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DERIVATIVES OF SASAKIAN METRIC ${}^{S}g$ ON COTANGENT BUNDLE

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ABSTRACT. In this paper, we define a Sasakian metric ${}^{S}g$ on cotangent bundle T^*M^n, which is completely determined by its action on complete lifts of vector fields. Later, we obtain the covariant and Lie derivatives applied to Sasakian metrics with respect to the complete and vertical lifts of vector and kovector fields, respectively.

1. Introduction

Riemannian manifolds and the tangent bundles of differentiable manifolds are very important in many areas of mathematics. This fields also studied a lot of authors [1, 2, 3, 9, 11, 15, 16]. The geometry of tangent bundles goes back to the fundamental paper [14] of Sasaki published in 1958. Sasakian metrics (diagonal lifts of metrics) on tangent bundles were also studied in [8, 9, 17]. In a more general case of tensor bundles of type $(1, q)$, $(0, q)$ and (p, q). Sasakian metrics and their geodesics are considered in [3, 12]. Cotangent bundle is dual of the tangent bundle. Because of this duality, some of the geometric results are similar to each other. The most significant difference between them is construction of lifts (see [17] for more details). In this paper, we define a Sasakian metric ${}^{S}g$ on cotangent bundle T^*M^n, which is completely determined by its action on complete lifts of vector fields. Later, we obtain the covariant and Lie derivatives applied to Sasakian metrics with respect to the complete and vertical lifts of vector and covector fields, respectively.

Let M^n be an $n-$dimensional Riemannian manifold of class C^∞ and with metric g, T^*M^n its cotangent bundle and π the natural projection $T^*M^n \to M^n$. A system of local coordinates $(U, x^i), i = 1, ..., n$ in M^n induces on T^*M^n. A system of local coordinates $(\pi^{-1}(U), x^i, x^{\bar{\imath}} = p_i), \bar{\imath} := n + i = n + 1, ..., 2n$, where $x^{\bar{\imath}} = p_i$ is the

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component of covectors p in each cotangent space $T^*_xM^n$, $x \in U$ with respect to the natural coframe $\{dx^i\}$.

We denote by $\Im^r_s(M^n)(\Im^r_s(T^*M^n))$ the module over $F(M^n)(F(T^*M^n))$ of C^∞ tensor fields of type (r,s), where $F(M^n)(F(T^*M^n))$ is the ring of real-valued C^∞ functions on $M^n(T^*M^n)$.

Let $X = X^i\frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be the local expressions in $U \subset M^n$ of a vector and a covector $(1-$form) fields $X \in \Im^1_0(M^n)$ and $\omega \in \Im^0_1(M^n)$, respectively. Then the complete and horizontal lifts $X^C, X^H \in \Im^1_0(T^*M^n)$ of $X \in \Im^1_0(M^n)$ and the vertical lift $\omega^V \in \Im^1_0(T^*M^n)$ of $\omega \in \Im^0_1(M^n)$ are given, respectively by

$$X^C = X^i\frac{\partial}{\partial x^i} - \sum_i p_h \partial_i X^h \frac{\partial}{\partial x^{\bar{i}}} \tag{1}$$

$$X^H = X^i\frac{\partial}{\partial x^i} - \sum_i p_h \Gamma^h_{ij} X^j \frac{\partial}{\partial x^{\bar{i}}} \tag{2}$$

$$\omega^V = \sum_i w_i \frac{\partial}{\partial x^{\bar{i}}} \tag{3}$$

with respect to the natural frame$\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}\right\}$, where Γ^h_{ij} are the components of the Levi-Civita connection ∇_g on M^n [10, 17].

For each $x \in M^n$, the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space $\pi^{-1}(x) = T^*_xM^n$ by

$$g^{-1}(\omega,\theta) = g^{ij}\omega_i\theta_j$$

for all $\omega, \theta \in \Im^0_1(M^n)$.

A Sasakian metric ${}^Sg$ is defined on T^*M^n by the following three equations:

$${}^Sg(\omega^V,\theta^V) = (g^{-1}(\omega,\theta))^V = g^{-1}(\omega,\theta)o\pi \tag{4}$$

$${}^Sg(\omega^V, Y^H) = 0 \tag{5}$$

$${}^Sg(X^H, Y^H) = (g(X,Y))^V = g(X,Y)o\pi \tag{6}$$

for any $X, Y \in \Im^1_0(M^n)$ and $\omega, \theta \in \Im^0_1(M^n)$. Since any tensor field of type $(0,2)$ on T^*M^n is completely determined by its action on vector fields of type X^H and ω^V (see p. 280 of [17]), it follows that ${}^Sg$ is completely determined by its equations (4),(5) and (6).

