Available online: September 18, 2018

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 68, Number 1, Pages 734–749 (2019)
DOI: 10.31801/cfsuasmas.464205
ISSN 1303–5991 E-ISSN 2618-6470
http://communications.science.ankara.edu.tr/index.php?series=A1

COMMUNICATIONS
SERIES A1

ON NEW HERMITE-HADAMARD-FEJÉR TYPE INEQUALITIES FOR HARMONICALLY QUASI CONVEX FUNCTIONS

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Abstract. In this paper, we give the theorems and results for the trapezoidal and midpoint type inequality of new Hermite-Hadamard-Fejér for harmonically-quasi convex functions via fractional integrals.

1. Introduction

Lots of inequalities have been established for convex functions but the most famous is the Hermite-Hadamard inequality, due to its rich importance and applications, which is stated as follows: Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then following double inequalities holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The inequalities (1.1) hold in reversed direction if f is concave.

Many researcher have studied on the Hermite-Hadamard inequalities for convex functions. (1.1) have been generalized and enhanced for many classes of convex functions.

In [4], İşcan have represented harmonically convex function and have proved inequalities related to its as follows

Definition 1. [4] *Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval. A function $f : I \longrightarrow \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{1.2}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.2) is reversed, then f is said to be harmonically concave.

Received by the editors: July 19, 2017; Accepted: April 09, 2018.
2010 *Mathematics Subject Classification.* Primary 26D15, 26D10; Secondary 26A51,26A15.
Key words and phrases. Harmonically-Quasi-convex, Hermite-Hadamard-Fejer type inequalities, fractional Integral.

Proposition 1. [4] *Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval and $f : I \longrightarrow \mathbb{R}$ is function, then:*

(1) *If $I \subset (0, \infty)$ and f is convex and nondecreasing function then f is harmonically convex.*
(2) *If $I \subset (0, \infty)$ and f is harmonically convex and nonincreasing function then f is convex.*
(3) *If $I \subset (-\infty, 0)$ and f is harmonically convex and nondecreasing function then f is convex.*
(4) *If $I \subset (-\infty, 0)$ and f is convex and nonincreasing function then f is harmonically convex.*

The following definitions and mathematical preliminaries of fractional calculus theory are used further in this paper.

Definition 2. [8] *Let $f \in L[a, b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a \tag{1.3}$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b \tag{1.4}$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\alpha} e^{-t}t^{\alpha-1}dt$ and $J^{0}_{a+}f(x) = J^{0}_{b-}f(x) = f(x)$.

Theorem 1. [4] *Let $I \subset \mathbb{R} \setminus \{0\}$ be a harmonically convex function on I^o, $a, b \in I^o$ with $a < b$. If $f \in L[a, b]$ then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{1.5}$$

The above inequalities are sharp.

In [9], Latif et al. showed the following definition:

Definition 3. *A function $g : [a, b] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ is said to be harmonically symmetric with respect to $2ab/(a + b)$, if*

$$g(x) = g\left(\frac{1}{\frac{1}{a} + \frac{1}{b} - \frac{1}{x}}\right)$$

holds for all $x \in [a, b]$.

In [3], İşcan and Wu have revealed Hermite-Hadamard' s inequalities for harmonically convex function via fractional integrals as follow

Theorem 2. [3] *Let $f : I \subseteq (0,\infty) \longrightarrow \mathbb{R}$ be a function such that $f \in L[a,b]$, where $a, b \in I$ with $a < b$. If f is a harmonically convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J^{\alpha}_{1/a^-}(f\circ g)(1/b)\right. \\ &+ \left. J^{\alpha}_{1/b^+}(f\circ g)(1/a)\right\} \leq \frac{f(a)+f(b)}{2} \end{aligned} \tag{1.6}$$

with $\alpha > 0$.

In [2], Chan and Wu represented Hermite-Hadamard-Fejér inequality for harmonically convex functions as follow

Theorem 3. [2] *Let $f : I \subseteq \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ and $g : [a,b] \subseteq \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ is nonnegative integrable and harmonically symmetric with respect to $2ab/(a+b)$ then*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx &\leq \int_a^b \frac{f(x)g(x)}{x^2}dx \\ &\leq \frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx. \end{aligned} \tag{1.7}$$

In [5], İşcan and Kunt showed Hermite-Hadamard-Fejér type inequality for harmonically convex functions in fractional integral forms and established following identity as follow

Theorem 4. [5] *Let $f : [a,b] \longrightarrow \mathbb{R}$ be harmonically convex function with $a < b$ and $f \in L[a,b]$. If $g : [a,b] \longrightarrow \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $2ab/(a+b)$ then the following inequalities for fractional integrals holds:*

$$\begin{aligned} & f\left(\frac{2ab}{a+b}\right)\left[J^{\alpha}_{1/b^+}(g\circ h)(1/a) + J^{\alpha}_{1/a^-}(g\circ h)(1/b)\right] \\ \leq & \left[J^{\alpha}_{1/b^+}(fg\circ h)(1/a) + J^{\alpha}_{1/a^-}(fg\circ h)(1/b)\right] \\ \leq & \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b^+}(g\circ h)(1/a) + J^{\alpha}_{1/a^-}(g\circ h)(1/b)\right] \end{aligned} \tag{1.8}$$

with $\alpha > 0$ and $h(x) = 1/x$, $x \in \left[\frac{1}{b}, \frac{1}{a}\right]$.

