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Statistical inference of stress-strength reliability for the exponential power (EP) distribution based on progressive type-II censored samples

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Abstract

Suppose that X represents the stress which is applied to a component and Y is strength of this component. Let X and Y have Exponential Power (EP) distribution with (α_1, β_1) and (α_2, β_2) parameters, respectively. In this case, stress-strength reliability (SSR) is shown by $P = P(X < Y)$. In this study, the SSR for EP distribution are obtained with numerical methods. Also maximum likelihood estimate (MLE) and approximate bayes estimates by using Lindley approximation method under squared-error loss function for SSR under progressive type-II censoring are obtained. Moreover, performances of these estimators are compared in terms of MSEs by using Monte Carlo simulation. Furthermore coverage probabilities of parametric bootstrap estimates are computed. Finally, real data analysis is presented.

Keywords: Maximum likelihood estimation, Bayes estimation, Exponential Power distribution, Lindley's approximation, Monte Carlo simulation, Bootstrap estimation.

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1. Introduction

In this article, it is considered estimation of $P = P(X < Y)$, when X and Y are independent exponential power random variables with different parameters. Stress-Strength model identifies the life of a component which is exposed to X stress with Y strength. The probability of $P = P(X < Y)$ is called as stress-strength reliability (SSR). Stress-Strength models can be used in variety of areas such as engineering, medicine, military. The reliability of a carbon fiber, a bridge, an elevator and so forth can be given as examples to SSR. When the failure time of all components in system may not always observe in reliability analysis and life test. In this case, censored data is obtained.

There are different types of censoring schemes such as type-I, type-II and progressive type-II right (PTR-II) censoring schemes in lifetesting experiments. In type-I censoring scheme, the experiment is stopped at a pre-fixed time. In type-II censoring scheme, the experiment is stopped whenever a fixed number of failures has been observed. One of the most widely used censoring schemes is progressive type-II right (PTR-II) censoring scheme. This scheme is described as follows. Assume that n identical units are placed on a test and m failures are going to be observed. At the time of the first failure, R_1 items are chosen at random and removed. Similarly, at the time of the second failure, R_2 of the remaining items are chosen at random and removed, and so on. Finally, at the time of the m^{th} failure, all the surviving items are censored. PTR-II censoring scheme is shown with $\mathbf{R} = (R_1, R_2, \ldots, R_m)$. In PTR-II censoring, Type-II censoring is obtained for $\mathbf{R} = (0, 0, \ldots, n-m)$. In this life time process, $\mathbf{X^R} = \left(X^{R_1}_{1:m:n}, X^{R_2}_{2:m:n}, ..., X^{R_m}_{m:m:n} \right)$ with $X^{R_1}_{1:m:n} < X^{R_2}_{2:m:n} < \cdots < X^{R_m}_{m:m:n}$ is called as PTR-II censored sample with $\mathbf{R} = (R_1, R_2, \ldots, R_m)$ scheme. The joint pdf of this censored sample is given by;

$$f_{X^{\mathbf{R}}_{1:m:n}, X^{\mathbf{R}}_{2:m:n}, \ldots, X^{\mathbf{R}}_{m:m:n}}(x_1, x_2, ..., x_m) = c \prod_{i=1}^{m} f(x_i) [1 - F(x_i)]^{R_i}, \quad -\infty < x_1 < x_2 < \cdots < x_m < \infty \tag{1.1}$$

where $c = n(n - R_1 - 1) \times \cdots \times (n - R_1 - R_2 - \cdots - R_{m-1} - m + 1)$. (Balakrishnan (2007) , Balakrishnan and Aggarwala (2000)).

Estimation problem of SSR under PTR-II censoring which reduces the cost and time of an experiment is fairly important.There are many studies about estimation of SSR for some distributions under PTR-II censoring. Among them are Saraçoğlu et al. (2012), Lio and Tsai (2012),Valıollahı et al. (2013), Lin and Ke (2013), Rezaei et al. (2015), Basirat et al. (2015).

The exponential power (EP) distribution introduced by Smith &Bain (1975) has been studied by many authors such as Leemis (1986), Rajarshi&Rajarshi (1988) and Chen (1999). In this study, it is assumed that X and Y are independent Exponential Power (EP) random variables with (α_1, β_1) and (α_2, β_2) parameters, where $\beta_1, \beta_2 > 0$ are shape parameters and $\alpha_1, \alpha_2 > 0$ are scale parameters, respectively. In this case, the probability density functions (pdfs) and cumulative distribution functions (cdfs) of X and Y are as follows;

$$f(x; \alpha_1, \beta_1) = \frac{\beta_1}{\alpha_1} \left(\frac{x}{\alpha_1} \right)^{\beta_1 - 1} \exp\left(\frac{x}{\alpha_1} \right)^{\beta_1} \exp\left[1 - \exp\left(\frac{x}{\alpha_1} \right)^{\beta_1} \right], \; x \geq 0 \tag{1.2}$$

$$F(x; \alpha_1, \beta_1) = 1 - \exp\left(1 - \exp\left(\frac{x}{\alpha_1} \right)^{\beta_1} \right) \tag{1.3}$$

$$f(y;\alpha_2,\beta_2)=\frac{\beta_2}{\alpha_2}\left(\frac{y}{\alpha_2}\right)^{\beta_2-1}\exp\left(\frac{y}{\alpha_2}\right)^{\beta_2}\exp\left[1-\exp\left(\frac{y}{\alpha_2}\right)^{\beta_2}\right],\ y\geq 0 \tag{1.4}$$

$$F(y;\alpha_2,\beta_2)=1-\exp\left(1-\exp\left(\frac{y}{\alpha_2}\right)^{\beta_2}\right) \tag{1.5}$$

The main objective of this manuscript is to obtained the approximate Bayes estimators for SSR of EP distribution based on PTR-II censored samples under squared error loss function and compare them with maximum likelihood estimators (MLEs) of SSR. The rest of the manuscript is organized as follows. In Section 2, SSR for EP distribution is given. In section 3, the MLEs and bootstrap confidence intervals for SSR are derived. In section 4, the approximate Bayes estimators under squared error loss function are derived by using Lindley's approximations. Using Monte Carlo simulation, The approximate Bayes estimation are compared with the maximum likelihood estimation in terms of Mean square error (MSE) and results are tabulated in section 5. In section 6, real data analysis is presented. Finally, conclusions are given in Section 7.

