



## ON THE SPECTRUM AND HILBERT SCHIMIDT PROPERTIES OF GENERALIZED RHALY MATRICES

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**ABSTRACT.** In this paper, we show that the multiplications of some generalized Rhaly operators are Hilbert Schmidt operators. We also calculate the spectrum and essential spectrum of a special generalized Rhaly matrices on the Hardy space  $H^2$ .

### 1. INTRODUCTION AND PRELIMINARIES

Let  $H(\mathbb{D})$  denotes the space of complex-valued analytic functions on the unit disk  $\mathbb{D}$ . Let  $H^p$  ( $1 \leq p < \infty$ ) denote the standard Hardy space on  $\mathbb{D}$ , and  $\ell_p$  denote the standard space of  $p$ -summable complex-valued sequences on the set of non-negative integers.

In [15], A.G. Siskakis gave the spectrum of Cesàro matrix on  $H^p$  by using the integral representation of Cesàro operator.

The Cesàro operator  $C$  is a paradigm of a noncompact operator in this class. Recall that  $C$  has the following form on  $H^2$ : If  $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2$ , then

$$C(f)(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} \left( \sum_{i=0}^n a_i \right) z^n. \quad (1)$$

We get the following integral representation of (1) by calculating the Taylor series

$$C(f)(z) = \frac{1}{z} \int_0^z \frac{f(t)}{1-t} dt. \quad (2)$$

In [19], S.W. Young generalized the Cesàro operator, by taking more general analytic function instead of the function  $1/(1-t)$  in equality (2) as follows:

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**Definition 1.** Let  $g$  be an analytic function on the unit disk. The generalized Cesàro operator on  $H^2$  with symbol  $g$  is defined by

$$C_g(f)(z) = \frac{1}{z} \int_0^z f(t)g(t) dt. \tag{3}$$

**Definition 2.** Let  $I$  be an arc of the unit circle  $\mathbb{T}$ , and let  $\varphi : \mathbb{T} \rightarrow \mathbb{C}$ . Then, let  $\varphi_I = \frac{1}{|I|} \int_I |\varphi|$ , where  $|I|$  denotes the arclength of  $I$ .  $\varphi$  is said to be of bounded mean oscillation if

$$\|\varphi\|_* = \sup_{I \subset \mathbb{T}} \frac{1}{|I|} \int_I |\varphi - \varphi_I| < \infty.$$

We denote the set of all functions of bounded mean oscillation by  $BMO$ . If we endow  $BMO$  with the norm  $\|\varphi\|_{BMO} = \|\varphi\|_* + |\varphi(0)|$ , then  $BMO$  is a Banach space (see [8]).

We say that  $g \in BMOA$  if  $g \in H^2$  and  $g(e^{i\theta}) \in BMO$ .

**Definition 3.** Let  $I$  be an arc of  $\mathbb{T}$ . We say that a function  $\varphi : \mathbb{T} \rightarrow \mathbb{C}$  is of vanishing mean oscillation if

$$\limsup_{\delta \rightarrow 0} \sup_{I \subset \delta} \frac{1}{|I|} \int_I |\varphi - \varphi_I| = 0.$$

We denote the set of all functions of vanishing mean oscillation by  $VMO$ .  $VMO$  is a closed subspace of  $BMO$ .

As with  $BMOA$ , we define  $VMOA$  as the set of  $g \in H^2$  such that  $g(e^{i\theta}) \in VMO$ .  $VMOA$  is a closed subspace of  $BMOA$  (see [8]).

We denote the spectrum of the linear operator  $T$  by  $\sigma(T)$ . That is,

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ not invertible}\}.$$

If  $G(z) = \int_0^z g(w) dw$ , then Pommerenke [12] showed that  $C_g$  is bounded on the Hilbert space  $H^2$  if and only if  $G \in BMOA$ . Aleman and Siskakis [2] extended Pommerenke’s result to the Hardy spaces  $H^p$  for  $1 \leq p < \infty$ , and show that  $C_g$  is compact on  $H^p$  if and only if  $G \in VMOA$ .

