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Nonexistence of global solutions to system of semi-linear fractional evolution equations

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Abstract

In this research we are interested to Cauchy problem for system of semi-linear fractional evolution equations.
Some authors were concerned with studying of global existence of solutions for the hyperbolic nonlinear equations with a damping term. Our goal is to extend some results obtained by the authors, by studying the system of semi-linear hyperbolic equations with fractional damping term and fractional Laplacian .
Thanks to the test functions method, we prove the nonexistence of nontrivial global weak solutions to the problem.

1. Introduction

in this paper we are concerned with the following Cauchy problem:

$$\begin{cases} u_{tt} + (-\Delta)^{\frac{\beta_1}{2}} u + D^{\alpha_1}_{0|t} u = f(t,x)\,|u|^{p_1}\,|v|^{q_1}, & (t,x) \in (0,+\infty)\times\mathbb{R}^N \\ v_{tt} + (-\Delta)^{\frac{\beta_2}{2}} v + D^{\alpha_2}_{0|t} v = g(t,x)\,|u|^{p_2}\,|v|^{q_2}, & (t,x) \in (0,+\infty)\times\mathbb{R}^N \end{cases} \tag{1.1}$$

subjected to the conditions

$$u(0,x) = u_0(x) \geq 0,, \quad u_t(0,x) = u_1(x) \geq 0,$$

$$v(0,x) = v_0(x) \geq 0,, \quad v_t(0,x) = v_1(x) \geq 0,.$$

where $p_1 \geq 0, q_2 \geq 0, p_2 > 1, q_1 > 1, 0 < \alpha_i < 1 \leq \beta_i \leq 2, i = 1,2$ are constants. $D^{\alpha_i}_{0/t}$ denotes the derivatives of order α_i in the sense of Caputo and $(-\Delta)^{\frac{\beta_i}{2}}$ is the fractional power of the $(-\Delta)$.
The integral representation of the fractional Laplacian in the N-dimensional space is

$$(-\Delta)^{\beta/2}\psi(x) = -c_N(\beta)\int_{\mathbb{R}^N} \frac{\psi(x+z)-\psi(x)}{|z|^{N+\beta}}dz, \quad \forall x \in \mathbb{R}^N, \tag{1.2}$$

where $c_N(\beta) = \Gamma((N+\beta)/2)/(2\pi^{N/2+\beta}\Gamma(1-\beta/2))$, and Γ denotes the gamma function (see [16]).
Note that The fractional Laplacian $\left((-\Delta)^{\beta/2}\right)$ with $1 \leq \beta \leq 2$ is a pseudo-differential operator defined by:

$$(-\Delta)^{\beta/2}u(x) = \mathscr{F}^{-1}\{|\zeta|^{\beta}\mathscr{F}(u)(\zeta)\}(x) \ \ \forall x \in \mathbb{R}^N,$$

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where $\mathscr{F}$ and $\mathscr{F}^{-1}$ are Fourier transform and inverse Fourier transform, respectively.
The functions f and g are non-negatives and assumed to satisfy the conditions

$$f(t,x) \geq C_1 t^{\nu_1} |x|^{\mu_1}, g(t,x) \geq C_2 t^{\nu_2} |x|^{\mu_2}, \text{ where } \nu_i \geq 0,\ \mu_i \geq 0,\ i=1,2. \tag{1.3}$$

The problem of global existence of solutions for nonlinear hyperbolic equations with a damping term have been studied by many researchers in several contexts (see [4], [8], [9] , [12], [18], [20]), for example, the following Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u + u_t = |u|^p, & (t,x) \in (0,\infty) \times \mathbb{R}^N \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