2. Stress-Strength Reliability

Let X and Y independent random variables have EP ditribution with (α_1,β_1) and (α_2,β_2) parameters, respectively. Suppose that Y represent the strength of a component exposed to X stress, then the SSR of this component is obtained as follows,

$$P=P(X<Y)=\int_{y=0}^{\infty} f_Y(y)F_X(y)\,dy \tag{2.1}$$

$$\begin{aligned}=&\int_{y=0}^{\infty}\frac{\beta_2}{\alpha_2}\left(\frac{y}{\alpha_2}\right)^{\beta_2-1}\exp\left(\frac{y}{\alpha_2}\right)^{\beta_2}\exp\left[1-\exp\left(\frac{y}{\alpha_2}\right)^{\beta_2}\right]\\&\times\left[1-\exp\left[1-\exp\left(\frac{y}{\alpha_1}\right)^{\beta_1}\right]\right]dy\end{aligned} \tag{2.2}$$

$$\begin{aligned}=&\ 1-\int_{y=0}^{\infty}\frac{\beta_2}{\alpha_2}\left(\frac{y}{\alpha_2}\right)^{\beta_2-1}\exp\left(\frac{y}{\alpha_2}\right)^{\beta_2}\exp\left[1-\exp\left(\frac{y}{\alpha_2}\right)^{\beta_2}\right]\times\\&\times\exp\left[1-\exp\left(\frac{y}{\alpha_1}\right)^{\beta_1}\right]dy\end{aligned} \tag{2.3}$$

This probability can be solved by using numerical methods.

3. Maximum Likelihood Estimation (MLE)

Let $\mathbf{X^R}=\left(X^{R_1}_{1:m_1:n_1},X^{R_2}_{2:m_1:n_1},...,X^{R_{m_1}}_{m_1:m_1:n_1}\right)$ denote PTR-II censored sample with $\mathbf{R}$ censoring scheme taken from $EP(\alpha_1,\beta_1)$ distribution having pdf and cdf defined in Eq.(1.2) and Eq.(1.3) and $\mathbf{Y^S}=\left(Y^{S_1}_{1:m_2:n_2},Y^{S_2}_{2:m_2:n_2},...,Y^{S_{m_2}}_{m_2:m_2:n_2}\right)$ denote PTR-II censored sample with $\mathbf{S}$ censoring scheme taken from $EP(\alpha_2,\beta_2)$ distribution having

pdf and cdf defined in Eq.(1.4) and Eq.(1.5). Then, the likelihood function is obtained as follows,

$$\begin{aligned} L(\alpha_1,\beta_1,\alpha_2,\,\beta_2|\,x^R,y^S) &= \left[c_1\prod_{i=1}^{m_1} f\left(x_{i:m_1:n_1}\right)\left[1-F\left(x_{i:m_1:n_1}\right)\right]^{R_i}\right] \\ &\quad\times\left[c_2\prod_{j=1}^{m_2} f\left(y_{j:m_2:n_2}\right)\left[1-F\left(y_{j:m_2:n_2}\right)\right]^{S_j}\right] \end{aligned}$$

$$\begin{aligned} (3.1)\quad L(\alpha_1,\beta_1,\alpha_2,\,\beta_2|\,x^R,y^S) &= c_1c_2\left(\frac{\beta_1}{\alpha_1}\right)^{m_1}\left(\frac{\beta_2}{\alpha_2}\right)^{m_2} \\ &\quad\times\prod_{i=1}^{m_1}\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1-1}\prod_{j=1}^{m_2}\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2-1} \end{aligned}$$

$$\times\exp\left(\sum_{i=1}^{m_1}\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}\right)\exp\left[\sum_{i=1}^{m_1}(1+R_i)\left(1-\exp\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}\right)\right]$$

$$\times\exp\left(\sum_{j=1}^{m_2}\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2}\right)\exp\left[\sum_{j=1}^{m_2}(1+S_j)\left(1-\exp\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2}\right)\right]$$

where

$$\begin{aligned} c_1 &= n_1(n_1-1-R_1)\times\cdots\times(\,n_1-m_1+1-R_1-\cdots-R_{m_1-1})\,, \\ c_2 &= n_2(n_2-1-S_1)\times\cdots\times(\,n_2-m_2+1-S_1-\cdots-S_{m_2-1})\,. \end{aligned}$$

Then, the log-likelihood function is given by,

$$(3.2)\quad \ell\left(\alpha_1,\beta_1,\alpha_2,\beta_2|\,x^R,y^S\right) = m_1\ln\beta_1-m_1\ln\alpha_1+m_2\ln\beta_2-m_2\ln\alpha_2+\ln\,c_1+\ln\,c_2$$

$$+(\beta_1-1)\sum_{i=1}^{m_1}\ln\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)+(\beta_2-1)\sum_{j=1}^{m_2}\ln\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)$$

$$+\sum_{i=1}^{m_i}\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}+\sum_{i=1}^{m_1}(1+R_i)\left[1-\exp\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}\right]$$

$$+\sum_{j=1}^{m_2}\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2}+\sum_{j=1}^{m_2}(1+S_j)\left[1-\exp\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2}\right]$$

By differentiating partially the log-likelihood function $\ell\left(\,\alpha_1,\beta_1,\alpha_2,\beta_2|\,x^R,y^S\right)$ with respect to α_1, β_1, α_2 and β_2 parameters and then equalizing them to zero we get,

$$(3.3)\quad \frac{\partial\ell}{\partial\alpha_1} = -\frac{m_1}{\alpha_1}+\sum_{i=1}^{m_1}\left[-\frac{\beta_1}{\alpha_1}\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}\right]$$

$$+\sum_{i=1}^{m_1}(1+R_i)\left[\frac{\beta_1}{\alpha_1}\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}\exp\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}\right]-\frac{(\beta_1-1)\,m_1}{\alpha_1}=0$$

$$\frac{\partial \ell}{\partial \beta_1} = \frac{m_1}{\beta_1} + \sum_{i=1}^{m_1} \left[\left(\frac{x_{i:m_1:n_1}}{\alpha_1} \right)^{\beta_1} \ln \left(\frac{x_{i:m_1:n_1}}{\alpha_1} \right) \right] + \sum_{i=1}^{m_1} \ln \left(\frac{x_{i:m_1:n_1}}{\alpha_1} \right) \tag{3.4}$$