In [10], Hardy, Littlewood, and Polya showed that  $C$  is bounded on  $\ell_p$ . In [3], Brown, Halmos and Shields obtained that the spectrum for  $C$  on  $\ell_p$ , is in the form  $\sigma(C) = \{z : |z - 1| \leq 1\}$ . Gonzalez [9] proved that  $\sigma_e(C, \ell_p) = \{z : |z - 1| = 1\}$ . Spectrums of  $C$  Cesàro matrix on some other sequence spaces are taken consideration in some studies like [11] and [1]. Given a sequence  $a = (a_n)$  of scalers, a terraced matrix  $R_a = (a_{nk})$  is the lower triangular matrix defined by  $a_{nk} = a_n$ ,  $k \leq n$  and  $a_{nk} = 0$ , otherwise. In [17], Yildirim calculated the spectrum of terraced matrices on  $\ell_p$ , such that if  $0 \neq L < \infty$  then  $\sigma(R_a) = \{z : |z - qL/2| \leq qL/2\}$  for  $p > 1$  and  $p^{-1} + q^{-1} = 1$ , where  $L = \lim_n (n + 1) a_n$  and  $S = \{a_n : n = 0, 1, 2, \dots\}$ . Later Valeryevna [16] showed that essentially spectrum of terraced matrices on  $\ell_p$  is  $\sigma_e(R_a) = \{\lambda : |\lambda - qL/2| = qL/2\}$ , where  $1 < p < \infty$ ,  $0 \neq L = \lim_n (n + 1) a_n$  and  $q = p/(p - 1)$ . If entries of  $R_a$  terraced matrix are taken as  $a_n = 1/(n + 1)$ ,

Cesàro matrix will be obtained and the spectrum results for this special terraced matrix will conflict with the previous results. A lower triangular matrix  $A$  is said to be factorable if  $a_{nk} = a_n b_k$  for all  $0 \leq k \leq n$ . Spectrum of the factorable matrix on  $\ell_p$  is studied in [14]. The choices  $a_n = 1/(n+1)$  and each  $b_k = 1$ ,  $a_n = a_n$  and each  $b_k = 1$  generate  $C$  (the Cesàro matrix of order one), terraced matrices defined by Rhaly [13], respectively. In [18], the fine spectrum for  $C_g$  on  $H^2$  was computed when  $g$  is a rational symbol such that, if  $g(z) = \sum_{i=1}^n \frac{a_i}{1-\beta_i z}$ , where  $|\beta_i| = 1$ ,  $\beta_i$ 's are distinct, and  $a_i \neq 0$  for each  $i$ , then for  $C_g$ ,

- (a)  $\sigma(C_g) = \bigcup_{i=1}^n \overline{D(a_i)} \cup F$  where  $F = \{g(0)/k\}_{k=1}^\infty \setminus \bigcup_{i=1}^n \overline{D(a_i)}$ ,
- (b)  $\sigma_e(C_g) = \bigcup_{i=1}^n \partial D(a_i)$ ,
- (c) For  $\lambda \notin \sigma_e(C_g)$ ,  $\text{ind}(C_g - \lambda) = -G(\lambda)$  where  $G = \sum_{i=1}^n \chi_{D(a_i)}$  and  $D(a) = \{z : |z - a| < |a|\}$  for any  $a \in \mathbb{C}$ . Furthermore, certain products of generalized Cesàro operators are shown to be a Hilbert-Schmidt operator.

In the next definition we will give generalized terraced matrix by using the generalized Cesàro matrix given in [7].

**Definition 4.** Let  $\{b_n\}$  be a scalar sequence and  $g(z) = \sum_{k=0}^\infty a_k z^k \in H(\mathbb{D})$ . The matrix

$$R_b^g = \begin{pmatrix} a_0 b_0 & & & & \\ a_1 b_1 & a_0 b_1 & & & \\ a_2 b_2 & a_1 b_2 & a_0 b_2 & & \\ a_3 b_3 & a_2 b_3 & a_1 b_3 & a_0 b_3 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (4)$$

is called generalized terraced matrix with symbol  $g$  on  $H^2$ .