We now see , from (1) and (2), that the complete lift X^C of $X \in \Im^1_0(M^n)$ is expressed by

$$X^C = X^H - (p(\nabla X))^V, \tag{7}$$

where $p(\nabla X) = p_i\left(\nabla_h X^i\right)dx^h$.

Using (4),(5),(6) and (7) , we have

$${}^Sg(X^C, Y^C) = (g(X,Y))^V + \left(g^{-1}(p(\nabla X), p(\nabla Y))\right)^V, \tag{8}$$

where $g^{-1}(p(\nabla X), p(\nabla Y)) = g^{ij}(p_l\nabla_i X^l)(p_k\nabla_j Y^k)$ [13].

Since the tensor field ${}^Sg \in \Im^0_2(T^*(M^n))$ is completely determined also by its action on vector fields of type X^C and Y^C (see p.237 of [17]), Sasakian metric ${}^Sg$

on $T^{*}(M^{n})$ is completely determined by the condition (8). Similarly, we get the following results

$$\begin{aligned} {}^{S}g(X^{C},\omega^{V}) &= {}^{S}g(X^{H}-(p\,(\nabla X))^{V},\omega^{V}) \\ &= {}^{S}g(X^{H},\omega^{V}) - {}^{S}g((p\,(\nabla X))^{V},\omega^{V}) \\ &= -\left(g^{-1}\left(p\,(\nabla X)\,,\omega\right)\right)^{V} \end{aligned} \tag{9}$$

$$\begin{aligned} {}^{S}g(\omega^{V},X^{C}) &= {}^{S}g(\omega^{V},X^{H}-(p\,(\nabla X))^{V}) \\ &= {}^{S}g(\omega^{V},X^{H}) - {}^{S}g(\omega^{V},(p\,(\nabla X))^{V}) \\ &= -\left(g^{-1}\left(\omega,p\,(\nabla X)\right)\right)^{V} \end{aligned} \tag{10}$$

$${}^{S}g(\omega^{V},\theta^{V}) = \left(g^{-1}\left(\omega,\theta\right)\right)^{V} \tag{11}$$

2. Main Results

2.1. Covariant Derivation of Sasakian metric ${}^{S}g$ with respect to vertical and complete lifts.

Definition 1. *Let M^{n} be an $n-$dimensional differentiable manifold. Differential transformation of algebra $T(M^{n})$, defined by $D=\nabla_{X}:T(M^{n})\rightarrow T(M^{n})$, $X\in\Im_{0}^{1}(M^{n})$,*

is called as covariant derivation with respect to vector field X if

$$\begin{aligned} \nabla_{fX+gY}t &= f\nabla_{X}t+g\nabla_{Y}t, \\ \nabla_{X}f &= Xf, \end{aligned} \tag{12}$$

where $\forall f,g\in\Im_{0}^{0}(M^{n}),\forall X,Y\in\Im_{0}^{1}(M^{n}),\forall t\in\Im(M^{n})$.

On the other hand, a transformation defined by

$$\nabla:\Im_{0}^{1}(M^{n})\times\Im_{0}^{1}(M^{n})\rightarrow\Im_{0}^{1}(M^{n}), \tag{13}$$

is called as an affine connection [11, 17].

Proposition 2. *Covariant differentiation with respect to the complete lift ∇^{C} of a symmetric affine connection ∇ in M^{n} to $T^{*}(M^{n})$ has the following properties:*

$$\begin{aligned} \nabla^{C}_{\omega^{V}}\theta^{V} &= 0,\ \nabla^{C}_{\omega^{V}}Y^{C}=-\gamma(\omega o(\nabla Y))=-(p(\omega o(\nabla Y)))^{V},\ \nabla^{C}_{X^{C}}\theta^{V}=(\nabla_{X}\theta)^{V}, \\ \nabla^{C}_{X^{C}}Y^{C} &= (\nabla_{X}Y)^{C}+\gamma(\nabla(\nabla_{X}Y+\nabla_{Y}X))-\gamma(\nabla_{X}\nabla Y+\nabla_{Y}\nabla X) \\ &= (\nabla_{X}Y)^{C}+(p((\nabla(\nabla_{X}Y+\nabla_{Y}X))-(\nabla_{X}\nabla Y+\nabla_{Y}\nabla X)))^{V} \end{aligned}$$

for $X,Y\in\Im_{0}^{1}(M^{n})$, $\theta,\omega\in\Im_{1}^{0}(M^{n})$ [17].