In [13], Zhang et al. defined the harmonically quasi-convex function and supplied several properties of this kind of functions.

Definition 4. [13] *A function $f : I \subseteq (0, \infty) \longrightarrow [0, \infty)$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \le \sup\{f(x), f(y)\} \tag{1.9}$$

for all $x, y \in I$ and $t \in [0, 1]$.

We would like to point out that any harmonically convex function on $I \subseteq (0, \infty)$ is a harmonically quasi-convex function, but not conversely. For example, the function

$$y = \begin{cases} 1 & x \in (0,1] \\ (x-2)^2 & x \in [1,4] \end{cases}$$

is harmonically quasi convex on (0,4], but it is not harmonically convex on (0,4].

In [10, 11], Park established inequalities Hermite-Hadamard-like type for differentiable harmonically convex function as follows

Theorem 5. [10] *Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \longrightarrow \mathbb{R}$ be differentiable function on I^o, the interior of an interval I, such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is harmonically quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \left(\frac{b-a}{2}\right) sup\{|f'(a)|, |f'(a)|\}. \tag{1.10}$$

Theorem 6. [11] *Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \longrightarrow \mathbb{R}$ be differentiable function on I^o, the interior of an interval I, such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is harmonically quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \left(\frac{b-a}{4}\right) \sup\{|f'(a)|, |f'(a)|\}. \tag{1.11}$$

In [6] İşcan and Kunt represented the following new theorem related to Hermite-Hadamard-Fejér type inequalities for harmonically quasi convex functions via fractional integrals

Theorem 7. [6] *Let $f : I \subset (0, \infty) \longrightarrow \mathbb{R}$ be a differentiable function on I^o such that $f' \in L[a, b]$, where $a, b \in I$ and $a < b$. If $|f'|^q$, $q \ge 1$, is harmonically quasi-convex on $[a, b]$, $g : [a, b] \longrightarrow \mathbb{R}$ is continuous and harmonically symmetric with respect to $2ab/a + b$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left|\frac{f(a) + f(b)}{2}\left[J^{\alpha}_{1/b^+}(g \circ h)(1/a) + J^{\alpha}_{1/a^-}(g \circ h)(1/b)\right]\right. \qquad (1.12) \\ &- \left.\left[J^{\alpha}_{1/b^+}(fg \circ h)(1/a) + J^{\alpha}_{1/a^-}(fg \circ h)(1/b)\right]\right| \\ &\le \frac{\|g\|_\infty ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha} C_2(\alpha)\left[sup\{|f'(a)|^q, |f'(a)|^q\right]^{\frac{1}{q}} \end{aligned}$$

where

$$\begin{aligned} C_2(\alpha) &= \frac{b^{-2}}{\alpha+1}\,{}_2F_1\left(2,1;\alpha+2;\frac{b-a}{b+a}\right)-\frac{b^{-2}}{\alpha+1}\,{}_2F_1\left(2,\alpha+1;\alpha+2;\frac{b-a}{b+a}\right) \\ &+ \frac{4(a+b)^{-2}}{(\alpha+1)}\,{}_2F_1\left(2,\alpha+1;\alpha+2;\frac{b-a}{b+a}\right) \end{aligned}$$

with $0<\alpha\leq 1$ *and* $h(x)=1/x$, $x\in\left[\frac{1}{b},\frac{1}{a}\right]$.

In [7], İşcan, Turhan and Maden gave identities for harmonically convex function as follows:

Lemma 1. [7] $f: I\subseteq\mathbb{R}\backslash\{0\}\longrightarrow\mathbb{R}$ *be a differentiable function on* I^o, $h:[a,b]\longrightarrow[0,\infty)$ *be differentiable function on* I, $a,b\in I$ *and* $a<b$. *If* $f'\in L[a,b]$ *then the following equality holds:*

$$\begin{aligned} &[h(b)-2h(a)]\frac{f(a)}{2}+h(b)\frac{f(b)}{2}-\int_a^b f(x)h'(x)dx \qquad (1.13) \\ &= \frac{b-a}{4ab}\left\{\int_0^1 \left[2h\left(L(t)\right)-h(b)\right]f'\left(L(t)\right)\left(L(t)\right)^2dt\right. \\ &+ \left.\int_0^1 \left[2h\left(U(t)\right)-h(b)\right]f'\left(U(t)\right)\left(U(t)\right)^2dt\right\} \end{aligned}$$

where $L(t)=\frac{aH}{tH+(1-t)a}$, $U(t)=\frac{bH}{tH+(1-t)b}$, $\forall t\in[0,1]$ *and* $H:=H(a,b)=\frac{2ab}{a+b}$.