The relation  $R_b^g = D_b C_g$  is valid similar to on Rhaly matrix, where  $D_b = \text{diag}\{(n+1)b_n\}_{n=0}^\infty$ . We recall that  $C_g = C_1$  for  $g(z) = \frac{1}{1-z}$ . Since  $g(z) = \sum_{k=0}^\infty z^k$ , which fixes then  $a_n = 1$  for all  $n \in \mathbb{N}$ . Thus, from (4), we get

$$R_b^g = \begin{pmatrix} b_0 & & & & \\ b_1 & b_1 & & & \\ b_2 & b_2 & b_2 & & \\ b_3 & b_3 & b_3 & b_3 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = R_b.$$

On the other hand  $R_{1/(n+1)}^g = C_g$ . Therefore this definition could be regarded as a two-way generalizing both for Rhaly and Cesàro operator.

In [6], Durna and Yildirim have investigated some topological results about the set of generalized Rhally matrices.

For this paper, we make the convention that whenever  $\beta$  or  $\beta_j$  is notated, it is assumed that  $|\beta| = |\beta_j| = 1$ . Furthermore,  $R_b^\beta = R_b^{1/(1-\beta^2)}$ .

2. HILBERT-SCHMIDT

We used the lemmas in the Young’s paper [20] to provide the spectrum of generalized terraced matrices. We have showed that  $R_b^{\beta_1} R_b^{\beta_2}$ ,  $(R_b^{\beta_1})^* R_b^{\beta_2}$  and  $R_b^{\beta_1} (R_b^{\beta_2})^*$  are Hilbert-Schmidt operator as  $\beta_1 \neq \beta_2$ , to enable the hypothesis of Young’s lemmas in this section.

**Lemma 5.** *If  $\alpha \in \mathbb{Z}$ ,  $B_{j-1} = 0$ ,  $B_k = \sum_{k=j}^n \beta^k$  for  $k = j, \dots, n$  and  $(a_k)$  is a positive decreasing sequence, then*

$$\left| \sum_{k=j}^n \frac{\beta^k}{a_k^\alpha} \right| \leq \frac{2}{|1-\beta|} \left( \frac{1}{a_n^\alpha} + \left| \frac{1}{a_j^\alpha} - \frac{1}{a_n^\alpha} \right| \right).$$

*Proof.* For each  $k$ ,

$$|B_k| = \left| \sum_{k=j}^n \beta^k \right| = \left| \beta^j \frac{1-\beta^{n-j+1}}{1-\beta} \right| \leq |\beta^j| \frac{1+|\beta^{n-j+1}|}{|1-\beta|} = \frac{2}{|1-\beta|}.$$

Then, we have

$$\begin{aligned} \sum_{k=j}^n \frac{\beta^k}{a_k^\alpha} &= \sum_{k=j}^n \frac{1}{a_k^\alpha} (B_k - B_{k-1}) \\ &= \sum_{k=j}^n \frac{B_k}{a_k^\alpha} - \sum_{k=j}^{n-1} \frac{B_k}{a_{k+1}^\alpha} \\ &= \frac{B_n}{a_n^\alpha} + \sum_{k=j}^{n-1} B_k \left( \frac{1}{a_k^\alpha} - \frac{1}{a_{k+1}^\alpha} \right). \end{aligned}$$

If we take absolute values in the last equations, we have

$$\begin{aligned} \left| \sum_{k=j}^n \frac{\beta^k}{a_k^\alpha} \right| &= \left| \frac{B_n}{a_n^\alpha} + \sum_{k=j}^{n-1} B_k \left( \frac{1}{a_k^\alpha} - \frac{1}{a_{k+1}^\alpha} \right) \right| \\ &\leq \left| \frac{B_n}{a_n^\alpha} \right| + \sum_{k=j}^{n-1} \left| B_k \left( \frac{1}{a_k^\alpha} - \frac{1}{a_{k+1}^\alpha} \right) \right| \\ &\leq \frac{2}{|1-\beta|} \left( \left| \frac{1}{a_n^\alpha} \right| + \sum_{k=j}^{n-1} \left| \frac{1}{a_k^\alpha} - \frac{1}{a_{k+1}^\alpha} \right| \right) \\ &\leq \frac{2}{|1-\beta|} \left( \frac{1}{a_n^\alpha} + \left| \frac{1}{a_j^\alpha} - \frac{1}{a_n^\alpha} \right| \right). \end{aligned}$$

□

**Theorem 6.** Let  $(b_n) \in \ell_2$  is a positive decreasing sequence. Then  $R_b^{\beta_1} R_b^{\beta_2}$  is a Hilbert-Schmidt operator if  $\beta_1 \neq \beta_2$ .