$$- \sum_{i=1}^{m_1} (1 + R_i) \left[\left(\frac{x_{i:m_1:n_1}}{\alpha_1} \right)^{\beta_1} \ln \left(\frac{x_{i:m_1:n_1}}{\alpha_1} \right) \exp \left(\frac{x_{i:m_1:n_1}}{\alpha_1} \right)^{\beta_1} \right] = 0$$

$$\frac{\partial \ell}{\partial \alpha_2} = -\frac{m_2}{\alpha_2} + \sum_{j=1}^{m_2} \left[-\frac{\beta_2}{\alpha_2} \left(\frac{y_{j:m_2:n_2}}{\alpha_2} \right)^{\beta_2} \right] \tag{3.5}$$

$$+ \sum_{j=1}^{m_2} (1 + S_j) \left[\frac{\beta_2}{\alpha_2} \left(\frac{y_{j:m_2:n_2}}{\alpha_2} \right)^{\beta_2} \exp \left(\frac{y_{j:m_2:n_2}}{\alpha_2} \right)^{\beta_2} \right] - \frac{(\beta_2 - 1)\, m_2}{\alpha_2} = 0$$

$$\frac{\partial \ell}{\partial \beta_2} = \frac{m_2}{\beta_2} + \sum_{j=1}^{m_2} \left(\frac{y_{j:m_2:n_2}}{\alpha_2} \right)^{\beta_2} \ln \left(\frac{y_{j:m_2:n_2}}{\alpha_2} \right) + \sum_{j=1}^{m_2} \ln \left(\frac{y_{j:m_2:n_2}}{\alpha_2} \right) \tag{3.6}$$

$$- \sum_{j=1}^{m_2} (1 + S_j) \left[\left(\frac{y_{j:m_2:n_2}}{\alpha_2} \right)^{\beta_2} \ln \left(\frac{y_{j:m_2:n_2}}{\alpha_2} \right) \exp \left(\frac{y_{j:m_2:n_2}}{\alpha_2} \right)^{\beta_2} \right] = 0$$

It is clear that the normal equations do not have explicit forms. MLEs of $\alpha_1, \alpha_2, \beta_1$ and β_2 parameters are obtained by using Newton Raphson method. By using Eq.(2.3) and invariant property of MLE, MLE for P is obtained as follows.

$$\widehat{P}_{MLE} = 1 - \int_{y=0}^{\infty} \frac{\hat{\beta}_2}{\hat{\alpha}_2} \left(\frac{y}{\hat{\alpha}_2} \right)^{\hat{\beta}_2 - 1} \exp \left(\frac{y}{\hat{\alpha}_2} \right)^{\hat{\beta}_2} \times \tag{3.7}$$

$$\times \exp \left[1 - \exp \left(\frac{y}{\hat{\alpha}_2} \right)^{\hat{\beta}_2} \right] \exp \left[1 - \exp \left(\frac{y}{\hat{\alpha}_1} \right)^{\hat{\beta}_1} \right] dy$$

Moreover, Bootstrap method is used to obtain confidence intervals of P.

3.1. Bootstrap Confidence Intervals. The bootstrap method that is widely used in practise is the percentile bootstrap (Boot-p) proposed by Efron (1982). It is illustrated shortly in following steps how to estimate parametric bootstrap confidence intervals of P using Boot-p method.

Step 1. Generate PTR-II censored samples $\left(x_{1:m_1:n_1}^{R_1}, x_{2:m_1:n_1}^{R_2}, ..., x_{m_1:m_1:n_1}^{R_{m_1}} \right)$ and $\left(y_{1:m_2:n_2}^{S_1}, y_{2:m_2:n_2}^{S_2}, ..., y_{m_2:m_2:n_2}^{S_{m_2}} \right)$ taken from EP distributions with (α_1, β_1) and (α_2, β_2) parameters, respectively.

Step 2. Estimate P , say $\widehat{P}_{MLE}$.

Step 3. Generate a bootstrap samples $(x_{1:m_1:n_1}^*, x_{2:m_1:n_1}^*, ..., x_{m_1:m_1:n_1}^*)$ and $(y_{1:m_2:n_2}^*, y_{2:m_2:n_2}^*, ..., y_{m_2:m_2:n_2}^*)$, using $\widehat{P}_{MLE}$, $R_1, R_2, \ldots, R_{m_1}$, $S_1, S_2, \ldots, S_{m_2}$.Obtain the bootstrap estimate of P , say $\widehat{P}_{MLE}^*$.

Step 4. Repeat Step 3 NBOOT times.

Step 5. Let $F^*(x) = P(\widehat{P}_{MLE}^* \leq x)$, be the cumulative distribution function of $\widehat{P}_{MLE}^*$. Define $\widehat{P}_{MLEBoot-p}(x) = F^{*-1}(x)$ for a given x. The approximate 100(1-γ)% confidence interval for P is given by

$$\left(\widehat{P}_{MLEBoot-p}\left(\gamma/2\right), \widehat{P}_{MLEBoot-p}(1 - \gamma/2) \right) \tag{3.8}$$

4. Bayes Estimation

Let $\left(X_{1:m_1:n_1}^{R_1}, X_{2:m_1:n_1}^{R_2}, ..., X_{m_1:m_1:n_1}^{R_{m_1}} \right)$ and $\left(Y_{1:m_2:n_2}^{S_1}, Y_{2:m_2:n_2}^{S_2}, ..., Y_{m_2:m_2:n_2}^{S_{m_2}} \right)$ are PTR-II censored samples taken from $EP(\alpha_1, \beta_1)$ and $EP(\alpha_2, \beta_2)$, respectively. For Bayesian estimation of SSR under PTR-II censored sample, we assume that the α_1, β_1 , α_2 and β_2 parameters have independent prior $Gamma(a_1, b_1)$, $Gamma(\ a_2, b_2)$, $Gamma(a_3, b_3)$ and $Gamma(\ a_4, b_4)$ distributions, respectively. These prior density functions are given as follows.