Todorova-Yordanov [18] showed that, if $p_c < p \leq \frac{n}{n-1}$, then (1.4) admits a unique global solution, and they proved that if $1 < p < 1 + \frac{2}{n}$, then the solution u blows up in a finite time.
Fino-Ibrahim and Wehbe [4] generalized the results of Ogawa-Takeda [12] by proving the blow-up of solutions of (1.4) under weaker assumptions on the initial data and they extended this results to the critical case $p_c = 1 + \frac{2}{n}$.
Qi. Zhang [20] studied the case $1 < p < 1 + \frac{2}{n}$, when $\int u_i(x)dx > 0, i = 0,1$, he proved that global solution of (1.4) does not exist. Therefore, he showed that $p = 1 + \frac{2}{n}$ belongs to the blow-up case.
A. Hakem [8] treated the same type of (1.4), then he extended this result to the case of a system :

$$\begin{cases} u_{tt} + -\Delta u + g(t)u_t = |v|^p, & (t,x) \in (0,+\infty) \times \mathbb{R}^N \\ v_{tt} + -\Delta v + f(t)v_t = |u|^q, & (t,x) \in (0,+\infty) \times \mathbb{R}^N \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x) & \\ v(0,x) = v_0(x), \ v_t(0,x) = v_1(x), & \end{cases} \tag{1.5}$$

$g(t)$ and $f(t)$ are functions behaving like t^β and t^α, respectively, where $0 \leq \beta, \alpha < 1$.
Hakem [8] showed that, if

$$\frac{N}{2} \leq \frac{1}{pq-1} \max\left[1-\beta+p(1-\alpha), 1-\alpha+q(1-\beta)\right] - \max\left(\alpha,\beta\right),$$

then the problem (1.5) has only the trivial solution.
By combining the works of the above authors with those of Kirane *et al.*[10] and Escobido *et al.*[2], we were able to prove a nonexistence result to (1.1) in the weak formulation.

2. Preliminaries

Let us start by introducing the definitions concerning fractional derivatives in the sense of Caputo and the weak local solution to problem (1.1).

Definition 2.1. *Let $0 < \alpha < 1$ and $\zeta' \in L^1(0,T)$. The left-sided and respectively right-sided Caputo derivatives of order α for ζ are defined as:*

$$D^\alpha_{0|t}\zeta(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\zeta'(s)}{(t-s)^\alpha} ds,$$

and

$$D^\alpha_{t|T}\zeta(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^T \frac{\zeta'(s)}{(s-t)^\alpha} ds,$$

where Γ denotes the gamma function (see [13] p 79).

Definition 2.2. *Let $Q_T = (0,T) \times \mathbb{R}^N$, $0 < T < +\infty$.
We say that $(u,v) \in \left(L^1_{loc}(Q_T)\right)^2$ is a local weak solution to problem (1.1) on Q_T,
if $(fu^{p_1}v^{q_1}, gu^{p_2}v^{q_2}) \in \left(L^1_{loc}(Q_T)\right)^2$, and it satisfies*

$$\begin{aligned} \int_{Q_T} f|u|^{p_1}|v|^{q_1}\zeta_1\,dx\,dt + \int_{\mathbb{R}^N} u_0(x)\zeta_1(0,x)\,dx + \int_{\mathbb{R}^N} u_1(x)\zeta_1(0,x)\,dx - \int_{\mathbb{R}^N} u_0(x)\zeta_{1_t}(0,x)\,dx \\ = \int_{Q_T} u\zeta_{1_{tt}}\,dx\,dt + \int_{Q_T} uD^{\alpha_1}_{t|T}\zeta_1\,dx\,dt + \int_{Q_T} u(-\Delta)^{\frac{\beta_1}{2}}\zeta_1\,dx\,dt. \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \int_{Q_T} g|u|^{p_2}|v|^{q_2}\zeta_2\,dx\,dt + \int_{\mathbb{R}^N} v_0(x)\zeta_2(0,x)\,dx + \int_{\mathbb{R}^N} v_1(x)\zeta_2(0,x)\,dx - \int_{\mathbb{R}^N} v_0(x)\zeta_{2_t}(0,x)\,dx \\ = \int_{Q_T} v\zeta_{2_{tt}}\,dx\,dt + \int_{Q_T} vD^{\alpha_2}_{t|T}\zeta_2\,dx\,dt + \int_{Q_T} v(-\Delta)^{\frac{\beta_2}{2}}\zeta_2\,dx\,dt. \end{aligned} \tag{2.2}$$

for all test function $\zeta_j \in C^{2,2}_{t,x}(Q_T)$ such as $\zeta_j \geq 0$ and $\zeta_j(T,x) = \zeta_{j_t}(T,x) = \zeta_{j_t}(0,x) = 0,\ j = 1,2$

(see [3] p 5501).