*Proof.*  $R_b^{\beta_1} R_b^{\beta_2} \in B_2(H^2) \Leftrightarrow U_{\beta_2} R_b^{\beta_1} U_{\beta_2} U_{\beta_2} R_b^{\beta_2} U_{\beta_2} \in B_2(H^2)$ . Since,  $U_{\beta_2} R_b^{\beta_2} U_{\beta_2} = R_b$  by [7, Theorem 5] then, we can assume  $\beta_2 = 1$ . Rewrite  $\beta_1$  as  $\beta$ . Since  $R_b^\beta = R_b^{1/(1-\beta z)}$ ,

$$\left(R_b^\beta\right)_{nj} = \begin{cases} \beta^{n-j} b_{n-1} & , \quad n \geq j \\ 0 & , \quad n < j \end{cases} \quad , \quad n, j = 1, 2, \dots$$

Since  $(R_b)_{nk} = \begin{cases} b_n & , \quad n \geq k \\ 0 & , \quad n < k \end{cases}$ , we have that

$$\begin{aligned} \left(R_b^\beta R_b\right)_{nj} &= \sum_{k=j}^n \left(R_b^\beta\right)_{nk} (R_b)_{kj} \\ &= \begin{cases} b_{n-1} \sum_{k=j}^n \beta^{n-k} b_{k-1} & , \quad n \geq j \\ 0 & , \quad n < j \end{cases} \quad , \quad n, j = 1, 2, \dots \\ &= \begin{cases} \beta^n b_{n-1} \sum_{k=j}^n \bar{\beta}^k b_{k-1} & , \quad n \geq j \\ 0 & , \quad n < j \end{cases} \quad , \quad n, j = 1, 2, \dots \end{aligned}$$

Now, we will calculate the Hilbert-Schmidt norm of  $R_b^\beta R_b$

$$\begin{aligned} \left\|R_b^\beta R_b\right\|_{H.S}^2 &= \sum_{n,j} \left| \left(R_b^\beta R_b\right)_{nj} \right|^2 \\ &= \sum_{n=1}^{\infty} \sum_{j=1}^n \left| \left(R_b^\beta R_b\right)_{nj} \right|^2 \\ &= \sum_{n=1}^{\infty} (b_{n-1})^2 \sum_{j=1}^n \left| \sum_{k=j}^n \bar{\beta}^k b_{k-1} \right|^2 . \end{aligned}$$

If we apply, Lemma 5 with  $\alpha = -1$ , we have

$$\left| \sum_{k=j}^n \bar{\beta}^k b_{k-1} \right| \leq \frac{2}{|1-\beta|} (b_{n-1} + |b_{j-1} - b_{n-1}|) \leq \frac{2b_{j-1}}{|1-\beta|}, \quad (5)$$

then by (5), we get

$$\begin{aligned} \|R_b^\beta R_b\|_{H.S}^2 &= \sum_{n=1}^\infty \sum_{j=1}^n (b_{n-1})^2 \left(\frac{2b_{j-1}}{|1-\beta|}\right)^2 \\ &\leq \frac{4}{|1-\beta|^2} \sum_{n=1}^\infty (b_{n-1})^2 \sum_{j=1}^n (b_{j-1})^2 \\ &\leq \frac{4}{|1-\beta|^2} \left(\sum_{n=1}^\infty (b_{n-1})^2\right)^2 < \infty. \end{aligned}$$

Thus  $R_b^\beta R_b$  is a Hilbert-Schmidt operator. □

**Remark 7.** *If we take  $b_n = 1/(n + 1)$  in Theorem 6, then we have [19, Theorem 3.2].*

We obtain the following results to complete our analysis.