$$\pi_1(\alpha_1) = \frac{\alpha_1^{a_1-1} \exp(\ -b_1\alpha_1)\, b_1^{a_1}}{\Gamma(a_1)} \quad a_1, b_1, \alpha_1 > 0 \tag{4.1}$$

$$\pi_2(\beta_1) = \frac{\beta_1^{a_2-1} \exp(\ -b_2\beta_1)\, b_2^{a_2}}{\Gamma(a_2)} \quad a_2, b_2, \beta_1 > 0 \tag{4.2}$$

$$\pi_3(\alpha_2) = \frac{\alpha_2^{a_3-1} \exp(\ -b_3\alpha_2)\, b_3^{a_3}}{\Gamma(a_3)} \quad a_3, b_3, \alpha_2 > 0 \tag{4.3}$$

$$\pi_4(\beta_2) = \frac{\beta_2^{a_4-1} \exp(\ -b_4\beta_2)\, b_4^{a_4}}{\Gamma(a_4)} \quad a_4, b_4, \beta_2 > 0 \tag{4.4}$$

Then, the joint prior density function of α_1, β_1 , α_2 and β_2 can be written as follows;

$$\pi(\alpha_1, \beta_1, \alpha_2, \beta_2) = \frac{\alpha_1^{a_1-1} b_1^{a_1} \beta_1^{a_2-1} b_2^{a_2} \alpha_2^{a_3-1} b_3^{a_3} \beta_2^{a_4-1} b_4^{a_4}}{\Gamma(\ a_1)\,\Gamma(a_2)\,\Gamma(a_3)\,\Gamma(a_4)} \times \tag{4.5}$$

$$\times \exp(-b_1\alpha_1) \exp(-b_2\beta_1) \exp(-b_3\alpha_2) \exp(-b_4\beta_2)$$

$$a_i, b_i, \alpha_1, \alpha_2, \beta_1, \beta_2 > 0\,, i = 1, 2, 3, 4$$

From (4.5), the log of joint prior density function is given by

$$G(\alpha_1, \beta_1, \alpha_2, \beta_2) = \ln \pi(\alpha_1, \beta_1, \alpha_2, \beta_2) \propto (a_1 - 1)\ln\alpha_1 + (a_2 - 1)\ln\beta_1 \tag{4.6}$$

$$+(a_3 - 1)\ln\alpha_2 + (a_4 - 1)\ln\beta_2 - b_1\alpha_1 - b_2\beta_1 - b_3\alpha_2 - b_4\beta_2$$

With the help of equation (3.1) and (4.5), the joint posterior density function of α_1, β_1 , α_2 and β_2 when data are given can be written as follows;

$$P(\alpha_1, \beta_1, \alpha_2, \beta_2 | X^R, Y^S) = \tag{4.7}$$

$$= \frac{\pi(\alpha_1, \beta_1, \alpha_2, \beta_2)\, k_1(x_{i:m_1:n_1}; \alpha_1, \beta_1)\, k_2(\ y_{j:m_2:n_2;}\alpha_2, \beta_2)}{\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \pi(\alpha_1, \beta_1, \alpha_2, \beta_2)\, k_1(x_{i:m_1:n_1}; \alpha_1, \beta_1)\, k_2(\ y_{j:m_2:n_2;}\alpha_2, \beta_2) d\alpha\beta}$$

where

$$d\alpha\beta = d\alpha_1 d\beta_1 d\alpha_2 d\beta_2$$

and

$$k_1(x_{i:m_1:n_1}; \alpha_1, \beta_1) = \left(\frac{\beta_1}{\alpha_1}\right)^{m_1} \exp\left(\sum_{i=1}^{m_1} \left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}\right) \times$$

$$\times \exp\left[\sum_{i=1}^{m_1} (1 + R_i) \left(1 - \exp\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}\right)\right] \prod_{i=1}^{m_1} \left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1 - 1}$$

and

$$k_2\left(y_{j:m_2:n_2};\alpha_2,\beta_2\right)=\left(\frac{\beta_2}{\alpha_2}\right)^{m_2}\exp\left(\sum_{j=1}^{m_2}\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2}\right)\times$$

$$\times\exp\left[\sum_{j=1}^{m_2}\left(1+S_j\right)\left(1-\exp\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2}\right)\right]\prod_{j=1}^{m_2}\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2-1}$$

Let any function of α_1, β_1 , α_2 and β_2 is $u\left(\alpha_1,\beta_1,\alpha_2,\beta_2\right)$. The bayes estimation of this function under squared error loss function can be written as follows,

$$\hat{u}_B\left(\alpha_1,\beta_1,\alpha_2,\beta_2\right)=E\left(u\left(\alpha_1,\beta_1,\alpha_2,\beta_2\right)|X^R,Y^S\right) \tag{4.8}$$

$$=\frac{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty u\left(\alpha_1,\beta_1,\alpha_2,\beta_2\right)\exp\left[\ell\left(\ \alpha_1,\beta_1,\alpha_2,\beta_2|x^R,y^S\right)+G\left(\ \alpha_1,\beta_1,\alpha_2,\beta_2\right)\right]d\alpha\beta}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\exp\left[\ell\left(\alpha_1,\beta_1,\alpha_2,\beta_2|x^R,y^S\right)+G\left(\ \alpha_1,\beta_1,\alpha_2,\beta_2\right)\right]d\alpha\beta}$$

In Eq.(4.8), if $u\left(\alpha_1,\beta_1,\alpha_2,\beta_2\right)$ is taken as P, Bayes estimator of SSR can be obtained. But, it is not possible to compute Eq.(4.8) in closed-form. Because of this reason, we have obtained the Bayes Estimators of SSR by using Lindley's approximation under the squared error loss function.

4.1. Lindley's Approximation. Lindley's approximation suggested by Lindley (1980) is an approximate method used to compute the ratio of two integrals as in Eq. (4.8) that can not be solved analytically. The Lindley's approximation formulas for case with four-parameters by using $(\theta_1,\theta_2,\theta_3,\theta_4)$ notation instead of $(\alpha_1,\beta_1,\alpha_2,\beta_2)$ can be written as follows.

$$u\left(\hat{\theta}\right)_{Bayes}=E\left[u\left(\theta\right)/X\right] \tag{4.9}$$

$$\approx\left\{u\left(\hat{\theta}_1,\hat{\theta}_2,\hat{\theta}_3,\hat{\theta}_4\right)+\frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}\left(u_{ij}+2u_ig_j\right)\sigma_{ij}+\frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4}\sum_{l=1}^{4}\ell_{ijk}\sigma_{ij}\sigma_{kl}u_l\right\}$$

where $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$ and $\hat{\theta}_4$ are the MLEs of θ_1 , θ_2, θ_3 and θ_4, respectively and other formulas are as follows.

