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A new class of unbiased linear estimators in systematic sampling

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Abstract

Use of auxiliary variables is very common in estimating various population parameters. In this study, we suggest a class of unbiased linear estimators for estimating the population mean of the study variate y using information on the auxiliary variate x in systematic sampling. The variance expressions of the suggested estimators are compared with usual unbiased estimator, Swain's (1964) ratio estimator and Shukla's (1971) product type estimator. It is demonstrated that the proposed estimators are more efficient than others.

Keywords: Systematic sampling, Ratio estimator, Product estimator, Unbiased, Efficiency.

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1. Introduction

In sample researches, auxiliary information is commonly used in order to improve efficiency and precision of estimators while calculating sum, mean and variance of population estimations. Auxiliary information is used in ratio, product, regression and spread estimators due to its simplicity and precision. These estimators are preferable regarding correlation between auxiliary variable and study variable, and in some conditions, give results that have smaller variance, which means more precise, compared to estimators based on simple means. Studies on ratio estimator, which is one of the basic estimation methods using auxiliary information, are being carried out further. In literature, new estimators for mean, sum and variance of population have been obtained by improving classical ratio estimation. Use of auxiliary information provided by auxiliary variable was analysed extensively by Cochran [3]. In 1956, Quenouille studied the bias reduction using ratio estimations obtained from two random halves [10]. Ratio estimator for estimation of population mean in systematic sampling was first obtained by Swain [9].

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In case of negative correlation between variable and auxiliary variable, Shukla suggested the product estimator in 1971 [6]. R. Singh and H. P. Singh, by weighting and summing ratio and product estimators, obtained new ratio and product estimators [8]. H. P. Singh et al. (2011) adapted the ratio and product type exponential estimators for systematic sampling [7].

1.1. Systematic Sampling. The method of systematic sampling, first studied by Madow and Madow in 1944 [5], is widely used in surveys of finite populations. When properly applied, the method pocks up any obvious or hidden stratification in the population and thus, can be more precise than random sampling. In addition, systematic sampling can be implemented easily and therefore, it reduces costs. In this variant of random sampling, only the first unit of the sample is randomly selected from the population. The subsequent units are then selected by following some definite rule.
After a simple random sampling is ranked randomly, a random unit is selected from the first k unit to take an n size sample. The sampling method where each following k unit is included in the sampling is called k-th systematic sampling.
The success of systematic sampling depends on the ranking of units. In case of availability of information on the units, ranking should be accordingly. Thus, one unit is selected from each group. In this method, the unit selected the first determines the sampling. This unit is called starting point. The starting point should be selected from the [1,k] interval according to k.
Systematic sampling is a special version of simple random sampling, which facilitates the selection of units to be included in sampling. It is possible to form the sampling more easily, in a shorter time and with less error than SRS. This is an important feature especially in terms of fieldworks. Systematic sampling is more fairly distributed in the population compared to simple random sampling.
yij denotes the j-th unit of i-th systematic sampling (i=1,2,...k and j=1,2,...,n). Sytematic sampling is considered a special case of simple random sampling and population mean can be calculated through simple random sampling.
To obtain the bias and mean square errors (MSE), let us define

$$e_0 = \frac{\overline{y}_{SYS} - \overline{Y}}{\overline{Y}}$$

and

$$e_1 = \frac{\overline{x}_{SYS} - \overline{X}}{\overline{X}}$$

[2] Such that,

$$E(e_0) = E(e_1) = 0$$

$$E\left(e_0^2\right) = \frac{N-1}{N}\frac{C_y^2}{n}\left[1 + (n-1)\rho_{yy}\right]$$

$$E\left(e_1^2\right) = \frac{N-1}{N}\frac{C_x^2}{n}\left[1 + (n-1)\rho_{xx}\right]$$