**Theorem 8.** *Let  $(b_n) \in \ell_2$  is a positive decreasing sequence. Then  $(R_b^{\beta_1})^* R_b^{\beta_2}$  and  $R_b^{\beta_1} (R_b^{\beta_2})^*$  are Hilbert-Schmidt operators as  $\beta_1 \neq \beta_2$ .*

*Proof.* Without loss of generality, let us take  $\beta_2 = 1$  and  $\beta_1 = \beta$ . Firstly, we will show that  $(R_b^{\beta_1})^* R_b^{\beta_2} \in B_2(H^2)$ .

$$(R_b^\beta)_{nj}^* = \begin{cases} 0 & , \quad n > j \\ \bar{\beta}^{j-n} b_{j-1} & , \quad n \leq j \end{cases} \quad , \quad n, j = 1, 2, \dots$$

is obtained from (4). Therefore

$$\left[ (R_b^\beta)^* R_b \right]_{nj} = \bar{\beta}^{-n} \sum_{k=\max\{n,j\}}^\infty \bar{\beta}^k b_{k-1}^2. \tag{6}$$

Now, it is sufficient to show that

$$\sum_{n,j=1}^\infty \left| \sum_{k=\max\{n,j\}}^\infty \bar{\beta}^k b_{k-1}^2 \right|^2$$

is convergent. We will only consider the case of  $n \geq j$ ; the case of  $j > n$  is similar. Let  $m \geq n$  be fixed natural number. From Lemma 5 with  $\alpha = -2$ , we have

$$\left| \sum_{k=n}^m \bar{\beta}^k b_{k-1}^2 \right| \leq \frac{2}{|1-\beta|} (b_{m-1}^2 + |b_{n-1}^2 - b_{m-1}^2|) = \frac{2b_{n-1}^2}{|1-\beta|}.$$

If  $m$  goes to infinity in last equality, we get

$$\left| \sum_{k=n}^{\infty} \bar{\beta}^k b_{k-1}^2 \right| \leq \frac{2b_{n-1}^2}{|1-\beta|}.$$

Hence

$$\begin{aligned} \left\| \left( R_b^\beta \right)^* R_b \right\|_{H.S} &= \sum_{n,j=1}^{\infty} \left| \sum_{k=j}^{\infty} \bar{\beta}^k b_{k-1}^2 \right|^2 \\ &\leq \frac{4}{|1-\beta|^2} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} b_{n-1}^4 \\ &\leq \frac{4}{|1-\beta|^2} \sum_{n=1}^{\infty} \sum_{j=1}^n b_{n-1}^2 b_{j-1}^2, \quad (n \geq j, b_n \leq b_j) \\ &= \frac{4}{|1-\beta|^2} \left( \sum_{n=1}^{\infty} b_{n-1}^2 \right)^2 < \infty. \end{aligned}$$

Thus  $\left( R_b^{\beta_1} \right)^* R_b^{\beta_2}$  is a Hilbert-Schmidt operator. Now, let us show that  $R_b^\beta (R_b)^* \in B_2(H^2)$ . We have

$$\begin{aligned} \left[ R_b^\beta (R_b)^* \right] &= \sum_{k=1}^{\min\{n,j\}} \beta^{n-k} b_{n-1} b_{j-1} \\ &= \beta^n b_{n-1} b_{j-1} \sum_{k=1}^{\min\{n,j\}} \bar{\beta}^k \end{aligned}$$

and so,

$$\begin{aligned} \left| \beta^n b_{n-1} b_{j-1} \sum_{k=1}^{\min\{n,j\}} \bar{\beta}^k \right| &= b_{n-1} b_{j-1} \left| \sum_{k=1}^{\min\{n,j\}} \bar{\beta}^k \right| \\ &= b_{n-1} b_{j-1} |\bar{\beta}| \frac{|1-\bar{\beta}^{\min\{n,j\}}|}{|1-\bar{\beta}|} \\ &\leq \frac{2b_{n-1} b_{j-1}}{|1-\bar{\beta}|}. \end{aligned}$$