Lemma 2. [12]$f: I\subseteq\mathbb{R}\backslash\{0\}\longrightarrow\mathbb{R}$ *be a differentiable function on* I^o, $h:[a,b]\longrightarrow[0,\infty)$ *be differentiable function on* I, $a,b\in I$ *and* $a<b$. *If* $f'\in L[a,b]$ *then the following equality holds:*

$$\begin{aligned} &\left(\frac{f(a)+f(b)}{2}\right)h(a)-f\left(H\right)h(b) \qquad (1.14) \\ &+ \frac{b-a}{4ab}\left\{\int_0^1 \begin{array}{c}\left[h'(L(t))\left(L(t)\right)^2+h'(U(t))\left(U(t)\right)^2\right] \\ \times\left[f(L(t))+f(U(t))\right]dt\end{array}\right\} \\ &= \frac{b-a}{4ab}\left\{\int_0^1 \left[h(L(t))-h(U(t))+h(b)\right]\times\left[-f'(L(t))\left(L(t)\right)^2\right.\right. \\ &+ \left.\left. f'(U(t))\left(U(t)\right)^2\right]dt\right\} \end{aligned}$$

where $L(t)=\frac{aH}{tH+(1-t)a}$, $U(t)=\frac{bH}{tH+(1-t)b}$, $\forall t\in[0,1]$ *and* $H:=H(a,b)=\frac{2ab}{a+b}$.

In this paper, we study both Fejér and Fejér fractional of new Hermite-Hadamard's inequalities related to both right and left of the inequalities for harmonically quasi convex functions.

2. Main Results

Throughout in this section, we will use the notations $L(t) = \frac{aH}{tH+(1-t)a}$, $U(t) = \frac{bH}{tH+(1-t)b}$ and $H = H(a,b) := \frac{2ab}{a+b}$.

Theorem 8. *Let $f : I \subseteq (0,\infty) \longrightarrow \mathbb{R}$ be differentiable mapping on I^o, $a,b \in I$ with $a < b$. If $h : [a,b] \longrightarrow [0,\infty)$ is a differentiable function and $|f'|^q$ is harmonically quasi convex on $[a,b]$ for $q \geq 1$, the following inequality holds*

$$\left| [h(b) - 2h(a)] \frac{f(a)}{2} + h(b)\frac{f(b)}{2} - \int_a^b f(x)h'(x)dx \right| \tag{2.1}$$

$$\leq \frac{b-a}{4ab} \left\{ \begin{array}{c} \left(\int_0^1 |2h(L(t)) - h(b)|\, dt \right)^{1-\frac{1}{q}} \\ \left(\int_0^1 |2h(L(t)) - h(b)|\, L^{2q}(t) \sup\{|f'(H)|^q, |f'(a)|^q\} dt \right)^{\frac{1}{q}} \\ + \left(\int_0^1 |2h(U(t)) - h(b)|\, dt \right)^{1-\frac{1}{q}} \\ \left(\int_0^1 |2h(U(t)) - h(b)|\, U^{2q}(t) \sup\{|f'(H)|^q, |f'(b)|^q\} dt \right)^{\frac{1}{q}} \end{array} \right\}.$$

Proof. Firstly, we use power mean inequality in (1.13), we get

$$\left| [h(b) - 2h(a)] \frac{f(a)}{2} + h(b)\frac{f(b)}{2} - \int_a^b f(x)h'(x)dx \right| \leq \frac{b-a}{4ab} \times \tag{2.2}$$

$$\left\{ \left(\int_0^1 |2h(L(t)) - h(b)|\, dt \right)^{1-\frac{1}{q}} \left(\int_0^1 |2h(L(t)) - h(b)|\, |f'(L(t))|^q L^{2q}(t) dt \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_0^1 |2h(U(t)) - h(b)|\, dt \right)^{1-\frac{1}{q}} \left(\int_0^1 |2h(U(t)) - h(b)|\, |f'(U(t))|^q U^{2q}(t) dt \right)^{\frac{1}{q}} \right\}.$$

Since $|f'|^q$, $q \geq 1$, is harmonically quasi convex function, it is obtained

$$\leq \frac{b-a}{4ab}\left\{\begin{array}{c}\left(\int\limits_0^1 |2h(L(t))-h(b)|\,dt\right)^{1-\frac{1}{q}} \\ \left(\int\limits_0^1 |2h(L(t))-h(b)|\,L^{2q}(t)\sup\{|f'(H)|^q,|f'(a)|^q\}dt\right)^{\frac{1}{q}} \\ +\left(\int\limits_0^1 |2h(U(t))-h(b)|\,dt\right)^{1-\frac{1}{q}} \\ \left(\int\limits_0^1 |2h(U(t))-h(b)|\,U^{2q}(t)\sup\{|f'(H)|^q,|f'(b)|^q\}dt\right)^{\frac{1}{q}}\end{array}\right\}.$$