Proposition 3. *Covariant differentiation with respect to the horizontal lift ∇^{H} of a symmetric affine connection ∇ in M^{n} to $T^{*}(M^{n})$ satisfies*

$$\begin{aligned} \nabla^{H}_{X^{H}}Y^{H} &= (\nabla_{X}Y)^{H},\ \nabla^{H}_{\theta^{V}}\omega^{V}=0, \\ \nabla^{H}_{X^{H}}\omega^{V} &= (\nabla_{X}\omega)^{V},\ \nabla^{H}_{\theta^{V}}Y^{H}=0, \end{aligned}$$

for any $X, Y \in \Im_0^1(M^n)$, $\theta, \omega \in \Im_1^0(M^n)$ [17].

Theorem 4. *Let* ${}^S g$ *be Sasakian metric, is defined by (8),(9),(10),(11) and the complete lift* ∇^C *of symmetric affine connection* ∇ *in* M^n *to* $T^*(M^n)$. *From proposition (2) and proposition (3), we get the following results*

$$
\begin{aligned}
i)\ (\nabla^C_{\omega^V}{}^S g)\left(\theta^V, \xi^V\right) &= 0, \\
ii)\ (\nabla^C_{Z^C}{}^S g)\left(\omega^V, \theta^V\right) &= \left(\left(\nabla_Z g^{-1}\right)(\omega, \theta)\right)^V, \\
iii)\ (\nabla^C_{\omega^V}{}^S g)\left(\theta^V, Y^C\right) &= (g^{-1}(\omega, p(\omega o(\nabla Y))))^V, \\
iv)\ (\nabla^C_{Z^C}{}^S g)\left(\omega^V, Y^C\right) &= -(Zg^{-1}(\omega, p(\nabla Y)))^V + \left(g^{-1}(\nabla_Z \omega, p\,(\nabla X))\right)^V \\
&\quad + \left(g^{-1}(\omega, p\,(\nabla(\nabla_Z Y)))\right)^V \\
&\quad -(g^{-1}(\omega, p(\nabla(\nabla_Z Y + \nabla_Y Z) - (\nabla_Z \nabla Y + \nabla_Y \nabla Z))))^V, \\
v)\ (\nabla^C_{\omega^V}{}^S g)\left(X^C, \theta^V\right) &= \left(g^{-1}\left(p\left(\omega o\left(\nabla X\right)\right), \theta\right)\right)^V, \\
vi)\ (\nabla^C_{Z^C}{}^S g)\left(X^C, \omega^V\right) &= -\left(Zg^{-1}\left(p\left(\nabla X\right), \omega\right)\right)^V + (g^{-1}(p(\nabla(\nabla_Z X)), \omega))^V \\
&\quad -(g^{-1}(p(\nabla(\nabla_Z X + \nabla_X Z) - (\nabla_Z \nabla X + \nabla_X \nabla Z)), \omega))^V \\
&\quad +(g^{-1}(p(\nabla X), \nabla_Z \omega))^V, \\
vii)\ (\nabla^C_{\omega^V}{}^S g)\left(X^C, Y^C\right) &= -(g^{-1}(p(\omega o(\nabla X)), p(\nabla Y)))^V - (g^{-1}(p(\nabla X), p(\omega o(\nabla Y))))^V, \\
viii)\ (\nabla^C_{Z^C}{}^S g)(X^C, Y^C) &= ((\nabla_Z g)\,(X, Y))^V + (Z(g^{-1}(p(\nabla X), P(\nabla Y)))^V \\
&\quad -(g^{-1}(p(\nabla(\nabla_Z X)), p(\nabla Y)))^V \\
&\quad +(g^{-1}(p(\nabla(\nabla_Z X)) - (\nabla_Z \nabla X + \nabla_X \nabla Z)), p(\nabla Y)))^V \\
&\quad -(g^{-1}(p(\nabla X), p(\nabla(\nabla_Z Y)))^V \\
&\quad +(g^{-1}(p(\nabla X), p(\nabla(\nabla_Z Y + \nabla_Y Z) - (\nabla_Z \nabla Y + \nabla_Y \nabla Z))))^V
\end{aligned}
$$

where the complete and horizontal lifts $X^C, X^H \in \Im_0^1(T^*M^n)$ *of* $X \in \Im_0^1(M^n)$ *and the vertical lift* $\omega^V \in \Im_0^1(T^*M^n)$ *of* $\omega \in \Im_1^0(M^n)$ *defined by (1),(2),(3) respectively.*