Remark 2.3. *To get the definition 2.2, we multiplying the first equation in (1.1) by ζ_1 and the second equation by ζ_2, integrating by parts on $Q_T = (0,T)\times\mathbb{R}^N$ and using the definition 2.1*

The integrals in the above definition are supposed to be convergent.
If in the definition $T = +\infty$, the solution (u,v) is called global.
Now, we recall the following integration by parts formula:

$$\int_0^T \phi(t)(D^\alpha_{0|t}\psi)(t)dt = \int_0^T (D^\alpha_{t|T}\phi)(t)\psi(t)dt,$$

(see [17], p 46).

3. Main results

We now in position to announce our result.

Theorem 3.1. *Let $p_2 > 1, q_1 > 1, 0 < \alpha_i < 1 \le \beta_i \le 2, i = 1,2$, and*

$$\mathscr{A} := \frac{\alpha_1 + \dfrac{\alpha_2}{p_2} - \left(1 - \dfrac{1}{p_2q_1}\right) - \dfrac{1}{p_2}\left(\mu_2\dfrac{\alpha_1}{\beta_1} + \nu_2\right) - \dfrac{1}{p_2q_1}\left(\mu_1\dfrac{\alpha_2}{\beta_2} + \nu_1\right)}{\dfrac{\alpha_1}{\beta_1\tilde{p}_2} + \dfrac{\alpha_2}{\beta_2p_2\tilde{q}_1}}$$

and

$$\mathscr{B} := \frac{\alpha_2 + \dfrac{\alpha_1}{q_1} - \left(1 - \dfrac{1}{p_2q_1}\right) - \dfrac{1}{q_1}\left(\mu_1\dfrac{\alpha_2}{\beta_2} + \nu_1\right) - \dfrac{1}{p_2q_1}\left(\mu_2\dfrac{\alpha_1}{\beta_1} + \nu_2\right)}{\dfrac{\alpha_2}{\beta_2\tilde{q}_1} + \dfrac{\alpha_1}{\beta_1q_1\tilde{p}_2}}$$

where $p_2\tilde{p}_2 = p_2 + \tilde{p}_2$, $q_1\tilde{q}_1 = q_1 + \tilde{q}_1$,
and the conditions (1.3) are fulfilled.
If

$$N \le \max\{\mathscr{A};\mathscr{B}\},$$

then the problem (1.1) admits no nontrivial global weak solutions.

Proof. We notice that, in all steps of proof , $C > 0$ is a real positive number which may change from line to line.
Set $\zeta_j(t,x) = \Phi\left(\dfrac{t^2+|x|^{2\theta_j}}{R^2}\right)$, $j = 1,2$ such as Φ is a decreasing function $C^2_0(\mathbb{R}^+)$, satisfies

$$0 \le \Phi \le 1 \text{ and } \Phi(r) = \begin{cases} 1 & \text{if } 0 \le r \le 1, \\ 0 & \text{if } r \ge 2. \end{cases}$$

Where $R > 0$, $\theta_1 = \beta_1/\alpha_1$ and $\theta_2 = \beta_2/\alpha_2$ (see [10]).
Multiplying the first equation of (1.1) by ζ_1 and integrating by parts on
$Q_T = (0,T)\times\mathbb{R}^N$, we get

$$\begin{aligned}\int_{Q_T} f|u|^{p_1}|v|^{q_1}\zeta_1\,dxdt + \int_{\mathbb{R}^N} u_0(x)\zeta_1(0,x)\,dx + \int_{\mathbb{R}^N} u_1(x)\zeta_1(0,x)\,dx - \int_{\mathbb{R}^N} u_0(x)\zeta_{1_t}(0,x)\,dx \\ = \int_{Q_T} u\zeta_{1_{tt}}\,dxdt - \int_{Q_T} uD^{\alpha_1}_{0|t}\zeta_1\,dxdt + \int_{Q_T} u(-\Delta)^{\frac{\beta_1}{2}}\zeta_1\,dxdt.\end{aligned} \tag{3.1}$$