So the proof is complete. □

Corollary 1. *Let $g:[a,b]\longrightarrow[0,\infty)$ be a positive continuous mapping and harmonically symmetric with respect to $\frac{2ab}{a+b}$, $a<b$. If $h(t)=\int\limits_{1/t}^{1/a}\psi(x)\,(g\circ\varphi)\,(x)dx$, $\psi(x)=\left[\left(x-\frac{1}{b}\right)^{\alpha-1}+\left(\frac{1}{a}-x\right)^{\alpha-1}\right]$ for all $t\in[a,b]$, $\alpha>0$ in Theorem 8, we obtain*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J^{\alpha}_{1/b^+}(g\circ\varphi)(1/a)+J^{\alpha}_{1/a^-}(g\circ\varphi)(1/b)\right]\right. \tag{2.3}$$
$$\left.-\left[J^{\alpha}_{1/b^+}(fg\circ\varphi)(1/a)+J^{\alpha}_{1/a^-}(fg\circ\varphi)(1/b)\right]\right|$$

$$\leq\left(\frac{b-a}{2ab}\right)^{\alpha+1}\frac{1}{\Gamma(\alpha+1)}\left(\frac{2^{\alpha+2}-4}{\alpha+1}\right)^{1-\frac{1}{q}}\left(A_1(t,\alpha;q)\sup\{|f'(H)|^q,|f'(a)|^q\}\right.$$
$$\left.+A_2(t,\alpha;q)\sup\{|f'(H)|^q,|f'(b)|^q\}\right)^{\frac{1}{q}}$$

where

$$A_1(t,\alpha;q) = \int\limits_0^1\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right]L^{2q}(t)dt,$$
$$A_2(t,\alpha;q) = \int\limits_0^1\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right]U^{2q}(t)dt.$$

Proof. If we use $h(t)=\int\limits_{1/t}^{1/a}\psi(x)\,(g\circ\varphi)\,(x)dx$, $\varphi(x)=\frac{1}{x}$, in (2.1), we get

$$\Gamma(\alpha)\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J^{\alpha}_{1/b^+}(g\circ\varphi)(1/a)+J^{\alpha}_{1/a^-}(g\circ\varphi)(1/b)\right]\right. \tag{2.4}$$
$$\left.-\left[J^{\alpha}_{1/b^+}(fg\circ\varphi)(1/a)+J^{\alpha}_{1/a^-}(fg\circ\varphi)(1/b)\right]\right|$$

$$\leq \frac{b-a}{4ab}\left\{\begin{array}{c}\left(\begin{array}{c}\int\limits_0^1\left|2\int\limits_{1/L(t)}^{1/a}\psi(x)\left(g\circ\varphi\right)(x)dx\right.\\ \left.-\int\limits_{1/b}^{1/a}\psi(x)\left(g\circ\varphi\right)(x)dx\right|dt\end{array}\right)^{1-\frac{1}{q}}\times\\ \left(\begin{array}{c}\int\limits_0^1\left|2\int\limits_{1/L(t)}^{1/a}\psi(x)\left(g\circ\varphi\right)(x)dx-\int\limits_{1/b}^{1/a}\psi(x)\left(g\circ\varphi\right)(x)dx\right|\times\\ L^{2q}\sup\{\left|f'(H)\right|^q,\left|f'(a)\right|^q\}dt\end{array}\right)^{\frac{1}{q}}\\ +\left(\begin{array}{c}\int\limits_0^1\left|2\int\limits_{1/U(t)}^{1/a}\psi(x)\left(g\circ\varphi\right)(x)dx\right.\\ \left.-\int\limits_{1/b}^{1/a}\psi(x)\left(g\circ\varphi\right)(x)dx\right|dt\end{array}\right)^{1-\frac{1}{q}}\times\\ \left(\begin{array}{c}\int\limits_0^1\left|2\int\limits_{1/L(t)}^{1/a}\psi(x)\left(g\circ\varphi\right)(x)dx-\int\limits_{1/b}^{1/a}\psi(x)\left(g\circ\varphi\right)(x)dx\right|\times\\ U^{2q}(t)\sup\{\left|f'(H)\right|^q,\left|f'(b)\right|^q\}dt\end{array}\right)^{\frac{1}{q}}\end{array}\right\}.$$

If we use g function that be harmonically symmetric (i.e $\frac{2ab}{a+b}$) in the simple calculation, we get

$$\begin{aligned}&\left|2\int_{1/L(t)}^{1/a}\psi(x)\left(g\circ\varphi\right)(x)dx-\int_{1/b}^{1/a}\psi(x)\left(g\circ\varphi\right)(x)dx\right| \\ =&\left|\int_{1/L(t)}^{1/U(t)}\psi(x)\left(g\circ\varphi\right)(x)dx\right|\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}&\left|2\int_{1/U(t)}^{1/a}\psi(x)\left(g\circ\varphi\right)(x)dx-\int_{1/b}^{1/a}\psi(x)\left(g\circ\varphi\right)(x)dx\right| \\ =&\left|\int_{1/L(t)}^{1/U(t)}\psi(x)\left(g\circ\varphi\right)(x)dx\right|.\end{aligned}\tag{2.6}$$

By using (2.5) and (2.6) in (2.4)