Where,

$$\rho_{yy} = \frac{2\sum_{i=1}^{k}\sum_{j<u}(y_{ij} - \overline{Y})(y_{iu} - \overline{Y})}{(n-1)(N-1)S_Y^2}$$

and

$$\rho_{xx} = \frac{2\sum_{i=1}^{k}\sum_{j<u}(x_{ij}-\overline{X})(x_{iu}-\overline{X})}{(n-1)(N-1)S_X^2}$$

is intraclass correlation between a pair of units within the systematic sample for the study variate y and auxiliary variate x, respectively;

$$\rho_{xy} = \frac{E\left(x_{ij}-\overline{X}\right)E\left(y_{ij}-\overline{Y}\right)}{\sqrt{E\left(x_{ij}-\overline{X}\right)^2 E\left(y_{ij}-\overline{Y}\right)^2}}$$

is the correlation coefficient between x and y [4];

$$\kappa = \rho_{xy}\left(C_y/C_x\right)$$

$$\rho^* = \frac{\left[1+(n-1)\rho_{yy}\right]^{1/2}}{\left[1+(n-1)\rho_{xx}\right]^{1/2}}$$

and Cx, Cy are the coefficients of variation of the variates x and y respectively. The mean of i-th systematic sampling can be found as,

$$\overline{y}_{SYS} = \frac{\sum_{i=1}^{n} y_i}{n}$$

It's not possible to say that the mean estimators in systematic sampling are always unbiased. When $N{=}nk$ is not true, estimators include a systematic error [1] The variance of the usual population mean estimator $\bar{y}_{SYS}$ is given by,

$$(1.1) \qquad V(\overline{y}_{SYS}) = \frac{(N-1)S_Y^2}{n\,N}\left[1+(n-1)\rho_{YY}\right]$$

Classical ratio and product estimators are respectively defined by Swain (1964) and Shukla (1971) as follows:

$$(1.2) \qquad \overline{y}_{r\,SYS} = \frac{\overline{y}_{SYS}}{\overline{x}_{SYS}}\overline{X}$$

$$(1.3) \qquad \overline{y}_{p_{SYS}} = \frac{\overline{x}_{SYS}}{\overline{y}_{SYS}}\overline{X}$$

The MSEs and biases of $\bar{y}_{r_{SYS}}$ and $\bar{y}_{p_{SYS}}$ to first degree approximation are found by,

$$(1.4) \qquad B(\overline{y}_{r_{SYS}}) = \frac{N-1}{N}\frac{\overline{Y}}{n}C_x^2\left[1+(n-1)\rho_{xx}\right]\left(1-\kappa\rho^*\right)$$

$$(1.5) \qquad B(\overline{y}_{p_{SYS}}) = \frac{N-1}{N}\frac{\overline{Y}}{n}\left[1+(n-1)\rho_{xx}\right]C_x^2\kappa\rho^*$$

$$(1.6) \qquad MSE(\overline{y}_{p_{SYS}}) = \frac{N-1}{Nn}\overline{Y}^2\left[1+(n-1)\rho_{xx}\right]\left[C_y^2\rho^{*2}+C_x^2(1+2\kappa\rho^*)\right]$$

$$(1.7) \qquad MSE(\overline{y}_{r\,SYS}) = \frac{N-1}{Nn}\overline{Y}^2\left[1+(n-1)\rho_{xx}\right]\left[C_y^2\rho^{*2}+C_x^2(1-2\kappa\rho^*)\right]$$

