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On fourth-order jacobsthal quaternions

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Abstract

In this paper, we present for the first time a sequence of quaternions of order 4 that we will call the fourth-order Jacobsthal and the fourth-order Jacobsthal-Lucas quaternions. In particular, we are interested in the generating function, Binet formula, explicit formula and some interesting results for fourth-order Jacobsthal quaternions and fourth-order Jacobsthal-Lucas quaternions. This generalizes some previous results given by Szynal-Liana and Włoch in [13], Torunbalci Aydin and Yüce in [14] and Cerda-Morales in [2].

1. Introduction

The Jacobsthal numbers have many interesting properties and applications in many fields of science (see, e.g. [1, 4, 12, 9]). In [1], Barry investigated a Jacobsthal decomposition of Pascal's triangle. In [4], Deveci et. al. defined the generalized order-k Jacobsthal sequences modulo m. In [12], Köken and Bozkurt showed that the Jacobsthal numbers are also generated by a special matrix. The Jacobsthal numbers J_n are defined [9] by the recurrence relation

$$J_0 = 0,\ J_1 = 1,\ J_{n+2} = J_{n+1} + 2J_n,\ n \geq 0. \tag{1.1}$$

Another important sequence is the Jacobsthal-Lucas sequence. This sequence is defined by the recurrence relation

$$j_0 = 2,\ j_1 = 1,\ j_{n+2} = j_{n+1} + 2j_n,\ n \geq 0. \tag{1.2}$$

In [3] the Jacobsthal recurrence relation is extended to higher order recurrence relations and the basic list of identities provided by Horadam [9] is expanded and extended to several identities for some of the higher order cases. Furthermore, the authors generalized the Jacobsthal recursion as

$$J_{n+r}^{(r)} = \sum_{s=1}^{r-1} J_{n+r-s}^{(r)} + 2J_n^{(r)}. \tag{1.3}$$

with $n \geq 0$ and initial conditions $J_0 = 0$ and $J_s = 1$ for $s = 1, \ldots, r-1$. For the n-th order-r Jacobsthal-Lucas numbers $j_n^{(r)}$ we use the same recursion with initial conditions $j_s^{(r)} = j_s^{(r-1)}$ for $s = 1, \ldots, r-1$.
In this work we consider the particular case $r = 4$, the fourth-order Jacobsthal numbers $\{J_n^{(4)}\}_{n\geq 0}$ and the fourth-order Jacobsthal-Lucas numbers $\{j_n^{(4)}\}_{n\geq 0}$ are defined by

$$J_{n+4}^{(4)} = J_{n+3}^{(4)} + J_{n+2}^{(4)} + J_{n+1}^{(4)} + 2J_n^{(4)},\ J_0^{(4)} = 0,\ J_1^{(4)} = J_2^{(4)} = J_3^{(4)} = 1 \tag{1.4}$$

and

$$j_{n+4}^{(4)} = j_{n+3}^{(4)} + j_{n+2}^{(4)} + j_{n+1}^{(4)} + 2j_n^{(4)},\ j_0^{(4)} = 2,\ j_1^{(4)} = 1,\ j_2^{(4)} = 5,\ j_3^{(4)} = 10, \tag{1.5}$$

respectively.

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The first fourth-order Jacobsthal numbers and fourth-order Jacobsthal-Lucas numbers are presented in the following table.

s	0	1	2	3	4	5	6	7	8	9	10	11	12	...
$J_s^{(4)}$	0	1	1	1	3	7	13	25	51	103	205	409	819	...
$j_s^{(4)}$	2	1	5	10	20	37	77	154	308	613	1229	2458	4916	...