Therefore

$$\begin{aligned} \left\| R_b^\beta (R_b)^* \right\|_{H.S} &= \sum_{n,j=1}^{\infty} \left| \beta^n b_{n-1} b_{j-1} \sum_{k=1}^{\min\{n,j\}} \bar{\beta}^k \right|^2 \\ &\leq \frac{4}{|1-\beta|^2} \left( \sum_{n=1}^{\infty} b_{n-1}^2 \sum_{j=1}^{\infty} b_{j-1}^2 \right) \\ &= \frac{4}{|1-\beta|^2} \left( \sum_{n=1}^{\infty} b_{n-1}^2 \right)^2 < \infty. \end{aligned}$$

Thus  $R_b^\beta (R_b)^* \in B_2(H^2)$  is a Hilbert-Schmidt operator. □

Denote by  $\pi$ , natural projection of  $B(H^2)$  onto the Calkin algebra  $Q(H^2)$ . From Theorem 6 and 8 the following equations are valid.

$$\pi(R_b^\beta R_b) = 0 \tag{7}$$

$$\pi\left(\left(R_b^\beta\right)^* R_b\right) = 0 \tag{8}$$

$$\pi\left(R_b^\beta (R_b)^*\right) = 0. \tag{9}$$

Let  $F(H)$  be the set of Fredholm operators on a Hilbert space  $H$ . Recall that  $T, S$  are Fredholm operators and  $K$  is a compact operator, then the following equations are true (see [5, Theorems XI.3.7 and XI.3.11]):

- (a)  $TS$  is Fredholm and  $\text{ind}(TS) = \text{ind}(T) + \text{ind}(S)$ ,
- (b)  $T + K$  is Fredholm and  $\text{ind}(T + K) = \text{ind}(T)$ .

### 3. SPECTRUM

In this section, we are going to calculate the spectrum of matrix  $R_b^g$ , where the symbol  $g$  is a rational function.

First, we will give the definition of a rational function. If  $P$  and  $Q$  are polynomials, then  $R(z) := P(z)/Q(z)$  is called a rational function.

We must use the rational function  $R$  that  $R_b^R$  is bounded by it, while we choose rational function in generalized Rhally matrices and calculate its spectrum.

Finally, we should mention that  $R$  has no pole inside the unit disk. Furthermore, if  $Q$  has zeros on  $\mathbb{T}$ , these zeros should be simple. If there is a double pole on the unit disk, then the growth of the primitive at the pole must be faster than logarithmic growth, hence this primitive can not in BMOA (for more information, see [4]).

In this section, we write  $R$  in its partial fractions decomposition relative to  $\mathbb{T}$ . That is,

$$R(z) = \sum_{i=1}^n \frac{\alpha_i}{1 - \beta_i z} + S(z),$$

where  $S$  is another rational function analytic on the closed unit disk. We note that  $\mathbb{T}$  is an unit circle.

**Lemma 9.** *If  $g \in \mathcal{B}$  and  $\{(n + 1) b_n\}$  is a bounded sequence, then  $R_b^g$  is bounded.*

*Proof.* Let  $(x_k) \in \ell_2$ . Then, we have

$$\begin{aligned} D_b x_n &= \begin{pmatrix} b_0 & & & \\ & 2b_1 & & \\ & & 3b_2 & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix} \\ &= (b_0 x_0, 2b_1 x_1, \dots, (n + 1) b_n, \dots). \end{aligned}$$



Since  $\{(n + 1) b_n\}$  is bounded sequence, there is a  $M > 0$  such that  $|(n + 1) b_n| \leq M$  for each  $n \in \mathbb{N}$ . Hence, we get

$$\sum_{n=0}^{\infty} |(n + 1) b_n|^2 |x_n|^2 \leq M^2 \sum_{n=0}^{\infty} |x_n|^2 = M^2 \|x\|^2,$$

i.e.,

$$\|D_b x_n\| \leq M \|x\|.$$

Thus  $D_b$  is bounded. Furthermore, since  $C_g$  is bounded for  $g \in \mathcal{B}$ ,  $R_b^g = D_b C_g$  is bounded.  $\square$