$$u_i=\frac{\partial u\left(\theta_1,\theta_2,\theta_3,\theta_4\right)}{\partial\theta_i}\ ,\ \ i=1,2,3,4$$

$$u_{ij}=\frac{\partial^2 u\left(\theta_1,\theta_2,\theta_3,\theta_4\right)}{\partial\theta_i\partial\theta_j}\ ,\ i,j=1,2,3,4$$

$$\ell_{ij}=\frac{\partial^2\ell\left(\theta_1,\theta_2,\theta_3,\theta_4\right)}{\partial\theta_i\partial\theta_j}\ ,\ i,j=1,2,3,4$$

$$\ell_{ijk}=\frac{\partial^3\ell\left(\theta_1,\theta_2,\theta_3,\theta_4\right)}{\partial\theta_i\partial\theta_j\partial\theta_k}\ ,\ i,j,k=1,2,3,4$$

$$\left[-\ell_{ij}\right]^{-1}=\left[\sigma_{ij}\right]\ ,i,j=1,2,3,4$$

and σ_{ij} is the $(i,\ j)$-th element of the matrix $[\sigma_{ij}]$. From (4.6), we have

$$g_1=\frac{\partial G\left(\alpha_1,\beta_1,\alpha_2,\beta_2\right)}{\partial\alpha_1}=\frac{a_1-1}{\alpha_1}-b_1,\quad g_2=\frac{\partial G\left(\alpha_1,\beta_1,\alpha_2,\beta_2\right)}{\partial\beta_1}=\frac{a_2-1}{\beta_1}-b_2$$

$$g_3 = \frac{\partial G\left(\alpha_1,\beta_1,\alpha_2,\beta_2\right)}{\partial \alpha_2} = \frac{a_3-1}{\alpha_2} - b_3 \quad , g_4 = \frac{\partial G\left(\alpha_1,\beta_1,\alpha_2,\beta_2\right)}{\partial \beta_2} = \frac{a_4-1}{\beta_2} - b_4$$

and then, we have the following values of ℓ_{ij} for i, j= 1,2,3,4 and ℓ_{ijk} for i, $j, k = 1, 2, 3, 4$.

$$\ell_{11} = \frac{m_1}{\alpha_1^2} + \frac{\left(\beta_1-1\right)m_1}{\alpha_1^2} + \sum_{i=1}^{m_1}\left(1+\beta_1\right) g_1\left(x_{i:m_1:n_1};\alpha_1,\beta_1\right) +$$

$$+\sum_{i=1}^{m_1}\left(1+R_i\right) h_1\left(\ x_{i:m_1:n_1};\alpha_1,\beta_1\right)\left(-\beta_1 - 1 - \beta_1\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}\right)$$

$$\ell_{12} = \sum_{i=1}^{m_1} g_1\left(x_{i:m_1:n_1};\alpha_1,\beta_1\right)\left(-\ln\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)\alpha_1 - \frac{\alpha_1}{\beta_1}\right) +$$

$$+\sum_{i=1}^{m} h_1\left(x_{i:m_1:n_1};\alpha_1,\beta_1\right)\alpha_1\left(1+R_i\right)\left[\ \ln\left(\tfrac{x_{i:m_1:n}}{\alpha_1}\right)\left(1+\left(\ \tfrac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}\right)+\left(\tfrac{1}{\beta_1}\right)\ \right] - \frac{m_1}{\alpha_1}$$

$$\ell_{13} = \ell_{31} = 0, \quad \ell_{14} = \ell_{41} = 0, \quad \ell_{21} = \ell_{12}$$

$$\ell_{22} = -\frac{m_1}{\beta_1^2} + \sum_{i=1}^{m_1}\left(\ \frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}\ln\left(\ \frac{x_{i:m_1:n_1}}{\alpha_1}\right)^2 -$$

$$-\sum_{i=1}^{m_1} h_1\left(x_{i:m_1:n_1};\alpha_1,\beta_1\right)\left(1+R_i\right)\frac{\alpha_1^2}{\beta_1}\ln\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^2\left[1+\left(\ \frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}\right]$$

$$\ell_{23} = \ell_{32} = 0, \quad \ell_{24} = \ell_{42} = 0$$

$$\ell_{33} = \frac{m_2}{\alpha_2^2} + \frac{\left(\beta_2-1\right)m_2}{\alpha_2^2} + \sum_{j=1}^{m_2}\left(1+\beta_2\right) g_2\left(y_{j:m_2:n_2};\alpha_2,\beta_2\right) +$$

$$+\sum_{j=1}^{m_2}\left(1+S_j\right) h_2\left(\ y_{j:m_2:n_2};\alpha_2,\beta_2\right)\left(-\beta_2 - 1 - \beta_2\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2}\right)$$

$$\ell_{34} = \sum_{j=1}^{m_2} g_1\left(y_{j:m_2:n_2};\alpha_2,\beta_2\right)\left(-\ln\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)\alpha_2 - \frac{\alpha_2}{\beta_2}\right) +$$

$$+\sum_{j=1}^{m_2} h_2\left(y_{j:m_2:n_2};\alpha_2,\beta_2\right)\alpha_2\left(1+S_j\right)\left[\ \ln\left(\tfrac{y_{j:m_2:n_2}}{\alpha_2}\right)\left(1+\left(\ \tfrac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2}\right)+\left(\tfrac{1}{\beta_2}\right)\ \right] - \frac{m_2}{\alpha_2}$$

$$\ell_{43} = \ell_{34}$$

$$\ell_{44} = -\frac{m_2}{\beta_2^2} + \sum_{j=1}^{m_2}\left(\ \frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2}\ln\left(\ \frac{y_{j:m_2:n_2}}{\alpha_2}\right)^2 -$$

$$-\sum_{j=1}^{m_2} h_2\left(y_{j:m_2:n_2};\alpha_2,\beta_2\right)\left(1+S_j\right)\frac{\alpha_2^2}{\beta_2}\ln\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^2\left[1+\left(\ \frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2}\right]$$

$$\ell_{111} = -\frac{2m_1}{\alpha_1^3} - \sum_{i=1}^{m_1} \frac{g_1\left(x_{i:m_1:n_1};\alpha_1,\beta_1\right)}{\alpha_1}\left[\beta_1^2+3\beta_1+2\right] + \frac{2\left(\beta_1-1\right)m_1}{\alpha_1^3} +$$