On the other hand, Horadam [7] introduced the n-th Fibonacci and the n-th Lucas quaternion as follows

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3} \tag{1.6}$$

and

$$Q_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3}, \tag{1.7}$$

respectively. Here F_n and L_n are the n-th Fibonacci and n-th Lucas numbers, respectively. Furthermore, the basis i, j, k satisface the following rules:

$$i^2 = j^2 = k^2 = -1,\ ijk = -1. \tag{1.8}$$

Furthermore, the rules (1.8) imply $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. In general, a quaternion is a hyper-complex number and is defined by $Q = q_r + iq_i + jq_j + kq_k$, where i, j, k are as in (1.8) and $\{q_r, q_i, q_j, q_k\} \subset \mathbb{R}$. Note that we can write $Q = q_r + V_Q$ where $V_Q = iq_i + jq_j + kq_k$. The conjugate of the quaternion Q is denoted by $\overline{Q} = q_r - V_Q$. The norm of a quaternion Q is defined by $Nr(Q) = Q\overline{Q} = q_r^2 + q_i^2 + q_j^2 + q_k^2 \in \mathbb{R}$.
Many interesting properties of Fibonacci and Lucas quaternions can be found in [5, 6, 7, 8, 10]. In [6], Halici investigated complex Fibonacci quaternions. In [8] Horadam mentioned the possibility of introducing Pell quaternions and generalized Pell quaternions. In [13], the authors defined the Jacobsthal quaternions and the Jacobsthal-Lucas quaternions. Recently, in [2] the author defined the third-order Jacobsthal quaternions and mentioned the possibility of introducing higher order Jacobsthal quaternions.
In this paper, we introduce and study the fourth-order Jacobsthal quaternions and the fourth-order Jacobsthal-Lucas quaternions. In particular, we give generating function, Binet formula and some interesting results for the fourth-order Jacobsthal quaternions and fourth-order Jacobsthal-Lucas quaternions.
For fourth-order Jacobsthal and fourth-order Jacobsthal-Lucas numbers some identities are given, see [3]. In this paper we need some of them.

$$j_n^{(4)} - 6J_n^{(4)} = \begin{cases} 2 & \text{if} \quad n \equiv 0 \pmod 4 \\ -5 & \text{if} \quad n \equiv 1 \pmod 4 \\ -1 & \text{if} \quad n \equiv 2 \pmod 4 \\ 4 & \text{if} \quad n \equiv 3 \pmod 4 \end{cases}, \tag{1.9}$$

$$6J_n^{(4)} + j_n^{(4)} - j_{n+1}^{(4)} = \begin{cases} 1 & \text{if} \quad n \equiv 0,2 \pmod 4 \\ 2 & \text{if} \quad n \equiv 1 \pmod 4 \\ -4 & \text{if} \quad n \equiv 3 \pmod 4 \end{cases}, \tag{1.10}$$

$$J_{n+2}^{(4)} - J_n^{(4)} - j_{n-1}^{(4)} = \begin{cases} 0 & \text{if} \quad n \equiv 0 \pmod 4 \\ -2 & \text{if} \quad n \equiv 1 \pmod 4 \\ 1 & \text{if} \quad n \equiv 2,3 \pmod 4 \end{cases} \quad (n \geq 1), \tag{1.11}$$

$$\sum_{s=0}^{n} J_s^{(4)} = \begin{cases} J_{n+1}^{(4)} - 1 & \text{if} \quad n \equiv 0 \pmod 4 \\ J_{n+1}^{(4)} & \text{if} \quad n \equiv 1,3 \pmod 4 \\ J_{n+1}^{(4)} + 1 & \text{if} \quad n \equiv 2 \pmod 4 \end{cases} \tag{1.12}$$

and

$$\sum_{s=0}^{n} j_s^{(4)} = \begin{cases} j_{n+1}^{(4)} - 2 & \text{if} \quad n \not\equiv 0 \pmod 3 \\ j_{n+1}^{(4)} + 1 & \text{if} \quad n \equiv 0 \pmod 3 \end{cases}. \tag{1.13}$$