It is clear that $\zeta_{j_t}(t,x) = 2R^{-2}t\Phi'\left(\dfrac{t^2+|x|^{2\theta_j}}{R^2}\right)$, consequently $\zeta_{j_t}(0,x) = 0$, thus

$$\begin{aligned}\int_{Q_T} f|u|^{p_1}|v|^{q_1}\zeta_1\,dxdt + \int_{\mathbb{R}^N} u_0(x)\zeta_1(0,x)\,dx + \int_{\mathbb{R}^N} u_1(x)\zeta_1(0,x)\,dx \\ = \int_{Q_T} u\zeta_{1_{tt}}\,dxdt + \int_{Q_T} uD^{\alpha_1}_{t|T}\zeta_1\,dxdt + \int_{Q_T} u(-\Delta)^{\frac{\beta_1}{2}}\zeta_1\,dxdt.\end{aligned} \tag{3.2}$$

Hence,

$$\int_{Q_T} f|u|^{p_1}|v|^{q_1}\zeta_1\,dxdt \le \int_{Q_T} |u|\,|\zeta_{1_{tt}}|\,dxdt + \int_{Q_T} |u|\left|D^{\alpha_1}_{t|T}\zeta_1\right|dxdt + \int_{Q_T} |u|\left|(-\Delta)^{\frac{\beta_1}{2}}\zeta_1\right|dxdt. \tag{3.3}$$

We have also

$$\int_{Q_T} g|u|^{p_2}|v|^{q_2}\zeta_2\,dxdt \le \int_{Q_T} |v|\,|\zeta_{2_{tt}}|\,dxdt + \int_{Q_T} |v|\left|D^{\alpha_2}_{t|T}\zeta_2\right|dxdt + \int_{Q_T} |v|\left|(-\Delta)^{\frac{\beta_2}{2}}\zeta_2\right|dxdt. \tag{3.4}$$

To estimate

$$\int_{Q_T} |u|\,|\zeta_{1tt}|\,dx\,dt,$$

we observe that it can be rewritten as

$$\int_{Q_T} |u|\,|\zeta_{1tt}|\,dx\,dt = \int_{Q_T} |u|\,(g\,|v|^{q_2}\,\zeta_2)^{\frac{1}{p_2}}\,|\zeta_{1tt}|\,(g\,|v|^{q_2}\,\zeta_2)^{\frac{-1}{p_2}}\,dx\,dt.$$

Using Hölder's inequality, we obtain

$$\int_{Q_T} |u|\,|\zeta_{1tt}|\,dx\,dt \leq \left(\int_{Q_T} |u|^{p_2}\,(g\,|v|^{q_2}\,\zeta_2)\,dx\,dt\right)^{\frac{1}{p_2}} \left(\int_{Q_T} |\zeta_{1tt}|^{\frac{p_2}{p_2-1}}\,(g\,|v|^{q_2}\,\zeta_2)^{\frac{-1}{p_2-1}}\,dx\,dt\right)^{\frac{p_2-1}{p_2}}.$$

Proceeding as above, we have

$$\begin{aligned}\int_{Q_T} |u|\left|D^{\alpha_1}_{t|T}\zeta_1\right|dx\,dt \leq &\left(\int_{Q_T} |u|^{p2}\,(g\,|v|^{q_2}\,\zeta_2)\,dx\,dt\right)^{\frac{1}{p_2}} \\ &\times\left(\int_{Q_T} \left|D^{\alpha_1}_{t|T}\zeta_1\right|^{\frac{p_2}{p_2-1}}\,(g\,|v|^{q_2}\,\zeta_2)^{\frac{-1}{p_2-1}}\,dx\,dt\right)^{\frac{p_2-1}{p_2}},\end{aligned}$$