$$+\sum_{i=1}^{m_1} \frac{h_1\left(x_{i:m_1:n_1};\alpha_1,\beta_1\right)}{\alpha_1}\left(1+R_i\right)\left[\begin{array}{c}\beta_1^2\left(1+3\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}+\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{2\beta_1}\right)\\ +3\beta_1\left(1+\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}\right)+2\end{array}\right]$$

$$\ell_{112} = \frac{m_1}{\alpha_1^2} + \sum_{i=1}^{m_1} g_1\left(x_{i:m_1:n_1};\alpha_1,\beta_1\right)\left[\ln\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)\left(1+\beta_1\right)+\left(2+\frac{1}{\beta_1}\right)\right] -$$

$$-\sum_{i=1}^{m_1} h_1\left(x_{i:m_1:n_1};\alpha_1,\beta_1\right)\left(1+R_i\right)\left[\beta_1\ln\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)\left(\begin{array}{c}1+3\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}\\ +\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{2\beta_1}\end{array}\right)\right] -$$

$$-\sum_{i=1}^{m_1} h_1\left(x_{i:m_1:n_1};\alpha_1,\beta_1\right)\left(1+R_i\right)\left[\begin{array}{c}\left(2+\ln\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)+2\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}\right)\\ +\ln\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}\end{array}\right] -$$

$$-\sum_{i=1}^{m_1} \frac{h_1\left(x_{i:m_1:n_1};\alpha_1,\beta_1\right)}{\beta_1}\left(1+R_i\right)$$

$$\begin{array}{rcll}
\ell_{112} & = & \ell_{121}=\ell_{211}, & \ell_{122}=\ell_{221}=\ell_{212}\\
\ell_{223} & = & \ell_{232}=\ell_{322}=0, & \ell_{224}=\ell_{242}=\ell_{422}=0\\
\ell_{233} & = & \ell_{332}=\ell_{323}=0, & \ell_{133}=\ell_{313}=\ell_{331}=0\\
\ell_{114} & = & \ell_{141}=\ell_{411}=0, & \ell_{144}=\ell_{441}=\ell_{414}=0\\
\ell_{244} & = & \ell_{442}=\ell_{424}=0, & \ell_{113}=\ell_{131}=\ell_{311}=0
\end{array}$$

$$\ell_{122} = \sum_{i=1}^{m_1} \frac{g_1\left(x_{i:m_1:n_1};\alpha_1,\beta_1\right)\alpha_1}{\beta_1}\left[-\beta_1\ln\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^2 - 2\ln\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)\right] +$$

$$+\sum_{i=1}^{m_1} \frac{h_1\left(x_{i:m_1:n_1};\alpha_1,\beta_1\right)\alpha_1}{\beta_1}\left(1+R_i\right)\left[\begin{array}{c}\beta_1\ln\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^2\left(\begin{array}{c}1+3\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}\\ +\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{2\beta_1}\end{array}\right)\\ +2\ln\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)\left(1+\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}\right)\end{array}\right]$$

$$\ell_{222} = \frac{2m_1}{\beta_1^3} + \sum_{i=1}^{m_1}\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}\ln\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^3 -$$

$$-\sum_{i=1}^{m_1} \frac{h_1\left(x_{i:m_1:n_1};\alpha_1,\beta_1\right)\alpha_1^2}{\beta_1}\left(1+R_i\right)\ln\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^3\left[\begin{array}{c}1+3\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}\\ +\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{2\beta_1}\end{array}\right]$$

$$\begin{aligned}
\ell_{123} &= \ell_{132} = \ell_{213} = \ell_{231} = \ell_{312} = \ell_{321} = 0 \\
\ell_{124} &= \ell_{142} = \ell_{214} = \ell_{241} = \ell_{412} = \ell_{421} = 0 \\
\ell_{134} &= \ell_{143} = \ell_{314} = \ell_{341} = \ell_{413} = \ell_{431} = 0 \\
\ell_{234} &= \ell_{243} = \ell_{324} = \ell_{342} = \ell_{423} = \ell_{432} = 0
\end{aligned}$$

$$\ell_{334} = \frac{m_2}{\alpha_2^2} + \sum_{j=1}^{m_2} g_2\left(y_{j:m_2:n_2};\alpha_2,\beta_2\right)\left[\ln\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)(1+\beta_2) + \left(2+\frac{1}{\beta_2}\right)\right] -$$

$$-\sum_{j=1}^{m_2} h_2\left(y_{j:m_2:n_2};\alpha_2,\beta_2\right)(1+S_j)\left[\beta_2 \ln\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)\left(\begin{array}{c} 1+3\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2} \\ +\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{2\beta_2} \end{array}\right)\right] -$$

$$-\sum_{j=1}^{m_2} h_2\left(y_{j:m_2:n_2};\alpha_2,\beta_2\right)(1+S_j)\left[\begin{array}{c} 2+\ln\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)+2\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2} \\ +\ln\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2} \end{array}\right] -$$

$$-\sum_{j=1}^{m_2} \frac{h_2\left(y_{j:m_2:n_2};\alpha_2,\beta_2\right)}{\beta_2}(1+S_j)$$

$$\ell_{334} = \ell_{433} = \ell_{343}, \quad \ell_{344} = \ell_{443} = \ell_{434}$$

$$\ell_{333} = -\frac{2m_2}{\alpha_2^3} - \sum_{j=1}^{m_2} \frac{g_2\left(y_{j:m_2:n_2};\alpha_2,\beta_2\right)}{\alpha_2}\left[\beta_2^2+3\beta_2+2\right] + \frac{2\left(\beta_2-1\right)m_2}{\alpha_2^3} +$$

$$+\sum_{j=1}^{m_2} \frac{h_2\left(y_{j:m_2:n_2};\alpha_2,\beta_2\right)}{\alpha_2}(1+S_j)\left[\begin{array}{c} \beta_2^2\left(1+3\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2}+\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{2\beta_2}\right) \\ +3\beta_2\left(1+\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2}\right)+2 \end{array}\right]$$

$$\ell_{444} = \frac{2m_2}{\beta_2^3} + \sum_{i=1}^{m_1}\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2}\ln\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^3 -$$

$$-\sum_{j=1}^{m_2} \frac{h_2\left(y_{j:m_2:n_2};\alpha_2,\beta_2\right)\alpha_2^2}{\beta_2}(1+S_j)\ln\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^3\left[\begin{array}{c} 1+3\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2} \\ +\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{2\beta_2} \end{array}\right]$$