Using standard techniques for solving recurrence relations, the auxiliary equation, and its roots are given by

$$x^4 - x^3 - x^2 - x - 2 = 0;\ x = 2,\ x = -1, \text{ and } x = \pm i.$$

Note that the latter two are the complex conjugate quartic roots of unity. Call them ω_1 and ω_2, respectively. Thus the Binet formulas can be written as

$$J_n^{(4)} = \frac{1}{5}\left(2^n - \left(\frac{1+3i}{2}\right)\omega_1^n - \left(\frac{1-3i}{2}\right)\omega_2^n\right) \tag{1.14}$$

and

$$j_n^{(4)} = \frac{3}{10}\left(2^{n+2} + \frac{5}{3}(-1)^n + \left(\frac{1+3i}{2}\right)\omega_1^n + \left(\frac{1-3i}{2}\right)\omega_2^n\right), \tag{1.15}$$

respectively.
Now, we use the notation

$$H_n^{(4)}(a,b)=\frac{A\omega_1^n-B\omega_2^n}{\omega_1-\omega_2}=\begin{cases} a & \text{if} \quad n\equiv 0 \pmod 4\\ b & \text{if} \quad n\equiv 1 \pmod 4\\ -a & \text{if} \quad n\equiv 2 \pmod 4\\ -b & \text{if} \quad n\equiv 3 \pmod 4\end{cases}, \tag{1.16}$$

where $A=b-a\omega_2$ and $B=b-a\omega_1$, in which ω_1 and ω_2 are the complex conjugate quartic roots of unity (i.e. $\omega_1^4=\omega_2^4=1$). Furthermore, note that for all $n\geq 0$ we have

$$H_{n+2}^{(4)}(a,b)=-H_n^{(4)}(a,b), \tag{1.17}$$

where $H_0^{(4)}(a,b)=a$ and $H_1^{(4)}(a,b)=b$.
From the Binet formulas (1.14), (1.15) and Eq. (1.16), we have

$$\begin{aligned} J_n^{(4)} &= \frac{1}{5}\left(2^n-V_n^{(4)}\right),\\ j_n^{(4)} &= \frac{3}{10}\left(2^{n+2}+\frac{5}{3}(-1)^n+V_n^{(4)}\right), \end{aligned} \tag{1.18}$$

where $V_n^{(4)}=H_n^{(4)}(1,-3)$.

2. The fourth-order jacobsthal quaternions

The n-th fourth-order Jacobsthal quaternion $JQ_n^{(4)}$ and the n-th fourth-order Jacobsthal-Lucas quaternion $jQ_n^{(4)}$ can be defined as

$$JQ_n^{(4)}=J_n^{(4)}+iJ_{n+1}^{(4)}+jJ_{n+2}^{(4)}+kJ_{n+3}^{(4)} \tag{2.1}$$

and

$$jQ_n^{(4)}=j_n^{(4)}+i\,j_{n+1}^{(4)}+j\,j_{n+2}^{(4)}+k\,j_{n+3}^{(4)},\ n\geq 0, \tag{2.2}$$

respectively. Here $J_n^{(4)}$ and $j_n^{(4)}$ are the n-th fourth-order Jacobsthal and n-th fourth-order Jacobsthal-Lucas numbers, respectively. Furthermore, the basis i, j, k satisface the rules in (1.8).
The function $G(t)=\sum_{n\geq 0}JQ_n^{(4)}t^n$ is called the generating function for the sequence $\{JQ_n^{(4)}\}$. In [3], the authors found a generating function for fourth-order Jacobsthal numbers. In the following theorem, we established the generating function for fourth-order Jacobsthal and fourth-order Jacobsthal-Lucas quaternions.

Theorem 2.1. *The generating function for fourth-order Jacobsthal-Lucas quaternion is*

$$\sum_{n\geq 0} jQ_n^{(4)}t^n=\frac{\left\{\begin{array}{c} 2+i+5j+10k+t\,(-1+4i+5j+10k)+t^2\,(2+4i+5j+7k)\\ +t^3\,(2+4i+2j+10k)\end{array}\right\}}{1-t-t^2-t^3-2t^4}. \tag{2.3}$$