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J^{\alpha}_{1/b^{+}}\left(g\circ\varphi\right)(1/a)+J^{\alpha}_{1/a^{-}}\left(g\circ\varphi\right)(1/b)\right]\right.\tag{2.7}$$
$$\left.-\left[J^{\alpha}_{1/b^{+}}\left(fg\circ\varphi\right)(1/a)+J^{\alpha}_{1/a^{-}}\left(fg\circ\varphi\right)(1/b)\right]\right|$$
$$\leq\frac{(b-a)\left\|g\right\|_{\infty}}{4ab\Gamma(\alpha)}\left\{\begin{array}{c}\left(\int_0^1\left|\int_{1/L(t)}^{1/U(t)}\psi(x)dx\right|dt\right)^{1-\frac{1}{q}}\times\\ \left(\begin{array}{c}\int_0^1\left|\int_{1/L(t)}^{1/U(t)}\psi(x)dx\right|\times\\ L^{2q}(t)\sup\{\left|f'(H)\right|^{q},\left|f'(a)\right|^{q}\}dt\end{array}\right)^{\frac{1}{q}}\\ \\ +\left(\int_0^1\left|\int_{1/L(t)}^{1/U(t)}\psi(x)dx\right|dt\right)^{1-\frac{1}{q}}\times\\ \left(\begin{array}{c}\int_0^1\left|\int_{1/L(t)}^{1/U(t)}\psi(x)dx\right|\times\\ U^{2q}\sup\{\left|f'(H)\right|^{q},\left|f'(b)\right|^{q}\}dt\end{array}\right)^{\frac{1}{q}}\end{array}\right\}.$$

If we write the following integral in (2.7)

$$\left|\int_{1/L(t)}^{1/U(t)}\psi(x)dx\right|=\frac{2^{1-\alpha}}{\alpha}\left(\frac{b-a}{ab}\right)^{\alpha}\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right],$$

we have

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J^{\alpha}_{1/b^{+}}\left(g\circ\varphi\right)(1/a)+J^{\alpha}_{1/a^{-}}\left(g\circ\varphi\right)(1/b)\right]\right.\tag{2.8}$$
$$\left.-\left[J^{\alpha}_{1/b^{+}}\left(fg\circ\varphi\right)(1/a)+J^{\alpha}_{1/a^{-}}\left(fg\circ\varphi\right)(1/b)\right]\right|$$
$$\leq\frac{(b-a)^{\alpha+1}\left\|g\right\|_{\infty}}{(2ab)^{\alpha+1}\Gamma(\alpha+1)}\left(\frac{2^{\alpha+1}-2}{\alpha+1}\right)^{1-\frac{1}{q}}\left\{\begin{array}{c}\left(\begin{array}{c}\int_0^1\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right]\times\\ L^{2q}\sup\{\left|f'(H)\right|^{q},\left|f'(a)\right|^{q}\}dt\end{array}\right)^{\frac{1}{q}}\\ \\ +\left(\begin{array}{c}\int_0^1\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right]\times\\ U^{2q}\sup\{\left|f'(H)\right|^{q},\left|f'(b)\right|^{q}\}dt\end{array}\right)^{\frac{1}{q}}\end{array}\right\}.$$

If we use $a^r+b^r\leq 2^{1-r}(a+b)^r$, $a,b>0$ inequality in (2.8), the proof is completed. □

Corollary 2.

i. *If we take* $q=1$, $\alpha=1$ *and* $|f'|$ *that be increasing function in* (2.3), *we get*

$$\left|\left[\frac{f(a)+f(b)}{2}\right]\int_a^b \frac{g(x)}{x^2}dx-\int_a^b f(x)\frac{g(x)}{x^2}dx\right| \tag{2.9}$$
$$\leq 2\left[\left(ln\left(\frac{2b}{a+b}\right)-\frac{b-a}{2b}\right)|f'(H)|+\left(ln\left(\frac{2a}{a+b}\right)+\frac{b-a}{2a}\right)|f'(b)|\right].$$

ii. *If we take* $q=1$, $g(x)=1$ *and* $|f'|$ *that be increasing function in* (2.3), *we get*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)-\frac{(ab)^{\alpha}\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{1/b^+}(f\circ\varphi)(1/a)\right.\right. \tag{2.10}$$
$$\left.\left.+J^{\alpha}_{1/a^-}(f\circ\varphi)(1/b)\right]\right|\leq\frac{b-a}{2^{\alpha+2}ab}\left(A_1(t,\alpha,1)|f'(H)|+A_2(t,\alpha,1)|f'(b)|\right).$$

iii. *If we take* $\alpha=1$, $g(x)=1$ *and* $|f'|$ *that be increasing function in* (2.3), *we get*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)-\frac{ab}{(b-a)}\int_a^b\frac{f(x)}{x^2}dx\right| \tag{2.11}$$
$$\leq\frac{2(b-a)^{1-\frac{2}{q}}ab}{(a+b)^{2-\frac{2}{q}}}\left\{a^{2q-2}\left[\frac{H^{2-2q}-a^{2-2q}}{2-2q}-a\left(\frac{H^{1-2q}-a^{1-2q}}{1-2q}\right)\right]|f'(H)|^q\right.$$
$$\left.+b^{2q-2}\left[\frac{H^{2-2q}-b^{2-2q}}{2-2q}-b\left(\frac{H^{1-2q}-b^{1-2q}}{1-2q}\right)\right]|f'(b)|^q\right\}^{\frac{1}{q}}.$$

In the second part of the study, the left side of Hermite-Hadamard inequality will be discussed.