Proof. i)

$$\begin{aligned}
(\nabla^{C}_{\omega^{V}}{}^{S}g)\left(\theta^{V},\xi^{V}\right) &= \nabla^{C}_{\omega^{V}}{}^{S}g\left(\theta^{V},\xi^{V}\right) -^{S} g(\nabla^{C}_{\omega^{V}}\ \theta^{V},\xi^{V}) -^{S} g\left(\theta^{V},\nabla^{C}_{\omega^{V}}\xi^{V}\right) \\
&= \nabla^{C}_{\omega^{V}}\left(g^{-1}(\theta,\xi)\right)^{V} \\
&= \omega^{V}\left(g^{-1}(\theta,\xi)\right)^{V} \\
&= 0
\end{aligned}$$

ii)

$$\begin{aligned}
(\nabla^{C}_{Z^{C}}{}^{S}g)\left(\omega^{V},\theta^{V}\right) &= \nabla^{C}_{Z^{C}}{}^{S}g\left(\omega^{V},\theta^{V}\right) -^{S} g\left(\nabla^{C}_{Z^{C}}\omega^{V},\theta^{V}\right) -^{S} g\left(\omega^{V},\nabla^{C}_{Z^{C}}\theta^{V}\right) \\
&= Z^{C}\left(g^{-1}(\omega,\theta)\right)^{V} - \left(g^{-1}(\nabla_{Z}\omega,\theta)\right)^{V} - \left(g^{-1}(\omega,(\nabla_{Z}\theta))\right)^{V} \\
&= \left(Zg^{-1}(\omega,\theta)\right)^{V} - \left(g^{-1}(\nabla_{Z}\omega,\theta)\right)^{V} - \left(g^{-1}(\omega,\nabla_{Z}\theta)\right)^{V} \\
&= \left(\nabla_{Z}g^{-1}(\omega,\theta)\right) - \left(g^{-1}(\nabla_{Z}\omega,\theta)\right)^{V} - \left(g^{-1}(\omega,\nabla_{Z}\theta)\right)^{V} \\
&= \left(\left(\nabla_{Z}g^{-1}\right)(\omega,\theta)\right)^{V}
\end{aligned}$$

iii)

$$\begin{aligned}
(\nabla^{C}_{\omega^{V}}{}^{S}g)\left(\theta^{V},Y^{C}\right) &= \nabla^{C}_{\omega^{V}}{}^{S}g\left(\theta^{V},Y^{C}\right) -^{S} g\left(\nabla^{C}_{\omega^{V}}\theta^{V},Y^{C}\right) -^{S} g\left(\omega^{V},\nabla^{C}_{\omega^{V}}Y^{C}\right) \\
&= -\nabla^{C}_{\omega^{V}}\left(g^{-1}(\theta,p(\theta Y))\right)^{V} +^{S} g\left(\omega^{V},\gamma(\omega o(\nabla Y))\right) \\
&= -\omega^{V}\left(g^{-1}(\theta,p(\nabla Y))\right)^{V} +^{S} g\left(\omega^{V},(p(\omega o(\nabla Y)))^{V}\right) \\
&= (g^{-1}(\omega,p(\omega o(\nabla Y))))^{V}
\end{aligned}$$

iv)

$$\begin{aligned}
(\nabla^{C}_{Z^{C}}{}^{S}g)\left(\omega^{V},Y^{C}\right) &= \nabla^{C}_{Z^{C}}{}^{S}g\left(\omega^{V},Y^{C}\right) -^{S} g\left(\nabla^{C}_{Z^{C}}\omega^{V},Y^{C}\right) -^{S} g\left(\omega^{V},\nabla^{C}_{Z^{C}}Y^{C}\right) \\
&= -\nabla^{C}_{Z^{C}}\left(g^{-1}(\omega,p(\nabla Y))\right)^{V} -^{S} g\left((\nabla_{Z}\omega)^{V},Y^{C}\right) \\
&\quad -^{S}g\left(\omega^{V},(\nabla_{Z}Y)^{C} + \gamma(\nabla(\nabla_{Z}Y+\nabla_{Y}Z)) - \gamma(\nabla_{Z}\nabla Y+\nabla_{Y}\nabla Z)\right) \\
&= -Z^{C}\left(g^{-1}(\omega,p(\nabla Y))\right)^{V} + \left(g^{-1}(\nabla_{Z}\omega,p(\nabla X))\right)^{V} -^{S} g\left(\omega^{V},(\nabla_{Z}Y)^{C}\right) \\
&\quad -^{S}g\left(\omega^{V},(p(\nabla(\nabla_{Z}Y+\nabla_{Y}Z) - (\nabla_{Z}\nabla Y+\nabla_{Y}\nabla Z)))^{V}\right) \\
&= -(Zg^{-1}(\omega,p(\nabla Y)))^{V} + \left(g^{-1}(\nabla_{Z}\omega,p(\nabla X))\right)^{V} + \left(g^{-1}(\omega,p(\nabla(\nabla_{Z}Y))\right)^{V} \\
&\quad -(g^{-1}(\omega,p(\nabla(\nabla_{Z}Y+\nabla_{Y}Z) - (\nabla_{Z}\nabla Y+\nabla_{Y}\nabla Z))))^{V}
\end{aligned}$$