Proof. Assuming that the generating function of the quaternion $\{jQ_n^{(4)}\}_{n\geq 0}$ has the form $G(t)=\sum_{n\geq 0} jQ_n^{(4)}t^n$, we obtain that

$$\begin{aligned}(1-t-t^2-t^3-2t^4)G(t) &= \left(jQ_0^{(4)}+jQ_1^{(4)}t+\cdots\right)-\left(jQ_0^{(4)}t+jQ_1^{(4)}t^2+\cdots\right)-\cdots\\ &= jQ_0^{(4)}+t\left(jQ_1^{(4)}-jQ_0^{(4)}\right)+t^2\left(jQ_2^{(4)}-jQ_1^{(4)}-jQ_0^{(4)}\right)+t^3\left(jQ_3^{(4)}-jQ_2^{(4)}-jQ_1^{(4)}-jQ_0^{(4)}\right),\end{aligned}$$

since $jQ_{n+4}^{(4)}=jQ_{n+3}^{(4)}+jQ_{n+2}^{(4)}+jQ_{n+1}^{(4)}+2jQ_n^{(4)}$ $(n\geq 0)$ and the coefficients of t^n for $n\geq 4$ are equal to zero. In equivalent form is

$$G(t)=\frac{\left\{\begin{array}{c} jQ_0^{(4)}+t\left(jQ_1^{(4)}-jQ_0^{(4)}\right)+t^2\left(jQ_2^{(4)}-jQ_1^{(4)}-jQ_0^{(4)}\right)\\ +t^3\left(jQ_3^{(4)}-jQ_2^{(4)}-jQ_1^{(4)}-jQ_0^{(4)}\right)\end{array}\right\}}{1-t-t^2-t^3-2t^4}.$$

Thus, the proof is completed. □

Thus, the Binet formula for $jQ_n^{(4)}$ can be given in the following theorem.

Theorem 2.2. *If $jQ_n^{(4)}=j_n^{(4)}+i\,j_{n+1}^{(4)}+j\,j_{n+2}^{(4)}+k\,j_{n+3}^{(4)}$ be the n-th fourth-order Jacobsthal-Lucas quaternion. Then,*

$$\begin{aligned} jQ_n^{(4)} &= \frac{3}{10}\left[2^{n+2}\alpha+\frac{5}{3}(-1)^n\beta+\left(\frac{1+3i}{2}\right)\omega_1^n\underline{\omega_1}+\left(\frac{1-3i}{2}\right)\omega_2^n\underline{\omega_2}\right]\\ &= \frac{3}{10}\left[2^{n+2}\alpha+\frac{5}{3}(-1)^n\beta+VQ_n^{(4)}\right], \end{aligned} \tag{2.4}$$

where ω_1, ω_2 are the complex conjugate quartic roots of unity. Furthermore, $\alpha=1+2i+4j+8k$, $\beta=1-i+j-k$ and $VQ_n^{(4)}=V_n^{(4)}+iV_{n+1}^{(4)}+jV_{n+2}^{(4)}+kV_{n+3}^{(4)}$.

Proof. Let $V_n^{(4)} = H_n^{(4)}(1,-3)$. Using the relation (1.18), we have

$$\begin{aligned}
\frac{10}{3}\cdot jQ_n^{(4)} &= \frac{10}{3}\left(j_n^{(4)} + i j_{n+1}^{(4)} + j j_{n+2}^{(4)} + k j_{n+3}^{(4)}\right)\\
&= \left(2^{n+2}+\frac{5}{3}(-1)^n + V_n^{(4)}\right) + i\left(2^{n+3}-\frac{5}{3}(-1)^n + V_{n+1}^{(4)}\right) + j\left(2^{n+4}+\frac{5}{3}(-1)^n + V_{n+2}^{(4)}\right) + k\left(2^{n+5}-\frac{5}{3}(-1)^n + V_{n+3}^{(4)}\right)\\
&= 2^{n+2}(1+2i+4j+8k) + \frac{5}{3}(-1)^n(1-i+j-k) + VQ_n^{(4)},
\end{aligned}$$

where $VQ_n^{(4)} = V_n^{(4)} + iV_{n+1}^{(4)} + jV_{n+2}^{(4)} + kV_{n+3}^{(4)}$. Furthermore,

