$$

$$+S(\xi,Z)\tilde{W}(U,Y)X$$

$$-S(X,Z)\tilde{W}(U,Y)\xi.$$

Now, if we consider the equation (16) in (25), then we find

$$\lambda\begin{bmatrix} -\eta(\tilde{W}(U,Y)Z)X+g(X,\tilde{W}(U,Y)Z)\xi \\ +g(X,U)\tilde{W}(\xi,Y)Z-\eta(U)\tilde{W}(X,Y)Z \\ +g(X,Y)\tilde{W}(U,\xi)Z-\eta(Y)\tilde{W}(U,X)Z \\ +g(X,Z)\tilde{W}(U,Y)\xi-\eta(Z)\tilde{W}(U,Y)X \end{bmatrix}=0. \tag{26}$$

Taking inner product with ξ and in view of (11) - (14) with (2), one can find

$$\lambda\begin{bmatrix} -\eta(\tilde{W}(U,Y)Z)\eta(X)+g(X,\tilde{W}(U,Y)Z) \\ +g(X,U)\eta(\tilde{W}(\xi,Y)Z)-\eta(U)\eta(\tilde{W}(X,Y)Z) \\ +g(X,Y)\eta(\tilde{W}(U,\xi)Z)-\eta(Y)\eta(\tilde{W}(U,X)Z) \\ +g(X,Z)\eta(\tilde{W}(U,Y)\xi)-\eta(Z)\eta(\tilde{W}(U,Y)X) \end{bmatrix}=0,$$

which yields

$$\lambda\begin{bmatrix} g(X,R(U,Y)Z)-g(Y,Z)g(U,X)+g(X,Y)g(U,Z) \\ -2\begin{pmatrix} 1+\lambda(r+s) \\ +\frac{t\tau}{2n+1}\left(\frac{1}{2n}+r+s\right) \end{pmatrix}\begin{pmatrix} g(U,Z)\eta(X)\eta(Y) \\ -g(Y,Z)\eta(X)\eta(U) \end{pmatrix} \end{bmatrix}=0.$$

(27)

Let $\{e_i,\phi e_i,\xi\}_{(i=1,\dots,n)}$ is a basis of M. Thus by a contraction of (27), we get

$$\lambda(S(Y,Z)-2ng(Y,Z))=0.$$

(28)

Theorem 3.3 A para-Sasakian manifold M satisfying $S(\xi,X)\cdot\tilde{W}=0$ admitting a non-steady Ricci soliton (g,V,λ) is an Einstein manifold.

Now, taking $Y=\xi=Z$ in (28) and using (16), we obtain

$$\lambda(\lambda+2n)=0,$$

which implies $\lambda=0$ or $\lambda=-2n$. Thus, we get:

Theorem 3.4 Let (g,V,λ) be a Ricci soliton on a para-Sasakian manifold. If the equation $S(\xi,X)\cdot\tilde{W}=0$ satisfies on M, the Ricci soliton is either steady or shrinking.

Take a para-Sasakian manifold satisfying the condition $S(\xi,X)\cdot R=0$. By definition we have

$$(S(\xi,X)\cdot R)(U,Y)Z=((X\wedge_S\xi)R)(U,Y)Z$$

$$=(X\wedge_S\xi)R(U,Y)Z+R((X\wedge_S\xi)U,Y)Z$$

$$+R(U,(X\wedge_S\xi)Y)Z+R(U,Y)(X\wedge_S\xi)Z,$$

where the endomorphisim

$$(U\wedge_S Y)Z=S(Y,Z)U-S(U,Z)Y,$$

we obtain

$$(S(X,\xi)\cdot R)(U,Y)Z=S(\xi,R(U,Y)Z)X$$
$$-S(X,R(U,Y)Z)\xi+S(\xi,U)R(X,Y)Z$$
$$-S(X,U)R(\xi,Y)Z+S(\xi,Y)R(U,X)Z$$
$$-S(X,Y)R(U,\xi)Z \quad (29)$$
$$+S(\xi,Z)R(U,Y)X$$
$$-S(X,Z)R(U,Y)\xi.$$

Now, if we consider the equation (16) in (29), then we get

$$\lambda\begin{bmatrix} -\eta(R(U,Y)Z)X+g(X,R(U,Y)Z)\xi \\ +g(X,U)R(\xi,Y)Z-\eta(U)R(X,Y)Z \\ +g(X,Y)R(U,\xi)Z-\eta(Y)R(U,X)Z \\ +g(X,Z)R(U,Y)\xi-\eta(Z)R(U,Y)X \end{bmatrix}=0.$$

(30)

By taking inner product with ξ and by use of (11)-(14) with (2), we find

$$\lambda\begin{bmatrix} -\eta(R(U,Y)Z)\eta(X)+g(X,R(U,Y)Z) \\ +g(X,U)\eta(R(\xi,Y)Z)-\eta(U)\eta(R(X,Y)Z) \\ +g(X,Y)\eta(R(U,\xi)Z)-\eta(Y)\eta(R(U,X)Z) \\ +g(X,Z)\eta(R(U,Y)\xi)-\eta(Z)\eta(R(U,Y)X) \end{bmatrix}=0,$$

(31)

from which

$$\lambda\begin{bmatrix} g(X,R(U,Y)Z)-g(Y,Z)g(U,X) \\ +g(X,Y)g(U,Z) \\ -2\begin{pmatrix} g(U,Z)\eta(X)\eta(Y) \\ -g(Y,Z)\eta(X)\eta(U) \end{pmatrix} \end{bmatrix}=0.$$

(32)

Let $\{e_i, \phi e_i, \xi\}_{(i=1,\ldots,n)}$ is an orthonormal basis of M. Thus by a contraction of (32), we have

$$\lambda \begin{bmatrix} S(Y,Z)-(2n-2)g(Y,Z) \\ -2\eta(Y)\eta(Z) \end{bmatrix} = 0. \quad (33)$$

Theorem 3.5 A para-Sasakian manifold satisfying $S(\xi, X)\cdot R = 0$ admitting a non-steady Ricci soliton (g,V,λ) is a $\eta-$ Einstein manifold.

Also, putting $Y = \xi = Z$ in (33) and in view of (16), we obtain

$\lambda(-\lambda - 2n) = 0,$

which implies $\lambda = 0$ or $\lambda = -2n$. Thus, we get:

Theorem 3.6 Let (g,V,λ) be a Ricci soliton on a para-Sasakian manifold M. If the equation $S(\xi, X)\cdot R = 0$ satisfies on M, then the Ricci soliton is either steady or shrinking.

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