$$\ell_{443} = \sum_{j=1}^{m_2} \frac{g_2\left(y_{j:m_2:n_2};\alpha_2,\beta_2\right)\alpha_2}{\beta_2}\left[-\beta_2\ln\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^2 - 2\ln\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)\right] +$$

$$+\sum_{j=1}^{m_2} \frac{h_2\left(y_{j:m_2:n_2};\alpha_2,\beta_2\right)\alpha_2}{\beta_2}(1+S_j)\left[\begin{array}{c} \beta_2\ln\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^2\left(\begin{array}{c} 1+\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{2\beta_2} \\ +3\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2} \end{array}\right) \\ +2\ln\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)\left(1+\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2}\right) \end{array}\right]$$

where $g_1(x_{i:m_1:n_1};\alpha_1,\beta_1)$, $h_1(x_{i:m_1:n_1};\alpha_1,\beta_1)$, $g_2(y_{j:m_2:n_2};\alpha_2,\beta_2)$ and $h_2(y_{j:m_2:n_2};\alpha_2,\beta_2)$ are given by

$$\begin{aligned} g_1(x_{i:m_1:n_1};\alpha_1,\beta_1) &= \frac{\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}\beta_1}{\alpha_1^2} \\ g_2(y_{j:m_2:n_2};\alpha_2,\beta_2) &= \frac{\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2}\beta_2}{\alpha_2^2} \\ h_1(x_{i:m_1:n_1};\alpha_1,\beta_1) &= \frac{\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}\beta_1\exp\left(\frac{x_{i:m_1:n_1}}{\alpha_1}\right)^{\beta_1}}{\alpha_1^2} \end{aligned}$$

and

$$h_2(y_{j:m_2:n_2};\alpha_2,\beta_2) = \frac{\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2}\beta_2\exp\left(\frac{y_{j:m_2:n_2}}{\alpha_2}\right)^{\beta_2}}{\alpha_2^2}$$

Finally, the approximate Bayes estimator under the squared error loss function of SSR for EP distribution based on PTR-II censored samples are obtained as follows,

$$\widehat{P}_{BS} = \widehat{P}_{MLE} + \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}(u_{ij}+2u_i g_j)\,\sigma_{ij} + \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4}\sum_{l=1}^{4}\ell_{ijk}\sigma_{ij}\sigma_{kl}u_l \tag{4.10}$$

where $\widehat{P}_{MLE}$ is defined in Eq.(3.7).

5. Simulation Study

In this section, a Monte Carlo simulation study is given to observe the performances of two estimation methods for different sample sizes and different censoring schemes for SSR based on EP distribution under PTR-II censoring. For different sample sizes and censoring schemes, by using the algorithm presented in Balakrishnan and Sandhu (1995), PTR-II censored samples are generated from EP distribution. This algorithm is as follows;

(1) $W_1, W_2, \ldots, W_{m_1}$ are m_1-sized samples generated from $Uniform(0,1)$ distribution.
(2) $V_i = W_i^{\left(i+\sum_{j=m_1-i+1}^{m_1} R_j\right)^{-1}}$ is defined for $i = 1,2,\ldots,m_1$.
(3) $U^R_{i:m_1:n_1} = 1 - V_{m_1}V_{m_1-1}\ldots V_{m_1-i+1}$ is obtained for $i = 1,2,\ldots,m_1$.

Thus $U^R_{1:m_1:n_1} < U^R_{2:m_1:n_1} < \cdots < U^R_{m:m_1:n_1}$ are obtained PTR-II censored samples with censoring scheme $\mathbf{R} = (R_1, R_2, \ldots, R_{m_1})$ taken from $Uniform(0,1)$ distribution. Finally, PTR-II censored i^{th}order statistics with censoring scheme $\mathbf{R} = (R_1, R_2, \ldots, R_{m_1})$ taken from $EP(\alpha_1,\beta_1)$ is obtained by transformation $X^R_{i:m_1:n_1} = \ln\left(-\ln\left(U^R_{i:m_1:n_1}+1\right)+1\right)^{\frac{1}{\beta_1}}\alpha_1$, $i = 1,2,\ldots,m_1$ and PTR-II censored j^{th}order statistics with censoring scheme $\mathbf{S} = (S_1, S_2, \ldots, S_{m_2})$ taken from $EP(\alpha_2,\beta_2)$ distribution is obtained by transformation

$Y^S_{j:m_2:n_2} = \ln\left(-\ln\left(U^S_{J:m_2:n_2}+1\right)+1\right)^{\frac{1}{\beta_2}}\alpha_2$, $j = 1,2,\ldots,m_2$. Then, the MSEs of the MLE of SSR are compared with MSEs of the approximate Bayes estimates obtained by Lindley approximation under the squared error loss function based on PTR-II censoring. For comparison purposes, the approximate Bayes estimates under informative priors $a_1 = 1; b_1 = 1; a_2 = 1; b_2 = 1; a_3 = 1; b_3 = 1; a_4 = 1; b_4 = 1$ are computed. We have reported MSE of the ML estimates and approximate bayes estimates of SSR over 5000

replications. The results are given in Table 1. Moreover, we have reported length and coverage probabilities based on bootstrap confidence intervals of SSR over 1000 bootstrap sample with 2000 replications. This results are given by Table 2.

Table 1. The MSEs of MLEs and Approximate Bayes Estimates for $\alpha_1 = 1, \beta_1 = 2, \alpha_2 = 1$ and $\beta_2 = 1$

(n_1, m_1)	(n_2, m_2)	*Censoring scheme*		$\widehat{R}_{MLE}$	$\widehat{R}_{BAYES}$
		R	S	MSE	MSE
5,5	5,5	$(5*0)$	$(5*0)$	0.0273	0.0217
10,5	10,5	$(4*0,5)$	$(4*0,5)$	0.0200	0.0153
10,5	10,5	$(5,4*0)$	$(5,4*0)$	0.0229	0.0199
10,5	10,5	$(5*1)$	$(5*1)$	0.0201	0.0169
10,10	10,10	$(10*0)$	$(10*0)$	0.0156	0.0171
20,10	20,10	$(9*0,10)$	$(9*0,10)$	0.0119	0.0118
20,10	20,10	$(10,9*0)$	$(10,9*0)$	0.0134	0.0145
20,10	20,10	$(10*1)$	$(10*1)$	0.0116	0.0116
20,20	20,20	$(20*0)$	$(20*0)$	0.0088	0.0098
30,10	30,10	$(9*0,20)$	$(9*0,20)$	0.0147	0.0103
30,10	30,10	$(20,9*0)$	$(20,9*0)$	0.0132	0.0145
30,10	30,10	$(10*2)$	$(10*2)$	0.0127	0.0103
30,30	30,30	$(30*0)$	$(30*0)$	0.0055	0.0059
100,50	100,50	$(49*0,50)$	$(49*0,50)$	0.0032	0.0027
100,50	100,50	$(50,49*0)$	$(50,49*0)$	0.0030	0.0031
100,50	100,50	$(50*1)$	$(50*1)$	0.0027	0.0023
100,100	100,100	$(100*0)$	$(100*0)$	0.0017	0.0017