R. Singh and H. P. Singh (1998) suggested ratio and product type estimators for estimating the population mean in systematic sampling as,

$$(1.8) \qquad d_r = \omega_1\overline{y}_{SYS}+\omega_2\overline{y}_{SYS}\frac{\overline{X}}{\overline{x}_{SYS}}+\omega_3\overline{y}_{SS}\left(\frac{\overline{X}}{\overline{x}_{SYS}}\right)^2$$

$$(1.9) \qquad d_p = \omega_1^*\overline{y}_{SYS}+\omega_2^*\overline{y}_{SYS}\frac{\overline{x}_{SYS}}{\overline{X}}+\omega_3^*\overline{y}_{SYS}\left(\frac{\overline{x}_{SYS}}{\overline{X}}\right)^2$$

where,

$$\sum_{i=1}^{3} \omega_i^* = 1$$

and

$$\sum_{i=1}^{3} \omega_i = 1$$

The minimum variances of d_r and d_p to first degree approximation are the same and found by,

$$V(d_r)_{MN} = V(d_p)_{MN} = \frac{N-1}{Nn}\left[1+(n-1)\rho_{yy}\right](1-\rho_{xy}^2)S_y^2 \tag{1.10}$$

H.P. Singh, R.Tailor, N.K. Jatwa (2011) proposed ratio and product type estimators as follows,

$$\overline{y}_{PE} = \overline{y}_{SYS}\exp\left(\frac{\overline{x}_{SYS}-\overline{X}}{\overline{X}+\overline{x}_{SYS}}\right) \tag{1.11}$$

$$\overline{y}_{RE} = \overline{y}_{SYS}\exp\left(\frac{\overline{X}-\overline{x}_{SYS}}{\overline{X}+\overline{x}_{SYS}}\right) \tag{1.12}$$

The biases and MSEs of $\overline{y}_{RE}$ and $\overline{y}_{PE}$ to first degree approximation are found by,

$$E\left(\overline{y}_{RE}-\overline{Y}\right) = \frac{N-1}{Nn}\overline{Y}\frac{C_x^2}{8}\left[1+(n-1)\rho_{xx}\right](3-4\kappa\rho^*) \tag{1.13}$$

$$E\left(\overline{y}_{PE}-\overline{Y}\right) = \frac{N-1}{Nn}\frac{\overline{Y}}{8}C_x^2\left[1+(n-1)\rho_{xx}\right](4\kappa\rho^*-1) \tag{1.14}$$

$$MSE\left(\overline{y}^*{}_{RE}\right) = \frac{N-1}{Nn}\overline{Y}^2\left[1+(n-1)\rho_{xx}\right]\left[C_y^2\rho^{*2}+C_x^2\left(\frac{1}{4}-\kappa\rho^*\right)\right] \tag{1.15}$$

$$MSE\left(\overline{y}^*{}_{PE}\right) = \frac{N-1}{Nn}\overline{Y}^2\left[1+(n-1)\rho_{xx}\right]\left[C_y^2\rho^{*2}+C_x^2\left(\frac{1}{4}+\kappa\rho^*\right)\right] \tag{1.16}$$

2. Suggested estimator in systematic sampling

Supposed $d_1 = \overline{y}_{SS}$, $d_2 = \overline{y}_{SS}\left(\frac{a\overline{x}_{SS}+b}{a\overline{X}+b}\right)^{\alpha}$ and $d_3 = \overline{y}_{SS}\left(\frac{a\overline{x}_{SS}+b}{a\overline{X}+b}\right)^{\beta}$ where $\alpha, \beta \in \mathbb{R}$. We proposed a new class of generalised and unbiased linear estimators based on ratio and product type estimators of R. Singh & H.P. Singh (1998) and also including these estimators is suggested as follows,

$$t_G = \sum_{i=1}^{3}\lambda_i d_i \tag{2.1}$$

where $\sum_{i=1}^{3}\lambda_i = 1$ for $\lambda \in \mathbb{R}$. and λ_i denotes the constant used for reducing the bias.
When this estimator is denominated with e terms for formulating the bias and mean square error,

$$\begin{aligned} t_G = {} & \lambda_1^*\overline{Y}(e_0+1) + \lambda_2^*\overline{Y}(e_0+1)\left[\frac{a\overline{X}(e_1+1)+b}{a\overline{X}+b}\right]^{\alpha} \\ & +\lambda_3^*\overline{Y}(e_0+1)\left[\frac{a\overline{X}(e_1+1)+b}{a\overline{X}+b}\right]^{\beta} \\ = {} & \overline{Y}(e_0+1)\left[\lambda_1+\lambda_2(1+\upsilon e_1)^{\alpha}+\lambda_3(1+\upsilon e_1)^{\beta}\right] \end{aligned}$$