and

$$\begin{aligned}\int_{Q_T} |u|\left|(-\Delta)^{\frac{\beta_1}{2}}\zeta_1\right|dx\,dt \leq &\left(\int_{Q_T} |u|^{p_2}\,(g\,|v|^{q_2}\,\zeta_2)\,dx\,dt\right)^{\frac{1}{p_2}} \\ &\times\left(\int_{Q_T} \left|(-\Delta)^{\frac{\beta_1}{2}}\zeta_1\right|^{\frac{p_2}{p_2-1}}\,(g\,|v|^{q_2}\,\zeta_2)^{\frac{-1}{p_2-1}}\,dx\,dt\right)^{\frac{p_2-1}{p_2}}.\end{aligned}$$

Finally, we infer

$$\int_{Q_T} f\,|u|^{p_1}\,|v|^{q_1}\,\zeta_1\,dx\,dt \leq \left(\int_{Q_T} |u|^{p_2}\,(g\,|v|^{q_2}\,\zeta_2)\,dx\,dt\right)^{\frac{1}{p_2}}\mathscr{K}_1, \tag{3.5}$$

where

$$\begin{aligned}\mathscr{K}_1 = &\left(\int_{Q_T} |\zeta_{1tt}|^{\frac{p_2}{p_2-1}}\,(g\,|v|^{q_2}\,\zeta_2)^{\frac{-1}{p_2-1}}\,dx\,dt\right)^{\frac{p_2-1}{p_2}} + \left(\int_{Q_T} \left|D^{\alpha_1}_{t|T}\zeta_1\right|^{\frac{p_2}{p_2-1}}\,(g\,|v|^{q_2}\,\zeta_2)^{\frac{-1}{p_2-1}}\,dx\,dt\right)^{\frac{p_2-1}{p_2}} \\ &+\left(\int_{Q_T} \left|(-\Delta)^{\frac{\beta_1}{2}}\zeta_1\right|^{\frac{p_2}{p_2-1}}\,(g\,|v|^{q_2}\,\zeta_2)^{\frac{-1}{p_2-1}}\,dx\,dt\right)^{\frac{p_2-1}{p_2}}.\end{aligned}$$

Arguing as above we have likewise

$$\int_{Q_T} g\,|u|^{p_2}\,|v|^{q_2}\,\zeta_2\,dx\,dt \leq \left(\int_{Q_T} |v|^{q_1}\,(f\,|u|^{p_1}\,\zeta_1)\,dx\,dt\right)^{\frac{1}{q_1}}\mathscr{K}_2, \tag{3.6}$$

where

$$\begin{aligned}\mathscr{K}_2 = &\left(\int_{Q_T} |\zeta_{2tt}|^{\frac{q_1}{q_1-1}}\,(f\,|u|^{p_1}\,\zeta_1)^{\frac{-1}{q_1-1}}\,dx\,dt\right)^{\frac{q_1-1}{q_1}} + \left(\int_{Q_T} \left|D^{\alpha_2}_{t|T}\zeta_2\right|^{\frac{q_1}{q_1-1}}\,(f\,|u|^{p_1}\,\zeta_1)^{\frac{-1}{q_1-1}}\,dx\,dt\right)^{\frac{q_1-1}{q_1}} \\ &+\left(\int_{Q_T} \left|(-\Delta)^{\frac{\beta_2}{2}}\zeta_2\right|^{\frac{q_1}{q_1-1}}\,(f\,|u|^{p_1}\,\zeta_1)^{\frac{-1}{q_1-1}}\,dx\,dt\right)^{\frac{q_1-1}{q_1}}.\end{aligned}$$

Using inequalities (3.5) and (3.6), it yield

$$\left(\int_{Q_T} f\,|u|^{p_1}\,|v|^{q_1}\,\zeta_1\,dx\,dt\right)^{\frac{q_1p_2-1}{q_1p_2}} \leq \mathscr{K}_1\mathscr{K}_2^{\frac{1}{p_2}}. \tag{3.7}$$

similarly, we get

$$\left(\int_{Q_T} g\,|u|^{p_2}\,|v|^{q_2}\,\zeta_2\,dx\,dt\right)^{\frac{q_1p_2-1}{q_1p_2}} \le \mathscr{K}_2\mathscr{K}_1^{\frac{1}{q_1}}. \tag{3.8}$$