v)

$$\begin{aligned}
\left(\nabla^{C}_{\omega^V}{}^{S}g\right)\left(X^C,\theta^V\right) &= \nabla^{C}_{\omega^V}{}^{S}g\left(X^C,\theta^V\right) -^{S} g\left(\nabla^{C}_{\omega^V}X^C,\theta^V\right) -^{S} g\left(X^C,\nabla^{C}_{\omega^V}\theta^V\right)\\
&= -\nabla^{C}_{\omega^V}\left(g^{-1}\left(p\left(\nabla X\right),\theta\right)\right)^V +^{S} g((p\left(\omega o\left(\nabla X\right)\right))^V,\theta^V)\\
&= -\omega^V\left(g^{-1}\left(p\left(\nabla X\right),\theta\right)\right)^V + \left(g^{-1}\left(p\left(\omega o\left(\nabla X\right)\right),\theta\right)\right)^V\\
&= \left(g^{-1}\left(p\left(\omega o\left(\nabla X\right)\right),\theta\right)\right)^V
\end{aligned}$$

vi)

$$\begin{aligned}
\left(\nabla^{C}_{Z^C}{}^{S}g\right)\left(X^C,\omega^V\right) &= \nabla^{C}_{Z^C}{}^{S}g\left(X^C,\omega^V\right) - {}^{S}g\left(\nabla^{C}_{Z^C}X^C,\omega^V\right) - {}^{S}g\left(X^C,\nabla^{C}_{Z^C}\omega^V\right)\\
&= -Z^C\left(g^{-1}\left(p\left(\nabla X\right),\omega\right)\right)^V -^{S} g((\nabla_Z X)^C,\omega^V)\\
&\quad -^{S}g((p(\nabla(\nabla_Z X+\nabla_X Z)-(\nabla_Z\nabla X+\nabla_X\nabla Z)))^V,\omega^V)\\
&\quad +(g^{-1}(p(\nabla X),\nabla_Z\omega))^V\\
&= -\left(Zg^{-1}\left(p\left(\nabla X\right),\omega\right)\right)^V + (g^{-1}(p(\nabla(\nabla_Z X)),\omega))^V\\
&\quad -(g^{-1}(p(\nabla(\nabla_Z X+\nabla_X Z)-(\nabla_Z\nabla X+\nabla_X\nabla Z)),\omega))^V\\
&\quad +(g^{-1}(p(\nabla X),\nabla_Z\omega))^V
\end{aligned}$$

vii)

$$\begin{aligned}
\left(\nabla^{C}_{\omega^V}{}^{S}g\right)\left(X^C,Y^C\right) &= \nabla^{C}_{\omega^V}{}^{S}g\left(X^C,Y^C\right) -^{S} g\left(\nabla^{C}_{\omega^V}\ X^C,Y^C\right) -^{S} g\left(X^C,\nabla^{C}_{\omega^V}Y^C\right)\\
&= \nabla^{C}_{\omega^V}(g(X,Y))^V + (g^{-1}(p(\nabla X),p(\nabla Y)))^V\\
&\quad -^{S}g(-\gamma(\omega o(\nabla X)),Y^C) -^{S} g(X^C,-\gamma(\omega o(\nabla Y)))\\
&= \omega^V(g(X,Y))^V + \omega^V(g^{-1}(p(\nabla X),p(\nabla Y)))^V\\
&\quad +^{S}g((p(\omega o(\nabla X)))^V,Y^C) +^{S} g(X^C,((p(\omega o(\nabla Y))^V)\\
&= {}^{S}g((p(\omega o(\nabla X)))^V,Y^C) +^{S} g(X^C,(p(\omega o(\nabla Y)))^V)\\
&= -(g^{-1}(p(\omega o(\nabla X)),p(\nabla Y)))^V - (g^{-1}(p(\nabla X),p(\omega o(\nabla Y))))^V
\end{aligned}$$

viii)