$$t_G \cong \overline{Y}\left\{1 + \upsilon e_1\lambda_G + \upsilon^2 e_1^2\left[\lambda_2\frac{\alpha(\alpha-1)}{2} + \lambda_3\frac{\beta(\beta-1)}{2}\right] + e_0 + \upsilon e_0 e_1\lambda_G\right\}$$

is obtained where $\lambda_G = \lambda_2\alpha + \lambda_3\beta$ and $\upsilon = \frac{a\overline{X}}{a\overline{X}+b}$. Then, to the first degree of approximation, the variance of t_G is given by,

$$MSE\left(t_G\right) = \frac{N-1}{N}\frac{\overline{Y}^2}{n}\left[1 + (n-1)\,\rho_{xx}\right]\left[C_y^2\rho^{*2} + \lambda_G\upsilon C_x^2\left(\lambda_G\upsilon + 2\kappa\rho^*\right)\right] \tag{2.2}$$

which is minimized for

$$[\lambda_{G_{MN}} = -\frac{\kappa\rho^*}{\upsilon} = -\frac{a\overline{X}+b}{a\overline{X}}\kappa\rho^* \tag{2.3}$$

$$MSE(t_G)_{MN} = \frac{N-1}{N}\frac{\overline{Y}^2}{n}\left[1 + (n-1)\,\rho_{xx}\right]\left[C_y^2\rho^{*2}\left(1-\rho_{xy}^2\right)\right] \tag{2.4}$$

From (2.1), (2.3), we have

$$\sum_{i=1}^{3}\lambda_i = 1 \tag{2.5}$$

and

$$\lambda_G = \lambda_2\alpha + \lambda_3\beta = -\frac{\kappa\rho^*}{\upsilon} \tag{2.6}$$

There are three unknown quantities to be determined (λ_1, λ_2 and λ_3) from only two equations. It is not possible to obtain unique values for the λ_i's. For obtaining λ_i's and making the estimator unbiased, we can write,

$$\sum_{i=1}^{3}B(d_i) = 0 \tag{2.7}$$

Where B(d_i) shows for the biases. Then we can write a matrix for solving Equation (2.5), (2.6) and (2.7) together,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & \alpha & \beta \\ B(d_1) & B(d_2) & B(d_3) \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{\kappa\rho^*}{\upsilon} \\ 0 \end{bmatrix} \tag{2.8}$$

where,

$$B(d_1) = B(\overline{y}_{SYS}) = 0$$

$$B(d_2) = B\left[\overline{y}_{SYS}\left(\frac{a\overline{x}_{SYS}+b}{a\overline{X}+b}\right)^{\alpha}\right]$$

$$= \frac{N-1}{Nn}\overline{Y}C_x^2\left[1 + (n-1)\rho_{xx}\right]\upsilon\alpha\left[\upsilon\frac{(\alpha-1)}{2} + \rho^*\kappa\right]$$

$$B(d_3) = B\left[\overline{y}_{SYS}\left(\frac{a\overline{x}_{SYS}+b}{a\overline{X}+b}\right)^{\beta}\right]$$

$$= \frac{N-1}{Nn}\overline{Y}C_x^2\left[1 + (n-1)\rho_{xx}\right]\upsilon\beta\left[\upsilon\frac{(\beta-1)}{2} + \rho^*\kappa\right]$$