Now, in $\mathscr{K}_1$ we consider the scale of variables:

$$t=\tau R,\quad x=yR^{\frac{\alpha_1}{\beta_1}},$$

while in $\mathscr{K}_2$ we use:

$$t=\tau R,\quad x=yR^{\frac{\alpha_2}{\beta_2}},$$

and use the fact that

$$dxdt=R^{\left(\frac{N\alpha_1}{\beta_1}+1\right)}dyd\tau,\ \zeta_{itt}=R^{-2}\zeta_{i\tau\tau},\ D^{\alpha_i}_{0|t}\zeta_{it}=R^{-\alpha_i}D^{\alpha_i}_{0|\tau R}\zeta_{i\tau},$$

$(-\Delta)_x^{\frac{\beta_i}{2}}\zeta_i=R^{-\alpha_i}(-\Delta)_y^{\frac{\beta_i}{2}}\zeta_i,\ i=1,2,$
we arrive at

$$\left(\int_{Q_T} f\,|u|^{p_1}\,|v|^{q_1}\,\zeta_1\,dx\,dt\right)^{\frac{q_1p_2-1}{q_1p_2}} \le C\Big[R^{\gamma_1}+R^{\gamma_2}+R^{\gamma_3}\Big]\times\Big[R^{\lambda_1}+R^{\lambda_2}+R^{\lambda_3}\Big]^{\frac{1}{p_2}}, \tag{3.9}$$

similarly, we have

$$\left(\int_{Q_T} g\,|u|^{p_2}\,|v|^{q_2}\,\zeta_2\,dx\,dt\right)^{\frac{q_1p_2-1}{q_1p_2}} \le C\Big[R^{\lambda_1}+R^{\lambda_2}+R^{\lambda_3}\Big]\times\Big[R^{\gamma_1}+R^{\gamma_2}+R^{\gamma_3}\Big]^{\frac{1}{q_1}}, \tag{3.10}$$

where
$$\begin{cases}\gamma_1=\left(\frac{N\alpha_1}{\beta_1}+1\right)\left(\frac{p_2-1}{p_2}\right)-2-\left(\frac{\mu_2\alpha_1}{\beta_1}+\nu_2\right)\frac{1}{p_2}\\[2mm] \gamma_2=\left(\frac{N\alpha_1}{\beta_1}+1\right)\left(\frac{p_2-1}{p_2}\right)-\alpha_1-\left(\frac{\mu_2\alpha_1}{\beta_1}+\nu_2\right)\frac{1}{p_2}\\[2mm] \gamma_3=\left(\frac{N\alpha_1}{\beta_1}+1\right)\left(\frac{p_2-1}{p_2}\right)-\alpha_1-\left(\frac{\mu_2\alpha_1}{\beta_1}+\nu_2\right)\frac{1}{p_2}\end{cases}$$
and
$$\begin{cases}\lambda_1=\left(\frac{N\alpha_2}{\beta_2}+1\right)\left(\frac{q_1-1}{q_1}\right)-2-\left(\frac{\mu_1\alpha_2}{\beta_2}+\nu_1\right)\frac{1}{q_1}\\[2mm] \lambda_2=\left(\frac{N\alpha_2}{\beta_2}+1\right)\left(\frac{q_1-1}{q_1}\right)-\alpha_2-\left(\frac{\mu_1\alpha_2}{\beta_2}+\nu_1\right)\frac{1}{q_1}\\[2mm] \lambda_3=\left(\frac{N\alpha_2}{\beta_2}+1\right)\left(\frac{q_1-1}{q_1}\right)-\alpha_2-\left(\frac{\mu_1\alpha_2}{\beta_2}+\nu_1\right)\frac{1}{q_1}\end{cases}$$
we observe that $\gamma_1<\gamma_2=\gamma_3$ and $\lambda_1<\lambda_2=\lambda_3$, hence