$$
\begin{aligned}
(\nabla^{C}_{Z^C}{}^{S}g)(X^C,Y^C) &= \nabla^{C}_{Z^C}{}^{S}g(X^C,Y^C) -^{S} g\left(\nabla^{C}_{Z^C}X^C,Y^C\right) -^{S} g\left(X^C,\nabla^{C}_{Z^C}Y^C\right) \\
&= \nabla^{C}_{Z^C}(g(X,Y))^V +^{S} g(((p\,(\nabla X))^V,(p\,(\nabla Y))^V) \\
&\quad -^{S}g\left(\nabla^{C}_{Z^C}X^C,Y^C\right) -^{S} g\left(X^C,\nabla^{C}_{Z^C}Y^C\right) \\
&= (Z(g(X,Y)))^V + (Z(g^{-1}(p(\nabla X),P(\nabla Y)))^V -^{S} g((\nabla_Z X)^C,Y^C) \\
&\quad -^{S}g((p(\nabla(\nabla_Z X+\nabla_X Z)-(\nabla_Z\nabla X+\nabla_X\nabla Z))^V,Y^C) \\
&\quad -^{S} g(X^C,(\nabla_Z Y)^C) \\
&\quad -^{S}g(X^C,(p(\nabla(\nabla_Z Y+\nabla_Y Z)-(\nabla_Z\nabla Y+\nabla_Y\nabla Z)))^V) \\
&= ((\nabla_Z g)(X,Y))^V + (Z(g^{-1}(p(\nabla X),P(\nabla Y)))^V \\
&\quad -(g^{-1}(p(\nabla(\nabla_Z X)),p(\nabla Y)))^V \\
&\quad +(g^{-1}(p(\nabla(\nabla_Z X))-(\nabla_Z\nabla X+\nabla_X\nabla Z)),p(\nabla Y)))^V \\
&\quad -(g^{-1}(p(\nabla X),p(\nabla(\nabla_Z Y)))^V \\
&\quad +(g^{-1}(p(\nabla X),p(\nabla(\nabla_Z Y+\nabla_Y Z)-(\nabla_Z\nabla Y+\nabla_Y\nabla Z))))^V
\end{aligned}
$$

□

2.2. **Lie Derivation of Sasakian metric ^{S}g with respect to vertical and complete lifts.**

Definition 5. *Let M^n be an $n-$dimensional differentiable manifold. Differential transformation $D = L_X$ is called as Lie derivation with respect to vector field $X \in \Im^1_0(M^n)$ if*

$$
\begin{aligned}
L_X f &= Xf, \forall f \in \Im^0_0(M^n), \\
L_X Y &= [X,Y], \forall X,Y \in \Im^1_0(M^n).
\end{aligned} \tag{14}
$$

$[X,Y]$ is called by Lie bracket. The Lie derivative $L_X F$ of a tensor field F of type $(1,1)$ with respect to a vector field X is defined by [4, 5, 6, 7, 17]

$$(L_X F)Y = [X,FY] - F[X,Y]. \tag{15}$$

Proposition 6. *If $X,Y \in \Im^1_0(M)$, $\omega,\theta \in \Im^1_0(M)$ and $F,G \in \Im^1_1(M)$, then*

$$
\begin{aligned}
\left[\omega^V,\theta^V\right] &= 0,\ \left[\omega^V,\gamma F\right] = (\omega o F)^V \\
[\gamma F,\gamma G] &= \gamma\,[F,G],\ \left[X^C,\omega^V\right] = (L_X\omega)^V \\
\left[X^C,\gamma F\right] &= \gamma(L_X F),\ \left[X^C,Y^C\right] = [X,Y]^C,
\end{aligned}
$$

where $\omega o F$ is a $1-$form defined by $(\omega o F)(Z) = \omega(FZ)$ for any $Z \in \Im^1_0(M)$ and L_X the operator of Lie derivation with respect to X [17].

Proposition 7. *If $X \in \Im_0^1(M)$, $\omega \in \Im_1^0(M)$ and $F, G \in \Im_1^1(M)$, then* [17]

$$\begin{aligned} F^C\omega^V &= (\omega o F)^V, \ F^C\gamma G = \gamma(GF), \\ F^C X^C &= (FX)^C + \gamma(L_X F). \end{aligned}$$