We get values of λ_1, λ_2 and λ_3 from solving matrix on (2.8) as follows,

Table 1. Some members of the class of estimators t_G

Estimators (λ_1, λ_2,λ_3 **are minimum values)**	α	β	**a**	**b**
$t_G = \lambda_1 \overline{y}_{SYS} + \lambda_2 \overline{y}_{SYS}\left(\frac{a\overline{x}_{SYS}+b}{a\overline{X}+b}\right)^{\alpha} + \lambda_3 \overline{y}_{SYS}\left(\frac{a\overline{x}_{SYS}+b}{a\overline{X}+b}\right)^{\beta}$	α	β	a	b
$t_{G_1} = \lambda_1 \overline{y}_{SYS} + (1-\lambda_1)\,\overline{y}_{SYS}\left(\frac{a\overline{x}_{SYS}+b}{a\overline{X}+b}\right)^{\alpha}$	α=β		a	b
$t_{G_2} = \overline{y}_{SYS}$	0	0	a	b
	α	β	0	1
$t_{G_3} = \lambda_1 \overline{y}_{SYS} + \lambda_2 \overline{y}_{SYS}\left(\frac{a\overline{x}_{SYS}+b}{a\overline{X}+b}\right) + \lambda_3 \overline{y}_{SYS}\left(\frac{a\overline{x}_{SYS}+b}{a\overline{X}+b}\right)^{2}$	1	2	a	b
$t_{G_4} = \lambda_1 \overline{y}_{SYS} + \lambda_2 \overline{y}_{SYS}\left(\frac{\overline{x}_{SYS}}{\overline{X}}\right)^{\alpha} + \lambda_3 \overline{y}_{SYS}\left(\frac{\overline{x}_{SYS}}{\overline{X}}\right)^{\beta}$	>0	>0	1	0
	α	β	C_x, ρ_{xy}, $\beta_2(x)$	0
$t_{G_5} = \lambda_1 \overline{y}_{SYS} + \lambda_2 \overline{y}_{SYS}\left(\frac{\overline{X}}{\overline{x}_{SYS}}\right)^{\alpha} + \lambda_3 \overline{y}_{SYS}\left(\frac{\overline{X}}{\overline{x}_{SYS}}\right)^{\beta}$	<0	<0	1	0
$t_{G_6} = \lambda_1 \overline{y}_{SYS} + \lambda_2 \overline{y}_{SYS}\left(\frac{\overline{X}}{\overline{x}_{SYS}}\right)^{\alpha} + \lambda_3 \overline{y}_{SYS}\left(\frac{\overline{x}_{SYS}}{\overline{X}}\right)^{\beta}$	<0	>0	1	0
$t_{G_7} = \lambda_1 \overline{y}_{SYS} + \lambda_2 \overline{y}_{SYS}\left(\frac{\overline{x}_{SYS}+C_x}{\overline{X}+C_x}\right)^{\alpha} + \lambda_3 \overline{y}_{SYS}\left(\frac{\overline{x}_{SYS}+C_x}{\overline{X}+C_x}\right)^{\beta}$	>0	>0	1	C_x
$t_{G_8} = \lambda_1 \overline{y}_{SYS} + \lambda_2 \overline{y}_{SYS}\left(\frac{\overline{x}_{SYS}+\rho_{xy}}{\overline{X}+\rho_{xy}}\right)^{\alpha} + \lambda_3 \overline{y}_{SYS}\left(\frac{\overline{x}_{SYS}+\rho_{xy}}{\overline{X}+\rho_{xy}}\right)^{\beta}$	>0	>0	1	ρ_{xy}
$t_{G_9} = \lambda_1 \overline{y}_{SYS} + \lambda_2 \overline{y}_{SYS}\left(\frac{\overline{x}_{SYS}+\beta_2(x)}{\overline{X}+\beta_2(x)}\right)^{\alpha} + \lambda_3 \overline{y}_{SYS}\left(\frac{\overline{x}_{SYS}+\beta_2(x)}{\overline{X}+\beta_2(x)}\right)^{\beta}$	>0	>0	1	$\beta_2(x)$
$t_{G_{10}} = \lambda_1 \overline{y}_{SYS} + \lambda_2 \overline{y}_{SYS}\left(\frac{\overline{X}+C_x}{\overline{x}_{SYS}+C_x}\right)^{\alpha} + \lambda_3 \overline{y}_{SYS}\left(\frac{\overline{X}+C_x}{\overline{x}_{SYS}+C_x}\right)^{\beta}$	<0	<0	1	C_x
$t_{G_{11}} = \lambda_1 \overline{y}_{SYS} + \lambda_2 \overline{y}_{SYS}\left(\frac{\overline{X}+\rho_{xy}}{\overline{x}_{SYS}+\rho_{xy}}\right)^{\alpha} + \lambda_3 \overline{y}_{SYS}\left(\frac{\overline{X}+\rho_{xy}}{\overline{x}_{SYS}+\rho_{xy}}\right)^{\beta}$	<0	<0	1	ρ_{xy}
$t_{G_{12}} = \lambda_1 \overline{y}_{SYS} + \lambda_2 \overline{y}_{SYS}\left(\frac{\overline{X}+\beta_2(x)}{\overline{x}_{SYS}+\beta_2(x)}\right)^{\alpha} + \lambda_3 \overline{y}_{SYS}\left(\frac{\overline{X}+\beta_2(x)}{\overline{x}_{SYS}+\beta_2(x)}\right)^{\beta}$	<0	<0	1	$\beta_2(x)$