$$\begin{aligned}
VQ_n^{(4)} &= \left(\left(\frac{1+3i}{2}\right)\omega_1^n + \left(\frac{1-3i}{2}\right)\omega_2^n\right) + i\left(\left(\frac{1+3i}{2}\right)\omega_1^{n+1} + \left(\frac{1-3i}{2}\right)\omega_2^{n+1}\right)\\
&+ j\left(\left(\frac{1+3i}{2}\right)\omega_1^{n+2} + \left(\frac{1-3i}{2}\right)\omega_2^{n+2}\right) + k\left(\left(\frac{1+3i}{2}\right)\omega_1^{n+3} + \left(\frac{1-3i}{2}\right)\omega_2^{n+3}\right)\\
&= \left(\frac{1+3i}{2}\right)\omega_1^n\underline{\omega_1} + \left(\frac{1-3i}{2}\right)\omega_2^n\underline{\omega_2},
\end{aligned}$$

with $\underline{\omega_1} = 1+\omega_1 i - j - \omega_1 k$ and $\underline{\omega_2} = 1+\omega_2 i - j - \omega_2 k$, since $\omega_1^2 = \omega_2^2 = -1$. So, the theorem is proved. □

In a similar way, using the Eqs. (2.3) and (2.4) one can easily prove the following theorem.

Theorem 2.3. *If $JQ_n^{(4)} = J_n^{(4)} + iJ_{n+1}^{(4)} + jJ_{n+2}^{(4)} + kJ_{n+3}^{(4)}$ be the n-th fourth-order Jacobsthal quaternion. Then,*

$$\sum_{n\geq 0} JQ_n^{(4)} t^n = \frac{\begin{pmatrix} i+j+k+t(1+2k) \\ +t^2(-i+j+3k)+t^3(-1+2j+2k)\end{pmatrix}}{1-t-t^2-t^3-2t^4}, \tag{2.5}$$

$$JQ_n^{(4)} = \frac{1}{5}\left[2^n\alpha - VQ_n^{(4)}\right], \tag{2.6}$$

where $\alpha = 1+2i+4j+8k$ and $VQ_n^{(4)} = V_n^{(4)} + iV_{n+1}^{(4)} + jV_{n+2}^{(4)} + kV_{n+3}^{(4)}$.

3. Some identities for the fourth-order jacobsthal quaternions

By some elementary calculations we find the following recurrence relations for the fourth-order Jacobsthal and fourth-order Jacobsthal-Lucas quaternions from (2.1) and (2.2):

$$\begin{aligned}
&JQ_{n+2}^{(4)} + JQ_{n+1}^{(4)} + JQ_n^{(4)} + 2JQ_{n-1}^{(4)}\\
&= (J_{n+2}^{(4)} + iJ_{n+3}^{(4)} + jJ_{n+4}^{(4)} + kJ_{n+5}^{(4)}) + (J_{n+1}^{(4)} + iJ_{n+2}^{(4)} + jJ_{n+3}^{(4)} + kJ_{n+4}^{(4)})\\
&\quad + (J_n^{(4)} + iJ_{n+1}^{(4)} + jJ_{n+2}^{(4)} + kJ_{n+3}^{(4)}) + 2(J_{n-1}^{(4)} + iJ_n^{(4)} + jJ_{n+1}^{(4)} + kJ_{n+2}^{(4)})\\
&= (J_{n+2}^{(4)} + J_{n+1}^{(4)} + J_n^{(4)} + 2J_{n-1}^{(4)}) + (J_{n+3}^{(4)} + J_{n+2}^{(4)} + J_{n+1}^{(4)} + 2J_n^{(4)})i\\
&\quad + (J_{n+4}^{(4)} + J_{n+3}^{(4)} + J_{n+2}^{(4)} + 2J_{n+1}^{(4)})j + 2(J_{n+5}^{(4)} + J_{n+4}^{(4)} + J_{n+3}^{(4)} + 2J_{n+2}^{(4)})k\\
&= J_{n+3}^{(4)} + iJ_{n+4}^{(4)} + jJ_{n+5}^{(4)} + kJ_{n+6}^{(4)}\\
&= JQ_{n+3}^{(4)}
\end{aligned} \tag{3.1}$$

and similarly $jQ_{n+3}^{(4)} = jQ_{n+2}^{(4)} + jQ_{n+1}^{(4)} + jQ_n^{(4)} + 2jQ_{n-1}^{(4)}$, for $n \geq 1$.