Table 2. Average length and coverage probabilities of bootstrap confidence intervals for P for $\alpha_1 = 1, \beta_1 = 2, \alpha_2 = 1$ and $\beta_2 = 1$

	(n_1, m_1)	(n_2, m_2)	Censoring scheme		Average	Coverage
			R	S	length	Probability
	20,20	20,20	$(20*0)$	$(20*0)$	0.4134	0.9855
(1)	100,50	100,50	$(49*0,50)$	$(49*0,50)$	0.2292	0.9540
	100,50	100,50	$(50,49*0)$	$(50,49*0)$	0.2219	0.9440
	100,50	100,50	$(50*1)$	$(50*1)$	0.2069	0.9345
	100,100	100,100	$(100*0)$	$(100*0)$	0.1587	0.9510

6. Data Analysis

To compute estimation methods of P , it is considered two datasets, originally reported by Bader and Priest (1982), on failure stresses (in GPa) of single carbonfibers of lengths 20 mm and 50 mm, respectively (Ghitany et.al. (2014)). The datasets are given in Table 3 and in Table 4.

Table 3. Length 20 mm: $(n_1 = 69)$

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944
1.958	1.966	1.997	2.006	2.021	2.027	2.055	2.063	2.098
2.140	2.179	2.224	2.240	2.253	2.270	2.272	2.274	2.301
2.301	2.359	2.382	2.382	2.426	2.434	2.435	2.478	2.490
2.511	2.514	2.535	2.554	2.566	2.570	2.586	2.629	2.633
2.642	2.648	2.684	2.697	2.726	2.770	2.773	2.800	2.809
2.818	2.821	2.848	2.880	2.954	3.012	3.067	3.084	3.090
3.096	3.128	3.233	3.433	3.585	3.585			

Table 4. Length 50 mm: ($n_2 = 65$)

1.339	1.434	1.549	1.574	1.589	1.613	1.746	1.753	1.764
1.807	1.812	1.840	1.852	1.852	1.862	1.864	1.931	1.952
1.974	2.019	2.051	2.055	2.058	2.088	2.125	2.162	2.171
2.172	2.180	2.194	2.211	2.270	2.272	2.280	2.299	2.308
2.335	2.349	2.356	2.386	2.390	2.410	2.430	2.431	2.458
2.471	2.497	2.514	2.558	2.577	2.593	2.601	2.604	2.620
2.633	2.670	2.682	2.699	2.705	2.735	2.785	3.020	3.042
3.116	3.174							

First, it is checked whether EP distribution can be used or not to analyze these data sets. The Kolmogorov-Smirnov (KS) distances have been used to check the goodness-of-fit. ML estimates, KS-Z and p- values based on above data are shown in Table 5.

Table 5. Kolmogorov-Smirnov Z and the corresponding p-values for MLE

Carbon Fiber	**MLE**		**Kolmogorov -Smirnov Z**	**p-values**
X (Length 20 mm)	$\hat{\alpha} = 2.9920$	$\hat{\beta} = 3.7061$	0.8519	0.4624
Y (Length 30 mm)	$\hat{\alpha} = 2.6964$	$\hat{\beta} = 4.0975$	0.6509	0.7906

Progressive censored samples based on X and Y for $m_1 = m_2 = 50$, $n_1 = 69$, $n_2 = 65$ are as follows:

Progressive censored samples for X (Length 20 mm)

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944
1.958	1.997	2.006	2.021	2.027	2.055	2.063	2.098	2.140
2.179	2.224	2.240	2.270	2.272	2.382	2.426	2.434	2.435
2.490	2.511	2.535	2.566	2.629	2.633	2.642	2.648	2.684
2.697	2.726	2.770	2.773	2.818	2.821	2.848	3.012	3.084
3.090	3.128	3.233	3.433	3.585				

Progressive censored samples for Y (Length 50 mm)

1.339	1.549	1.574	1.589	1.613	1.746	1.753	1.764	1.807
1.812	1.840	1.852	1.852	1.862	1.864	1.952	1.974	2.051
2.058	2.088	2.125	2.171	2.172	2.211	2.270	2.272	2.299
2.308	2.356	2.390	2.410	2.430	2.431	2.458	2.497	2.558
2.577	2.593	2.601	2.620	2.633	2.670	2.682	2.705	2.735
2.785	3.020	3.042	3.116	3.174				

In this case, ML and approximate Bayes estimates for SSR under progressive type II right Censoring are as follows.

		Censoring scheme			
(n_1, m_1)	(n_2, m_2)	R	S	$\widehat{P}_{MLE}$	$\widehat{P}_{BAYES}$
69,50	65,50	$(19, 49 * 0)$	$(15, 49 * 0)$	0.4158	0.4187

Moreover, bootstrap confidence intervals for SSR are (0.2765,0.4649) and bootstrap estimate of SSR is 0.3706.

7. Corollary

In this study, we have considered the estimation problem of SSR for the EP distribution under PTR-II censored sample.It is observed that the maximum likelihood estimators of SSR can be obtained by using some numerical methods. We have obtained the approximate Bayes estimators of SSR by using the idea of Lindley's approximation under the squared-error loss function. We have compared the MSEs of the approximate Bayes estimators and the MLEs through Monte Carlo simulations, and it has been observed that the performances of the approximate bayes estimates are more satisfactory than ML estimators. When m_1/n_1 and m_2/n_2 rate increased, MSEs of the MLE and approximate Bayes estimates tended to decrease. The best estimates have been obtained when $n_1 = m_1$ and $n_2 = m_2$ (complete sample case).

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