$$
\begin{aligned}
(2.9)\quad \lambda_1 &= 1 - \frac{\kappa\rho^*}{\upsilon^2}\frac{1}{\alpha\beta(\alpha-\beta)}\left\{\beta\left[\upsilon(\beta-1)+2\rho^*\kappa\right] - \alpha\left[\upsilon(\alpha-1)+2\rho^*\kappa\right]\right\} \\
\lambda_2 &= \frac{\kappa\rho^*}{\upsilon^2}\left[\upsilon(\beta-1)+2\rho^*\kappa\right] \\
\lambda_3 &= \frac{-\kappa^2\rho^{*2}}{\upsilon^2}\frac{\left[\upsilon(\alpha-1)+2\rho^*\kappa\right]}{\beta(\alpha-\beta)}
\end{aligned}
$$

Equations of (2.9) yield in Equation (2.1) and (2.2), we have class of unbiased linear estimator for $\overline{Y}$ as,

$$
\begin{aligned}
t_G &= 1 - \frac{\kappa\rho^*}{\upsilon^2}\frac{1}{\alpha\beta(\alpha-\beta)}\left\{\beta\left[\upsilon(\beta-1)+2\rho^*\kappa\right] - \alpha\left[\upsilon(\alpha-1)+2\rho^*\kappa\right]\right\}\overline{y}_{SYS} - \\
&\quad - \frac{\kappa\rho^*}{\upsilon^2}\left[\upsilon(\beta-1)+2\rho^*\kappa\right]\overline{y}_{SYS}\left(\frac{a\overline{x}_{SYS}+b}{a\overline{X}+b}\right)^{\alpha} - \\
&\quad - \frac{\kappa^2\rho^{*2}}{\upsilon^2}\frac{\left[\upsilon(\alpha-1)+2\rho^*\kappa\right]}{\beta(\alpha-\beta)}\overline{y}_{SYS}\left(\frac{a\overline{x}_{SYS}+b}{a\overline{X}+b}\right)^{\beta} \qquad (2.10)
\end{aligned}
$$

With the variance,

$$
(2.11)\quad MSE(t_G)_{MN} = \frac{N-1}{N}\frac{\overline{Y}^2}{n}\left[1+(n-1)\rho_{xx}\right]\left[C_y^2\rho^{*2}\left(1-\rho_{xy}^2\right)\right]
$$

Some unbiased members of the class of estimators t_G are shown in Table 1.