Now, we give some interesting results for the fourth-order Jacobsthal quaternions $\{JQ_n^{(4)}\}_{n\geq 0}$ and the fourth-order Jacobsthal-Lucas quaternions $\{jQ_n^{(4)}\}_{n\geq 0}$.

Theorem 3.1. *Let $n \geq 0$ integer. Then, we have*

$$jQ_n^{(4)} - 6JQ_n^{(4)} = \begin{cases} 2-5i-j+4k & \text{if} \quad n\equiv 0 \pmod 4 \\ -5-i+4j+2k & \text{if} \quad n\equiv 1 \pmod 4 \\ -1+4i+2j-5k & \text{if} \quad n\equiv 2 \pmod 4 \\ 4+2i-5j-k & \text{if} \quad n\equiv 3 \pmod 4 \end{cases}. \tag{3.2}$$

Proof. To prove Eq. (3.2) we need the Eq. (1.9). In fact, it suffices to take the Binet's formula of $J_n^{(4)}$ and $j_n^{(4)}$ in (1.18). Then,

$$\begin{aligned}
j_n^{(4)} - 6J_n^{(4)} &= \frac{3}{10}\left(2^{n+2}+\frac{5}{3}(-1)^n + V_n^{(4)}\right) - \frac{6}{5}\left(2^n - V_n^{(4)}\right)\\
&= \frac{1}{2}\left((-1)^n + 3V_n^{(4)}\right).
\end{aligned}$$

For definitions (2.1) and (2.2), we have $JQ_n^{(4)} = J_n^{(4)} + iJ_{n+1}^{(4)} + jJ_{n+2}^{(4)} + kJ_{n+3}^{(4)}$ and $jQ_n^{(4)} = j_n^{(4)} + ij_{n+1}^{(4)} + jj_{n+2}^{(4)} + kj_{n+3}^{(4)}$. Then, if we consider $n \equiv 0 (\text{mod } 4)$, we obtain

$$\begin{aligned} jQ_n^{(4)} - 6JQ_n^{(4)} &= \left(j_n^{(4)} + ij_{n+1}^{(4)} + jj_{n+2}^{(4)} + kj_{n+3}^{(4)}\right) - 6\left(J_n^{(4)} + iJ_{n+1}^{(4)} + jJ_{n+2}^{(4)} + kJ_{n+3}^{(4)}\right) \\ &= \left(j_n^{(4)} - 6J_n^{(4)}\right) + i\left(j_{n+1}^{(4)} - 6J_{n+1}^{(4)}\right) + j\left(j_{n+2}^{(4)} - 6J_{n+2}^{(4)}\right) + k\left(j_{n+3}^{(4)} - 6J_{n+3}^{(4)}\right) \\ &= 2 - 5i - j + 4k, \end{aligned}$$

since $j_{n+1}^{(4)} - 6J_{n+1}^{(4)} = -5$, $j_{n+2}^{(4)} - 6J_{n+2}^{(4)} = -1$ and $j_{n+3}^{(4)} - 6J_{n+3}^{(4)} = 4$. The other identities are clear from equations (1.9) and (1.18). □

Theorem 3.2. *Let $n \geq 0$ integer. Then,*

$$Nr(JQ_n^{(4)}) = \begin{cases} \frac{1}{5}\left(17 \cdot 2^{2n} - 6 \cdot 2^n + 4\right) & \text{if} \quad n \equiv 0,1 \pmod 4 \\ \frac{1}{5}\left(17 \cdot 2^{2n} + 6 \cdot 2^n + 4\right) & \text{if} \quad n \equiv 2,3 \pmod 4 \end{cases}. \tag{3.3}$$