Theorem 9. *Let* $f:I\subseteq(0,\infty)\longrightarrow\mathbb{R}$ *be differentiable mapping on* I^o, $a,b\in I$ *with* $a<b$. *If* $h:[a,b]\longrightarrow[0,\infty)$ *is a differentiable function and* $|f'|^q$,$q\geq1$, *is harmonically quasi convex on* $[a,b]$, *the following inequality holds*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)h(a)-h(b)f(\frac{2ab}{a+b})+\frac{1}{2}\left[\int_a^b f(x)h'(x)dx\right.\right. \tag{2.12}$$
$$\left.\left.+\int_a^b f(x)h'(\frac{Hx}{2x-H})\left(\frac{H}{2x-H}\right)^2dx\right]\right|$$

$$\leq \frac{b-a}{4ab}\left\{\begin{array}{c}\left(\int_0^1 |h(L(t))-h(U(t))+h(b)|\,dt\right)^{1-\frac{1}{q}}\times \\ \left(\int_0^1 |h(L(t))-h(U(t))+h(b)|\,L^2(t)\sup\{|f'(H)|,|f'(a)|\}dt\right)^{\frac{1}{q}} \\ \\ +\left(\int_0^1 |h(L(t))-h(U(t))+h(b)|\,dt\right)^{1-\frac{1}{q}}\times \\ \left(\int_0^1 |h(L(t))-h(U(t))+h(b)|\,U^2(t)\sup\{|f'(H)|,|f'(b)|\}dt\right)^{\frac{1}{q}}\end{array}\right\}.$$

Proof. We take absolute value and then use power mean inequality in (1.14),

$$\left|\left(\frac{f(a)+f(b)}{2}\right)h(a)-h(b)f(\frac{2ab}{a+b})+\frac{1}{2}\left[\int_a^b f(x)h'(x)dx\right.\right. \tag{2.13}$$

$$\left.\left.+\int_a^b f(x)h'(\frac{Hx}{2x-H})\left(\frac{H}{2x-H}\right)^2 dx\right]\right|$$

$$\leq \frac{b-a}{4ab}\left\{\begin{array}{c}\left(\int_0^1 |h(L(t))-h(U(t))+h(b)|\,dt\right)^{1-\frac{1}{q}}\times \\ \left(\int_0^1 |h(L(t))-h(U(t))+h(b)|\left|f'\left(L(t)\right)L^{2q}(t)\right|dt\right)^{\frac{1}{q}} \\ \\ +\left(\int_0^1 |h(L(t))-h(U(t))+h(b)|\,dt\right)^{1-\frac{1}{q}}\times \\ \left(\int_0^1 |h(L(t))-h(U(t))+h(b)|\left|f'\left(U(t)\right)U^{2q}(t)\right|dt\right)^{\frac{1}{q}}\end{array}\right\}.$$

By using $|f'|$ that is harmonically quasi convex, the proof is completed. □

Corollary 3. *Let $g:[a,b]\longrightarrow[0,\infty)$ be a positive continuous mapping and harmonically symmetric with respect to $\frac{2ab}{a+b}$, $a<b$. If $h(t)=\int_{1/t}^{1/a}\psi(x)\left(g\circ\varphi\right)(x)dx$ $\psi(x)=\left[\left(x-\frac{1}{b}\right)^{\alpha-1}+\left(\frac{1}{a}-x\right)^{\alpha-1}\right]$ for all $t\in[a,b]$, $\alpha>0$ in Theorem 9, we obtain*

$$\begin{aligned} & \Big|\Big[J^{\alpha}_{1/b^{+}}\left(fg\circ\varphi\right)(1/a)+J^{\alpha}_{1/a^{-}}\left(fg\circ\varphi\right)(1/b)\Big] \\ & -f\left(\frac{2ab}{a+b}\right)\Big[J^{\alpha}_{1/b^{+}}\left(g\circ\varphi\right)(1/a)+J^{\alpha}_{1/a^{-}}\left(g\circ\varphi\right)(1/b)\Big]\Big| \\ & \leq \frac{(b-a)^{\alpha+1}\left\|g\right\|_{\infty}}{2(ab)^{\alpha+1}\Gamma\left(\alpha+1\right)}\left[2-\frac{4}{\alpha+1}+\frac{2^{2-\alpha}}{\alpha+1}\right]^{1-\frac{1}{q}} \end{aligned} \tag{2.14}$$

$$\left[B_1(t,\alpha;q)\sup\{\left|f'(H)\right|^q,\left|f'(a)\right|^q\}+B_2(t,\alpha;q)\sup\{\left|f'(H)\right|^q,\left|f'(b)\right|^q\}\right]^{\frac{1}{q}}$$

where

$$\begin{aligned} B_1(t,\alpha;q) &= \int_0^1\left[1-\left(\frac{1+t}{2}\right)^{\alpha}+\left(\frac{1-t}{2}\right)^{\alpha}\right]L^{2q}(t)dt, \\ B_2(t,\alpha;q) &= \int_0^1\left[1-\left(\frac{1+t}{2}\right)^{\alpha}+\left(\frac{1-t}{2}\right)^{\alpha}\right]U^{2q}(t)dt. \end{aligned}$$