Theorem 8. *Let ${}^Sg$ be Sasakian metric ${}^Sg$, is defined by (8),(9),(10),(11) and L_X the operator Lie derivation with respect to X. From proposition (6) and proposition (7), we get the following results*

$$\begin{aligned}
i)\ (L_{\omega^V}{}^Sg)\left(\theta^V, \xi^V\right) &= 0, \\
ii)\ (L_{Z^C}{}^Sg)\left(\omega^V, \theta^V\right) &= \left(\left(L_Z g^{-1}\right)(\omega, \theta)\right)^V, \\
iii)\ (L_{\omega^V}{}^Sg)\left(\theta^V, Y^C\right) &= (g^{-1}(\theta, L_Y\omega))^V, \\
iv)\ (L_{Z^C}{}^Sg)\left(\omega^V, Y^C\right) &= -(L_Z g^{-1}(\omega, p(\nabla Y)))^V + (g^{-1}(L_Z\omega, p(\nabla Y)))^V \\
&\quad +(g^{-1}(\omega, p(\nabla(L_Z Y))))^V, \\
v)\ (L_{\omega^V}{}^Sg)\left(X^C, \theta^V\right) &= (g^{-1}(L_X\omega, \theta))^V, \\
vi)\ (L_{Z^C}{}^Sg)\left(X^C, \omega^V\right) &= -(L_Z g^{-1}(p(\nabla X), \omega))^V + (g^{-1}(p(\nabla(L_Z X)), \omega))^V \\
&\quad +(g^{-1}(p(\nabla X), L_Z\omega)), \\
vii)\ (L_{\omega^V}{}^Sg)\left(X^C, Y^C\right) &= -(g^{-1}(L_X\omega, p\,(\nabla Y)))^V - (g^{-1}(p(\nabla X), L_Y\omega))^V, \\
viii)\ (L_{Z^C}{}^Sg)(X^C, Y^C) &= ((L_Z g)(X, Y))^V + (L_Z g^{-1}(p(\nabla X), p(\nabla Y)) \\
&\quad -g^{-1}(p(\nabla(L_Z X)), p(\nabla Y)) - g^{-1}(p(\nabla X), p(\nabla(L_Z Y))))^V,
\end{aligned}$$

where the complete and horizontal lifts $X^C, X^H \in \Im_0^1(T^*M^n)$ of $X \in \Im_0^1(M^n)$ and the vertical lift $\omega^V \in \Im_0^1(T^*M^n)$ of $\omega \in \Im_1^0(M^n)$ defined by (1),(2),(3), respectively.

Proof. i)

$$\begin{aligned}
(L_{\omega^V}{}^Sg)(\theta^V, \xi^V) &= L_{\omega^V}{}^Sg(\theta^V, \xi^V) -{}^S g(L_{\omega^V}\theta^V, \xi^V) -{}^S g(\theta^V, L_{\omega^V}\xi^V) \\
&= L_{\omega^V}(g^{-1}(\theta, \xi))^V \\
&= \omega^V(g^{-1}(\theta, \xi))^V \\
&= 0
\end{aligned}$$

ii)

$$\begin{aligned}
(L_{Z^C}{}^Sg)(\omega^V, \theta^V) &= L_{Z^C}{}^Sg(\omega^V, \theta^V) -{}^S g(L_{Z^C}\omega^V, \theta^V) -{}^S g(\omega^V, L_{Z^C}\theta^V) \\
&= Z^C(g^{-1}(\omega, \theta))^V -{}^S g((L_Z\omega)^V, \theta^V) -{}^S g(\omega^V, (L_Z\theta)^V) \\
&= (L_Z g^{-1}(\omega, \theta))^V - (g^{-1}(L_Z\omega, \theta))^V - (g^{-1}(\omega, L_Z\theta))^V \\
&= ((L_Z g^{-1})(\omega, \theta))^V
\end{aligned}$$

iii)

$$\begin{aligned}
(L_{\omega^V}{}^{S}g)(\theta^V, Y^C) &= L_{\omega^V}{}^{S}g(\theta^V, Y^C) -^{S} g(L_{\omega^V}\theta^V, Y^C) -^{S} g(\theta^V, L_{\omega^V}Y^C) \\
&= -L_{\omega^V}(g^{-1}(\theta, p(\nabla Y)))^V +^{S} g(\theta^V, (L_Y\omega)^V) \\
&= -\omega^V(g^{-1}(\theta, p\,(\nabla Y)))^V + (g^{-1}(\theta, L_Y\omega))^V \\
&= (g^{-1}(\theta, L_Y\omega))^V
\end{aligned}$$

iv)

$$\begin{aligned}
(L_{Z^C}{}^{S}g)(\omega^V, Y^C) &= L_{Z^C}{}^{S}g(\omega^V, Y^C) -^{S} g(L_{Z^C}\omega^V, Y^C) -^{S} g(\omega^V, L_{Z^C}Y^C) \\
&= -L_{Z^C}(g^{-1}(\omega, p(\nabla Y)))^V -^{S} g((L_Z\omega)^V, Y^C) \\
&\quad -^{S}g(\omega^V, (L_ZY)^C) \\
&= -(L_Z g^{-1}(\omega, p(\nabla Y)))^V + (g^{-1}(L_Z\omega, p(\nabla Y)))^V \\
&\quad +(g^{-1}(\omega, p(\nabla(L_ZY))))^V
\end{aligned}$$