Proof. To prove Eq. (3.3), we use definition of norm for the fourth-order Jacobsthal quaternion $JQ_n^{(4)}$,

$$Nr(JQ_n^{(4)}) = \left(J_n^{(4)}\right)^2 + \left(J_{n+1}^{(4)}\right)^2 + \left(J_{n+2}^{(4)}\right)^2 + \left(J_{n+3}^{(4)}\right)^2.$$

Then, by the Binet formula (1.18) we have

$$\begin{aligned} Nr(JQ_n^{(4)}) &= \frac{1}{25}\left(\begin{array}{c} \left(2^n - V_n^{(4)}\right)^2 + \left(2^{n+1} - V_{n+1}^{(4)}\right)^2 \\ + \left(2^{n+2} - V_{n+2}^{(4)}\right)^2 + \left(2^{n+3} - V_{n+3}^{(4)}\right)^2 \end{array}\right) \\ &= \frac{1}{25}\left(\begin{array}{c} 85 \cdot 2^{2n} - 2^{n+1}\left(V_n^{(4)} + 2V_{n+1}^{(4)} + 4V_{n+2}^{(4)} + 8V_{n+3}^{(4)}\right) \\ + \left(V_n^{(4)}\right)^2 + \left(V_{n+1}^{(4)}\right)^2 + \left(V_{n+2}^{(4)}\right)^2 + \left(V_{n+3}^{(4)}\right)^2 \end{array}\right) \\ &= \frac{1}{25}\left(85 \cdot 2^{2n} + 3 \cdot 2^{n+1}\left(V_n^{(4)} + 2V_{n+1}^{(4)}\right) + 20\right) \\ &= \frac{1}{5}\left(17 \cdot 2^{2n} + 3 \cdot 2^{n+1} U_{n+1}^{(4)} + 4\right), \end{aligned} \tag{3.4}$$

where $U_n^{(4)} = H_n^{(4)}(1,-1)$. Then, if $n \equiv 0,1 (\text{mod } 4)$, we obtain $U_{n+1}^{(4)} = -1$ and $Nr(JQ_n^{(4)}) = \frac{1}{5}\left(17 \cdot 2^{2n} - 3 \cdot 2^{n+1} + 4\right)$. The other identities are clear from equations (3.4) and (1.16). □

In a similar way, using the Eqs. (1.10) and (1.11) one can easily prove the following theorem.

Theorem 3.3. *Let $n \geq 0$ integer. Then,*

$$6JQ_n^{(4)} - jQ_n^{(4)} - jQ_{n+1}^{(4)} = \begin{cases} 1 + 2i + j - 4k & \text{if} \quad n \equiv 0 \pmod 4 \\ 2 + i - 4j + k & \text{if} \quad n \equiv 1 \pmod 4 \\ 1 - 4i + j + 2k & \text{if} \quad n \equiv 2 \pmod 4 \\ -4 + i + 2j + k & \text{if} \quad n \equiv 3 \pmod 4 \end{cases}, \tag{3.5}$$

$$JQ_{n+2}^{(4)} - JQ_n^{(4)} - jQ_{n-1}^{(4)} = \begin{cases} -2i + j + k & \text{if} \quad n \equiv 0 \pmod 4 \\ -2 + i + j & \text{if} \quad n \equiv 1 \pmod 4 \\ 1 + i - 2k & \text{if} \quad n \equiv 2 \pmod 4 \\ 1 - 2j + k & \text{if} \quad n \equiv 3 \pmod 4 \end{cases}, \ (n \geq 1). \tag{3.6}$$

The following is a result for the sum of fourth-order Jacobsthal quaternions.