$$\left(\int_{Q_T} f\,|u|^{p_1}\,|v|^{q_1}\,\zeta_1\,dx\,dt\right)^{\frac{q_1p_2-1}{q_1p_2}} \le CR^{\gamma_2+\frac{\lambda_2}{p_2}} \tag{3.11}$$

and

$$\left(\int_{Q_T} g\,|u|^{p_2}\,|v|^{q_2}\,\zeta_2\,dx\,dt\right)^{\frac{q_1p_2-1}{q_1p_2}} \le CR^{\lambda_2+\frac{\gamma_2}{q_1}}. \tag{3.12}$$

with the fact that

$$\frac{1}{p_2}+\frac{1}{\tilde{p}_2}=1\ \text{ and }\ \frac{1}{q_1}+\frac{1}{\tilde{q}_1}=1 \tag{3.13}$$

by a simple computation,

$$\gamma_2+\frac{\lambda_2}{p_2}=N\Big(\frac{\alpha_1}{\beta_1\tilde{p}_2}+\frac{\alpha_2}{\beta_2p_2\tilde{q}_1}\Big)-\Big(\alpha_1+\frac{\alpha_2}{p_2}\Big)+\frac{1}{\tilde{p}_2}+\frac{1}{p_2\tilde{q}_1}+\frac{1}{p_2}\Big(\mu_2\frac{\alpha_1}{\beta_1}+\nu_2\Big)+\frac{1}{p_2q_1}\Big(\mu_1\frac{\alpha_2}{\beta_2}+\nu_1\Big)$$

and

$$\lambda_2+\frac{\gamma_2}{q_1}=N\Big(\frac{\alpha_2}{\beta_2\tilde{q}_1}+\frac{\alpha_1}{\beta_1q_1\tilde{p}_2}\Big)-\Big(\alpha_2+\frac{\alpha_1}{q_1}\Big)+\frac{1}{\tilde{q}_1}+\frac{1}{q_1\tilde{p}_2}+\frac{1}{q_1}\Big(\mu_1\frac{\alpha_2}{\beta_2}+\nu_1\Big)+\frac{1}{p_2q_1}\Big(\mu_2\frac{\alpha_1}{\beta_1}+\nu_2\Big)$$

also, using (3.13) we have

$$\frac{1}{\tilde{p_2}}+\frac{1}{p_2\tilde{q_1}}=1-\frac{1}{p_2}+\frac{1}{p_2\tilde{q_1}}=1-\frac{1}{p_2}\left(1-\frac{1}{\tilde{q_1}}\right)=1-\frac{1}{p_2q_1}$$

and

$$\frac{1}{\tilde{q_1}}+\frac{1}{q_1\tilde{p_2}}=1-\frac{1}{q_1}+\frac{1}{q_1\tilde{p_2}}=1-\frac{1}{q_1}\left(1-\frac{1}{\tilde{p_2}}\right)=1-\frac{1}{p_2q_1}$$

we obtain

$$\gamma_2+\frac{\lambda_2}{p_2}=N\Big(\frac{\alpha_1}{\beta_1\tilde{p_2}}+\frac{\alpha_2}{\beta_2p_2\tilde{q_1}}\Big)-\Big(\alpha_1+\frac{\alpha_2}{p_2}\Big)+1-\frac{1}{p_2q_1}+\frac{1}{p_2}\Big(\mu_2\frac{\alpha_1}{\beta_1}+\nu_2\Big)+\frac{1}{p_2q_1}\Big(\mu_1\frac{\alpha_2}{\beta_2}+\nu_1\Big)$$

and

$$\lambda_2+\frac{\gamma_2}{q_1}=N\Big(\frac{\alpha_2}{\beta_2\tilde{q_1}}+\frac{\alpha_1}{\beta_1q_1\tilde{p_2}}\Big)-\Big(\alpha_2+\frac{\alpha_1}{q_1}\Big)+1-\frac{1}{p_2q_1}+\frac{1}{q_1}\Big(\mu_1\frac{\alpha_2}{\beta_2}+\nu_1\Big)+\frac{1}{p_2q_1}\Big(\mu_2\frac{\alpha_1}{\beta_1}+\nu_2\Big)$$