Proof. If we use $h(t)=\int\limits_{1/t}^{1/a}\psi(x)\left(g\circ\varphi\right)(x)dx$, $\varphi(x)=\frac{1}{x}$ in (2.13), we get

$$\begin{aligned} & \left|\frac{1}{2}\left[\int_a^b\left[\left(\frac{1}{a}-x\right)^{\alpha-1}+\left(x-\frac{1}{b}\right)^{\alpha-1}\right]\frac{f(x)g(x)}{x^2}dx\right.\right. \\ & \left.+\int_a^b\left[\left(\frac{1}{a}-x\right)^{\alpha-1}+\left(x-\frac{1}{b}\right)^{\alpha-1}\right]\frac{f(x)g(\frac{Hx}{2x-H})}{x^2}dx\right] \\ & \left.-f(\frac{2ab}{a+b})\int_a^b\left[\left(\frac{1}{a}-x\right)^{\alpha-1}+\left(x-\frac{1}{b}\right)^{\alpha-1}\right]\frac{g(x)}{x^2}dx\right| \end{aligned} \tag{2.15}$$

and from $g(x)$ is harmonically symmetric function with respect to $x=2ab/a+b$

$$=\Gamma(\alpha)\left|\begin{array}{c}\left[J^{\alpha}_{1/b^{+}}\left(fg\circ\varphi\right)(1/a)+J^{\alpha}_{1/a^{-}}\left(fg\circ\varphi\right)(1/b)\right] \\ -\left[J^{\alpha}_{1/b^{+}}g\circ\varphi(1/a)+J^{\alpha}_{1/a^{-}}g\circ\varphi(1/b)\right]f\left(\frac{2ab}{a+b}\right)\end{array}\right|. \tag{2.16}$$

On the other hand,

$$\leq \frac{b-a}{4ab}\left(\int_0^1 \left| \begin{array}{c} \int_{1/L(t)}^{1/a} \psi(x)\left(g\circ\varphi\right)(x)dx \\ -\int_{1/U(t)}^{1/a} \psi(x)\left(g\circ\varphi\right)(x)dx \\ +\int_{1/b}^{1/a} \psi(x)\left(g\circ\varphi\right)(x)dx \end{array} \right| dt\right)^{1-\frac{1}{q}} \times \tag{2.17}$$

$$\left\{\left(\int_0^1 \left| \begin{array}{c} \int_{1/L(t)}^{1/a} \psi(x)\left(g\circ\varphi\right)(x)dx \\ -\int_{1/U(t)}^{1/a} \psi(x)\left(g\circ\varphi\right)(x)dx \\ +\int_{1/b}^{1/a} \psi(x)\left(g\circ\varphi\right)(x)dx \end{array} \right| \sup\{\left|f'(H)\right|^q, \left|f'(a)\right|^q\} L^{2q}(t)dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 \left| \begin{array}{c} \int_{1/L(t)}^{1/a} \psi(x)\left(g\circ\varphi\right)(x)dx \\ -\int_{1/U(t)}^{1/a} \psi(x)\left(g\circ\varphi\right)(x)dx \\ +\int_{1/b}^{1/a} \psi(x)\left(g\circ\varphi\right)(x)dx \end{array} \right| \sup\{\left|f'(H)\right|^q, \left|f'(b)\right|^q\} U^{2q}(t)dt\right)^{\frac{1}{q}}\right\}.$$

Since g is harmonically symmetric function with respect to $\frac{2ab}{a+b}$, we have

$$\left| \begin{array}{c} \left[J^{\alpha}_{1/b^+}\left(fg\circ\varphi\right)(1/a) + J^{\alpha}_{1/a^-}\left(fg\circ\varphi\right)(1/b)\right] \\ -\left[J^{\alpha}_{1/b^+} g\circ\varphi(1/a) + J^{\alpha}_{1/a^-} g\circ\varphi(1/b)\right] f\left(\frac{2ab}{a+b}\right) \end{array} \right| \tag{2.18}$$

$$\leq \frac{b-a}{2ab\Gamma(\alpha)}\left(\int_0^1 \left| \int_{1/L(t)}^{1/a} \psi(x)\left(g\circ\varphi\right)(x)dx \right| dt\right)^{1-\frac{1}{q}} \times$$

$$\left\{\left(\int_0^1 \left| \int_{1/L(t)}^{1/a} \psi(x)\left(g\circ\varphi\right)(x)dx \right| \sup\{\left|f'(H)\right|^q, \left|f'(a)\right|^q\} L^{2q}(t)dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1\left|\int_{1/L(t)}^{1/a}\psi(x)\left(g\circ\varphi\right)(x)dx\right|\sup\{\left|f'(H)\right|^q,\left|f'(b)\right|^q\}U^{2q}(t)dt\right)^{\frac{1}{q}}\Bigg\}$$