3. Theorical comparison

Minimum variance of proposed estimator is always smaller than the variance of Swain's ratio type, Shukla's product type estimators and H. P. Singh and etc. (2011) ratio and

product type estimators except special situations as follows:

The inequality,

$$\begin{aligned}
V(t_G)_{MIN} &< V(\overline{y}_{SYS}) \\
\frac{N-1}{Nn}\left[1+(n-1)\rho_{xx}\right]\left[\rho^{*2}S_y^2(1-\rho_{xy}^2)\right] &< \frac{N-1}{Nn}\left[1+(n-1)\rho_{yy}\right]S_Y^2 \\
\frac{N-1}{Nn}\left[1+(n-1)\rho_{yy}\right]S_y^2 - \frac{N-1}{Nn}\left[1+(n-1)\rho_{yy}\right]S_y^2\rho_{xy}^2 &< \frac{N-1}{Nn}\left[1+(n-1)\rho_{yy}\right]S_Y^2 \\
\left[1+(n-1)\rho_{yy}\right] &> 0
\end{aligned}$$

is always true when $\rho_{yy} > -1/(n-1)$ condition is true.

The inequality,

$$\begin{aligned}
V(t_G)_{MIN} &< MSE(\overline{y}_{r\,SYS}) \\
\frac{N-1}{Nn}\left[1+(n-1)\rho_{xx}\right]\overline{Y}^2\left[\rho^{*2}C_y^2(1-\rho_{xy}^2)\right] &< \frac{N-1}{Nn}\left[1+(n-1)\rho_{xx}\right] \\
&\quad \overline{Y}^2\left[\rho^{*2}C_y^2 + C_x^2(1-2\kappa\rho^*)\right] \\
0 &< C_x^2 - 2C_x^2\kappa\rho^* + \rho^{*2}C_y^2\rho_{xy}^2 \\
0 &< \left(\kappa\rho^* - 1\right)^2
\end{aligned}$$

is always true except when $\kappa\rho^* > 1$.

The inequality,

$$\begin{aligned}
V(t_G)_{MIN} &< HKO(\overline{y}_{p\,SYS}) \\
\frac{N-1}{Nn}\left[1+(n-1)\rho_{xx}\right]\overline{Y}^2\left[\rho^{*2}C_y^2(1-\rho_{xy}^2)\right] &< \frac{N-1}{Nn}\left[1+(n-1)\rho_{xx}\right] \\
&\quad \overline{Y}^2\left[\rho^{*2}C_y^2 + C_x^2(1+2\kappa\rho^*)\right] \\
0 &< C_x^2 + 2C_x^2\kappa\rho^* + \rho^{*2}C_y^2\rho_{xy}^2 \\
0 &< \left(\kappa\rho^* + 1\right)^2
\end{aligned}$$

is always true except when $\kappa\rho^* > -1$

The inequality,

$$\begin{aligned}
V(t_G)_{MIN} &< HKO(\overline{y}_{RE}) \\
\frac{N-1}{Nn}\left[1+(n-1)\rho_{xx}\right]\overline{Y}^2\left[\rho^{*2}C_y^2(1-\rho_{xy}^2)\right] &< \frac{N-1}{Nn}\left[1+(n-1)\rho_{xx}\right] \\
&\quad \overline{Y}^2\left[\rho^{*2}C_y^2 + C_x^2(\frac{1}{4} - \kappa\rho^*)\right] \\
-\rho^{*2}C_y^2\rho_{xy}^2 &< C_x^2(\frac{1}{4} - \kappa\rho^*) \\
0 &< C_x^2\left(\frac{1}{4} - \rho_{xy}\frac{C_y}{C_x}\rho^* + \rho^{*2}\frac{C_y^2}{C_x^2}\rho_{xy}^2\right) \\
\frac{1}{4} &> \kappa\rho^*\left(1-\kappa\rho^*\right)
\end{aligned}$$

is always true when the condition is true.