We conclude that

- If $\gamma_2+\dfrac{\lambda_2}{p_2}<0$, it yield

$$N<\frac{\alpha_1+\dfrac{\alpha_2}{p_2}-\Big(1-\dfrac{1}{p_2q_1}\Big)-\dfrac{1}{p_2}\Big(\mu_2\dfrac{\alpha_1}{\beta_1}+\nu_2\Big)-\dfrac{1}{p_2q_1}\Big(\mu_1\dfrac{\alpha_2}{\beta_2}+\nu_1\Big)}{\dfrac{\alpha_1}{\beta_1\tilde{p_2}}+\dfrac{\alpha_2}{\beta_2p_2\tilde{q_1}}}.$$

  Then the right hand side of (3.11) goes to 0, when R tends to infinity, while the left hand side converge to

$$\left(\int_{Q_T}f\,|u|^{p_1}\,|v|^{q_1}\,dx\,dt\right)^{\frac{q_1p_2-1}{q_1p_2}}.$$

  This implies that $v\equiv 0$ or $u\equiv 0$.
  Similarly, if $\lambda_2+\dfrac{\gamma_2}{q_1}<0$, it yield

$$N<\frac{\alpha_2+\dfrac{\alpha_1}{q_1}-\Big(1-\dfrac{1}{p_2q_1}\Big)-\dfrac{1}{q_1}\Big(\mu_1\dfrac{\alpha_2}{\beta_2}+\nu_1\Big)-\dfrac{1}{p_2q_1}\Big(\mu_2\dfrac{\alpha_1}{\beta_1}+\nu_2\Big)}{\dfrac{\alpha_2}{\beta_2\tilde{q_1}}+\dfrac{\alpha_1}{\beta_1q_1\tilde{p_2}}},$$

  by using also (3.12) to proceeding as above, we obtain $u\equiv 0$ or $v\equiv 0$.
- If $\gamma_2+\frac{\lambda_2}{p_2}=0$, we get

$$\int_{\mathbb{R}^+\times\mathbb{R}^N}f\,|u|^{p_1}\,|v|^{q_1}\,dx\,dt<+\infty.$$

  Using again Hölder's inequality, we obtain

$$\int_{Q_T}g\,|u|^{p_2}\,|v|^{q_2}\,\zeta_2\,dx\,dt\leq\left(\int_{B_R}|v|^{q_1}\,(f\,|u|^{p_1}\,\zeta_1)\,dx\,dt\right)^{\frac{1}{q_1}}\mathscr{K}_2,$$

  where

$$B_R=\Big\{(t,x)\in\mathbb{R}^+\times\mathbb{R}^N;\;R^2\leq t^2+|x|^{2\theta_1}\leq 2R^2\Big\}.$$

  Since,

$$\int_{\mathbb{R}^+\times\mathbb{R}^N}f\,|u|^{p_1}\,|v|^{q_1}\,dx\,dt<+\infty,$$

  we get

$$\lim_{R\to+\infty}\int_{B_R}f\,|u|^{p_1}\,|v|^{q_1}\,dx\,dt=0,$$

  hence, we infer that

$$\int_{\mathbb{R}^+\times\mathbb{R}^N}g\,|u|^{p_2}\,|v|^{q_2}\,dx\,dt=0,$$

  this implies that $v\equiv 0$ or $u\equiv 0$.
  Similarly, if $\lambda_2+\frac{\gamma_2}{q_1}=0$, proceeding as above, we infer that $u\equiv 0$ or $v\equiv 0$.
  We deduce that no global weak solution is possible other than the trivial one, which ends the proof.

□

Remark 3.2. *In the case* $\alpha_i=1,\ \beta_i=2,\nu_i=\mu_i=0,p_1=q_2=0,$
$i=1,2$, *we recover the case who studied by A. Hakem (see [8]), when* $\alpha=\beta=0$.

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