$$\leq\frac{(b-a)\left\|g\right\|_\infty}{2ab\Gamma(\alpha)}\left(\int_0^1\left|\int_{1/L(t)}^{1/a}\psi(x)dx\right|dt\right)^{1-\frac{1}{q}}\times$$

$$\Bigg\{\left(\int_0^1\left(\int_{1/L(t)}^{1/a}\left|\psi(x)\right|dx\right)\sup\{\left|f'(H)\right|^q,\left|f'(a)\right|^q\}L^{2q}(t)dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\left(\int_{1/L(t)}^{1/a}\left|\psi(x)\right|dx\right)\sup\{\left|f'(H)\right|^q,\left|f'(b)\right|^q\}U^{2q}(t)dt\right)^{\frac{1}{q}}\Bigg\}.$$

By using $a^r+b^r\leq 2^{1-r}(a+b)^r$, $r\leq 1$ and $a,b>0$ inequality, we get inequality as follows

$$\left|\begin{array}{c}\left[J^\alpha_{1/b^+}\left(fg\circ\varphi\right)(1/a)+J^\alpha_{1/a^-}\left(fg\circ\varphi\right)(1/b)\right]\\-\left[J^\alpha_{1/b^+}g\circ\varphi(1/a)+J^\alpha_{1/a^-}g\circ\varphi(1/b)\right]f\left(\frac{2ab}{a+b}\right)\end{array}\right|\tag{2.19}$$

$$\leq\frac{(b-a)\left\|g\right\|_\infty}{2ab\Gamma(\alpha)}\left(2\int_0^1\left|\int_{1/L(t)}^{1/a}\psi(x)dx\right|dt\right)^{1-\frac{1}{q}}\times$$

$$\Bigg\{\left(\int_0^1\left(\int_{1/L(t)}^{1/a}\left|\psi(x)\right|dx\right)\sup\{\left|f'(H)\right|^q,\left|f'(a)\right|^q\}L^{2q}(t)dt\right.$$

$$\left.+\int_0^1\left(\int_{1/L(t)}^{1/a}\left|\psi(x)\right|dx\right)\sup\{\left|f'(H)\right|^q,\left|f'(b)\right|^q\}U^{2q}(t)dt\right)^{\frac{1}{q}}\Bigg\}.$$

In the last inequality, we simply calculate integral as follows

$$\int_{1/L(t)}^{1/a} \psi(x)dx = \frac{(b-a)^{\alpha}}{\alpha(ab)^{\alpha}}\left[1-\left(\frac{1+t}{2}\right)^{\alpha}+\left(\frac{1-t}{2}\right)^{\alpha}\right]. \tag{2.20}$$

If we use (2.20) with in (2.19) inequality, the proof is completed. □

Corollary 4.

i. *I we take $q = 1$, $\alpha = 1$ and and $|f'|$ that is increasing function in* (2.14) *inequality, we get*

$$\left|\int_a^b f(x)\frac{g(x)}{x^2}dx - f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx\right| \tag{2.21}$$

$$\leq \|g\|_{\infty}\left[\left(\frac{b-a}{a+b}+ln\left(\frac{a+b}{2b}\right)\right)\sup\{|f'(H)|,|f'(a)|\}\right.$$
$$\left.+\left(ln\left(\frac{a+b}{2a}\right)-\frac{b-a}{a+b}\right)\sup\{|f'(H)|,|f'(b)|\}\right].$$

ii. *If we take $g(x) = 1$, $q = 1$ and $|f'|^q$ that is increasing function in* (2.14) *inequality, we obtain*

$$\left|\frac{(ab)^{\alpha}\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{1/b^{+}}(f\circ\varphi)(1/a)+J^{\alpha}_{1/a^{-}}(f\circ\varphi)(1/b)\right]-f\left(\frac{2ab}{a+b}\right)\right| \tag{2.22}$$
$$\leq \frac{b-a}{4ab}\left[B_1(t,\alpha;1)|f'(H)|+B_2(t,\alpha;1)|f'(b)|\right].$$

iii. *If we take $\alpha = 1$, $g(x) = 1$ and $|f'|^q$ that is increasing function, we get*

$$\left|\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx - f\left(\frac{2ab}{a+b}\right)\right| \tag{2.23}$$
$$\leq \frac{(b-a)^{1-\frac{2}{q}}ab}{(a+b)^{2-\frac{2}{q}}}\left\{a^{2q-2}\left[H\left(\frac{H^{1-2q}-a^{1-2q}}{1-2q}\right)-\frac{H^{2-2q}-a^{2-2q}}{2-2q}\right]|f'(H)|^q\right.$$
$$\left.+b^{2q-2}\left[H\left(\frac{H^{1-2q}-b^{1-2q}}{1-2q}\right)-\frac{H^{2-2q}-b^{2-2q}}{2-2q}\right]|f'(b)|^q\right\}^{\frac{1}{q}}.$$

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