**Remark 10.** *If we take  $b_n = 1/(n + 1)$  in Lemma 9, we obtain [19, expression in page 130].*

**Lemma 11** ([20, Lemma 3.2]). *If  $a_1, a_2, \dots, a_n$  are elements of a  $C^*$ -algebra  $A$  such that  $a_i a_j = a_j a_i = 0$  and  $a_i^* a_j = a_j a_i^* = 0$  for  $i \neq j$  and  $\alpha_i \in \mathbb{C}$ , then*

$$\{0\} \cup \sigma \left( \sum_{i=1}^n \alpha_i a_i \right) = \bigcup_{i=1}^n \sigma(\alpha_i a_i). \tag{10}$$

We note that, the  $\{0\}$  in (10) is necessary because the sum of non-invertible elements can be invertible. For example, let  $P$  be a projection;  $PP^\perp = P^\perp P = 0$ , but  $P + P^\perp = I$ .

**Lemma 12** ([20, Lemma 3.3]). *If  $T_1, \dots, T_n \in B(H)$  such that  $T_i T_j$  is compact for  $i \neq j$  and  $\alpha_i \in \mathbb{C}$ , then for  $\lambda \notin \bigcup_{i=1}^n \sigma_e(T_i)$ , we have that*

$$\text{ind} \left( \sum_{i=1}^n \alpha_i T_i - \lambda \right) = \sum_{i=1}^n \text{ind}(\alpha_i T_i - \lambda). \tag{11}$$

**Lemma 13** ([16, Theorem 2.2]). *Let  $1 < p < \infty$ ,  $0 \neq L = \lim_n (n + 1) a_n$  and  $q = p/(p - 1)$ , then*

$$\sigma_e(R_a) = \left\{ \lambda : \left| \lambda - \frac{qL}{2} \right| = \frac{qL}{2} \right\}. \tag{12}$$

The final lemma is well-known and can easily be derived.

**Lemma 14.** *If  $T$  is a bounded linear operator on a Hilbert space such that it has a lower triangular matrix representation in an orthonormal basis  $\mathcal{B}$  with diagonal entries  $\{\alpha_n\}_{n=0}^\infty$ , then  $\sigma_p(T) \subseteq \{\alpha_n\}_{n=0}^\infty$ .*

**Theorem 15.** *Let  $(b_k) \in \ell_2$  is a positive strictly decreasing sequence and  $\lim_k (k + 1) b_k = L$ ,  $0 \neq L < \infty$ . If  $g(z) = \sum_{i=1}^n \frac{\alpha_i}{1 - \beta_i z}$ , where  $\beta_i$  are distinct and  $\alpha_i \neq 0$  for each*

*i. Then the following equations hold for  $R_b^g$ :*

- (i)  $\sigma(R_b^g) = \bigcup_{i=1}^n \overline{D(\alpha_i L)} \cup E$ , where  $E = \{g(0) b_k\}_{k=0}^\infty \setminus \bigcup_{i=1}^n \overline{D(\alpha_i L)}$ ,
- (ii)  $\sigma_e(R_b^g) = \bigcup_{i=1}^n \partial D(\alpha_i L)$ ,
- (iii) For  $\lambda \notin \sigma_e(R_b^g)$ ,  $\text{ind}(R_b^g - \lambda) = -G(\lambda)$ , where  $G = \sum_{i=1}^n \chi_{D(\alpha_i L)}$ .

*Proof.* Since  $g(z) = \sum_{i=1}^n \frac{\alpha_i}{1 - \beta_i z}$ , we have

$$R_b^g = \sum_{i=1}^n \alpha_i R_b^{\beta_i}. \tag{13}$$

From (7), (8), (9) and Lemma 11, we obtain that

$$\begin{aligned} \{0\} \cup \sigma(\pi(R_b^g)) &= \{0\} \cup \sigma\left(\sum_{i=1}^n \pi(\alpha_i R_b^{\beta_i})\right) \\ &= \bigcup_{i=1}^n \sigma(\pi(\alpha_i R_b^{\beta_i})). \end{aligned}$$