The inequality,

$$\begin{aligned} V(t_G)_{MIN} &< HKO(\overline{y}_{PE}) \\ \frac{N-1}{Nn}\left[1+(n-1)\rho_{xx}\right]\overline{Y}^2\left[\rho^{*2}C_y^2(1-\rho_{xy}^2)\right] &< \frac{N-1}{Nn}\left[1+(n-1)\rho_{xx}\right] \\ &\quad \overline{Y}^2\left[\rho^{*2}C_y^2+C_x^2(\frac{1}{4}+\kappa\rho^*)\right] \\ -\rho^{*2}C_y^2\rho_{xy}^2 &< C_x^2(\frac{1}{4}+\kappa\rho^*) \\ 0 &< C_x^2\left(\frac{1}{4}+\rho_{xy}\frac{C_y}{C_x}\rho^*+\rho^{*2}\frac{C_y^2}{C_x^2}\rho_{xy}^2\right) \\ -\frac{1}{4} &< \kappa\rho^*\left(1+\kappa\rho^*\right) \end{aligned}$$

is always true when the condition is true.

4. Numerical Study

This chapter, in order to be able to make digital impressions at 2013, the amount of particulate matter and Nitric oxide measurements which recorded at air quality monitoring station in Istanbul's Kadikoy district , were used. Airborne particulate matter like emoticon study variable, is the amount of nitrogen oxide (x) auxiliary variables has been accepted. The correlation between the amount of airborne particulate matter and amount of Nitric oxide was found to be $\rho_{xy} = 0.833$ as well as seen in Table 2. Due to the positive high relationship between these two variables in the estimates used to estimation of population mean values of ratio estimation, as well, and MSEs was calculated.

Table 2. Population parameters

	N	**Mean**	**Std. Deviation**	**Correlation**
PM10 (Y)	351	51,57	32,71	0,833
NO (X)	351	36,11	52,74	

We ranked the population according to the dates and drew a sample with n=117 and k=3 from the population with systematic sampling. Table 3 shows the information of sample. Table 4 gives solution of the calculation of estimates, biases and MSEs.

Table 3. Summaring of the sample data

	n	**Mean**	**Std. Deviation**
PM10 (Y)	351	51,57	32,71
NO (X)	351	36,11	52,74

5. Conclusion

Table 4 shows that the performance of the proposed a class of estimator t_G is better than usual anbiased estimator , Swain's (1964) ratio estimator and H.P. Singh and etc. (2011) ratio estimator . d_r and t_G estimators have minimum and the same variance value. Suggested estimator is independent from alpha and beta's negativity so estimates are not depend on type of estimator. Thus we don't need to check positive or negative correlation between study and auxiliary variable.

Table 4. Estimates, biases ans MSEs values

Estimators	**Estimate**	**Bias**	**MSE**
$\overline{y}_{SYS}$	52,9316	-	8,1453
$\overline{y}_{r_{SYS}} = \frac{\overline{y}^*_{SYS}}{\overline{x}^*_{SYS}}\overline{X}$	50,2763	0,1344	2,7909
$d_r = \omega_1\overline{y}_{SYS} + \omega_2\overline{y}_{SYS}\frac{\overline{X}}{\overline{x}_{SYS}} + \omega_3\overline{y}_{SYS}\left(\frac{\overline{X}}{\overline{x}_{SYS}}\right)^2$	50,7472	-	2,4931
$\overline{y}_{RE} = \overline{y}_{SYS}\exp\left(\frac{\overline{X}-\overline{x}_{SYS}}{\overline{X}+\overline{x}_{SYS}}\right)$	51,5872	-0,0054	3,3389
$t_G = \lambda_1\overline{y}_{SYS} + \lambda_2\overline{y}_{SYS}\left(\frac{a\overline{x}_{SYS}+b}{a\overline{X}+b}\right)^{\alpha} + \lambda_3\overline{y}_{SYS}\left(\frac{a\overline{x}_{SYS}+b}{a\overline{X}+b}\right)^{\beta}$	50,7514	-	2,4931

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