v)

$$\begin{aligned}
(L_{\omega^V}{}^{S}g)(X^C, \theta^V) &= L_{\omega^V}{}^{S}g(X^C, \theta^V) -^{S} g(L_{\omega^V}X^C, \theta^V) -^{S} g(X^C, L_{\omega^V}\theta^V) \\
&= -L_{\omega^V}(g^{-1}(p(\nabla X), \theta))^V +^{S} g((L_X\omega)^V, \theta^V) \\
&= -\omega^V(g^{-1}(p(\nabla X), \theta))^V + (g^{-1}(L_X\omega, \theta))^V \\
&= (g^{-1}(L_X\omega, \theta))^V
\end{aligned}$$

vi)

$$\begin{aligned}
(L_{Z^C}{}^{S}g)(X^C, \omega^V) &= L_{Z^C}{}^{S}g(X^C, \omega^V) -^{S} g(L_{Z^C}X^C, \omega^V) -^{S} g(X^C, L_{Z^C}\omega^V) \\
&= -L_{Z^C}(g^{-1}(p(\nabla X), \omega))^V -^{S} g((L_ZX)^C, \omega^V) \\
&\quad -^{S}g(X^C, (L_Z\omega)^V) \\
&= -(L_Z g^{-1}(p(\nabla X), \omega))^V + (g^{-1}(p(\nabla(L_ZX)), \omega))^V \\
&\quad +(g^{-1}(p(\nabla X), L_Z\omega))
\end{aligned}$$

vii)

$$\begin{aligned}
(L_{\omega^V}{}^{S}g)(X^C, Y^C) &= L_{\omega^V}{}^{S}g(X^C, Y^C) -^{S} g(L_{\omega^V}X^C, Y^C) -^{S} g(X^C, L_{\omega^V}Y^C) \\
&= L_{\omega^V}(g(X,Y))^V + (g^{-1}(p(\nabla X), p(\nabla Y)))^V \\
&\quad +^{S}g((L_X\omega)^V, Y^C) +^{S} g(X^C, (L_Y\omega)^V) \\
&= \omega^V(g(X,Y))^V + \omega^V(g^{-1}(p(\nabla X), p(\nabla Y)))^V \\
&\quad -(g^{-1}(L_X\omega, p(\nabla Y)))^V - (g^{-1}(p(\nabla X), L_Y\omega))^V \\
&= -(g^{-1}(L_X\omega, p\,(\nabla Y)))^V - (g^{-1}(p(\nabla X), L_Y\omega))^V
\end{aligned}$$

viii)

$$
\begin{aligned}
(L_{Z^C}{}^S g)(X^C, Y^C) &= L_{Z^C}{}^S g(X^C, Y^C) -^S g(L_{Z^C} X^C, Y^C) -^S g(X^C, L_{Z^C} Y^C) \\
&= L_{Z^C}(g\,(X,Y))^V + (g^{-1}(p(\nabla X), p(\nabla Y)))^V \\
&\quad -^S g((L_Z X)^C, Y^C) -^S g(X^C, (L_Z Y)^C) \\
&= (Zg\,(X,Y))^V + (Zg^{-1}(p(\nabla X), p(\nabla Y)))^V \\
&\quad -(g(L_Z X, Y))^V - (g^{-1}(p(\nabla(L_Z X)), p(\nabla Y)))^V \\
&\quad -(g(X, L_Z Y))^V - (g^{-1}(p(\nabla X), p(\nabla(L_Z Y))))^V \\
&= (L_Z g(X,Y) - g(L_Z X, Y) - g(X, L_Z Y))^V \\
&\quad +(L_Z g^{-1}(p(\nabla X), p(\nabla Y)))^V - (g^{-1}(p(\nabla(L_Z X)), p(\nabla Y)))^V \\
&\quad -(g^{-1}(p(\nabla X), p(\nabla(L_Z Y))))^V \\
&= ((L_Z g)(X,Y))^V + (L_Z g^{-1}(p(\nabla X), p(\nabla Y)))^V \\
&\quad -(g^{-1}(p(\nabla(L_Z X)), p(\nabla Y)))^V - (g^{-1}(p(\nabla X), p(\nabla(L_Z Y))))^V
\end{aligned}
$$

□

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