Since the essential spectrum of the operator  $T$  is the spectrum of  $\pi(T)$  coset in the Calkin algebra, i.e.,  $\sigma_e(T) = \sigma(\pi(T))$  and from Lemma 13, we get

$$\{0\} \cup \sigma_e(R_b^g) = \bigcup_{i=1}^n \sigma_e(\alpha_i R_b^{\beta_i}) = \bigcup_{i=1}^n \partial D(\alpha_i L). \tag{14}$$

However, we know that 0 is a limit point of the right side from (14),  $0 \in \sigma_e(R_b^g)$ . Therefore

$$\sigma_e(R_b^g) = \bigcup_{i=1}^n \partial D(\alpha_i L). \tag{15}$$

Similarly,  $a_i R_b^{\beta_i} a_j R_b^{\beta_j} \in \mathcal{K}(H^2)$  since  $R_b^{\beta_i} R_b^{\beta_j} \in B_2(H^2) \subset \mathcal{K}(H^2)$  and  $\mathcal{K}(H^2)$  is a vector space where

$$\mathcal{K}(H^2) = \{K : K : H^2 \rightarrow H^2 \text{ is a compact linear operator}\}.$$

Therefore, from Lemma 12, we have  $\lambda \notin \sigma_e(\sum_{i=1}^n \alpha_i R_b^{\beta_i})$  for  $\lambda \notin \bigcup_{i=1}^n \sigma_e(R_b^g)$  and  $\text{ind}(R_b^g - \lambda) = \sum_{i=1}^n \text{ind}(\alpha_i R_b^{\beta_i} - \lambda)$ . Since  $\text{ind}(\alpha_i R_b^{\beta_i} - \lambda) = -\chi_{D(\alpha_i L)}$ , we get  $\text{ind}(\alpha_i R_b^{\beta_i} - \lambda) = -1$  for  $\lambda \in D(\alpha_i L)$  and  $\text{ind}(\alpha_i R_b^{\beta_i} - \lambda) = 0$  for  $\lambda \notin \overline{D(\alpha_i L)}$ . From Lemma 12, we know that index for  $R_b^g$  is additive. Thus, we have

$$\text{ind}(R_b^g - \lambda) = -\sum_{i=1}^n \chi_{D(\alpha_i L)} = -G(\lambda) \tag{16}$$

for  $\lambda \notin \sigma_e(R_b^g) = \bigcup_{i=1}^n \partial D(\alpha_i L)$  by using Theorem 15-(iii). From (11), if  $\text{ind}(R_b^g - \lambda) = 0$  then  $\lambda \notin \overline{D(\alpha_i L)}$  for every  $i$ .

Now, we observe points  $\lambda \in \sigma(R_b^g)$  such that  $\text{ind}(R_b^g - \lambda) = 0$ . Firstly, we indicate that  $\lambda \in \sigma_p(R_b^g)$  since  $\text{ind}(R_b^g - \lambda) = 0$ . From definition of set  $E$ ,  $E \cap \bigcup_{i=1}^n \overline{D(\alpha_i L)} = \emptyset$ , we define the set  $E := \{\lambda \in \sigma(R_b^g) : \text{ind}(R_b^g - \lambda) = 0\}$ . Since  $R_b^g$  is a lower triangular matrix, from Lemma 14, we know that the only possible eigenvalues of  $R_b^g$  are the diagonal elements  $\{g(0) b_k\}_{k=1}^\infty$ . We can easily

show that  $\{\overline{g(0)b_k}\}_{k=1}^{\infty} \subseteq \sigma_p((R_b^g)^*)$ , because they are diagonal elements of upper triangular operator. Therefore,  $E \subseteq \{g(0)b_k\}_{k=1}^{\infty}$ . Thus,

$$\sigma(R_b^g) = \bigcup_{i=1}^n \overline{D(La_i)} \cup E.$$

□

**Remark 16.** *If we take  $b_n = 1/(n+1)$  in Theorem 15, we obtain [19, Theorem 3.3].*

**Remark 17.** *If we take  $g(z) = 1/(1-z)$  in Theorem 15, we obtain [17, Theorem 3.3] and [16, Theorem 2.2].*

**Remark 18.** *If we take  $b_n = 1/(n+1)$  and  $g(z) = 1/(1-z)$  in Theorem 15, we obtain [3, Theorem 2] and [9, Theorem 4].*

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