Theorem 3.4. *Let $n \geq 0$ integer. Then,*

$$\sum_{s=0}^{n} JQ_s^{(4)} = \begin{cases} JQ_{n+1}^{(4)} - (1 + 2k) & \text{if} \quad n \equiv 0 \pmod 4 \\ JQ_{n+1}^{(4)} + (i - j - 3k) & \text{if} \quad n \equiv 1 \pmod 4 \\ JQ_{n+1}^{(4)} + (1 - 2j - 2k) & \text{if} \quad n \equiv 2 \pmod 4 \\ JQ_{n+1}^{(4)} - (i + j + k) & \text{if} \quad n \equiv 3 \pmod 4 \end{cases}. \tag{3.7}$$

Proof. Using equality (1.12), we have

$$\sum_{s=0}^{n} J_s^{(4)} = \begin{cases} J_{n+1}^{(4)} - 1 & \text{if} \quad n \equiv 0 \pmod 4 \\ J_{n+1}^{(4)} & \text{if} \quad n \equiv 1,3 \pmod 4 \\ J_{n+1}^{(4)} + 1 & \text{if} \quad n \equiv 2 \pmod 4 \end{cases}.$$

Furthermore, if $n\equiv 0(\text{mod } 4)$, $\sum_{s=0}^{n}J_s^{(4)}=J_{n+1}^{(4)}-1$, $\sum_{s=0}^{n+1}J_s^{(4)}=J_{n+2}^{(4)}$, $\sum_{s=0}^{n+2}J_s^{(4)}=J_{n+3}^{(4)}+1$ and $\sum_{s=0}^{n+3}J_s^{(4)}=J_{n+4}^{(4)}$. Then,

$$\begin{aligned}\sum_{s=0}^{n}JQ_s^{(4)} &= \sum_{s=0}^{n}J_s^{(4)}+i\sum_{s=0}^{n}J_{s+1}^{(4)}+j\sum_{s=0}^{n}J_{s+2}^{(4)}+k\sum_{s=0}^{n}J_{s+3}^{(4)}\\ &= \sum_{s=0}^{n}J_s^{(4)}+i\left(\sum_{s=0}^{n+1}J_s^{(4)}\right)+j\left(\sum_{s=0}^{n+2}J_s^{(4)}-1\right)+k\left(\sum_{s=0}^{n+3}J_s^{(4)}-2\right)\\ &= \left(J_{n+1}^{(4)}-1\right)+i\left(J_{n+2}^{(4)}\right)+j\left(J_{n+3}^{(4)}\right)+k\left(J_{n+4}^{(4)}-2\right)\\ &= JQ_{n+1}^{(4)}-(1+2k).\end{aligned}$$

If $n\equiv 1(\text{mod } 4)$, we have $\sum_{s=0}^{n+1}J_s^{(4)}=J_{n+2}^{(4)}+1$, $\sum_{s=0}^{n+2}J_s^{(4)}=J_{n+3}^{(4)}$ and $\sum_{s=0}^{n+3}J_s^{(4)}=J_{n+4}^{(4)}-1$, then $\sum_{s=0}^{n}JQ_s^{(4)}=JQ_{n+1}^{(4)}+(i-j-3k)$. The proof is similar for the cases $n\equiv 2,3(\text{mod } 4)$. Thus, the proof is completed. □

There are three well-known identities for Fibonacci numbers, namely, Catalan's, Cassini's, and d'Ocagne's identities. The proofs of these identities are based on Binet formulas. We can obtain these types of identities for fourth-order Jacobsthal quaternions using the Binet formulas derived above. We use the notation

$$\begin{aligned}HQ_n^{(4)}(a,b) &= \frac{A\omega_1^n\underline{\omega_1}-B\omega_2^n\underline{\omega_2}}{\omega_1-\omega_2}\\ &= \begin{cases} a+bi-aj-bk & \text{if} \quad n\equiv 0 \pmod 4\\ b-ai-bj+ak & \text{if} \quad n\equiv 1 \pmod 4\\ -a-bi+aj+bk & \text{if} \quad n\equiv 2 \pmod 4\\ -b+ai+bj-ak & \text{if} \quad n\equiv 3 \pmod 4\end{cases},\end{aligned} \tag{3.8}$$

where $A=b-a\omega_2$ and $B=b-a\omega_1$, in which $\underline{\omega_1}=1+\omega_1 i-j-\omega_1 k$ and $\underline{\omega_2}=1+\omega_2 i-j-\omega_2 k$ are the complex conjugate quartic roots of unity (i.e. $\omega_1^2=\omega_2^2=-1$). Furthermore, note that for all $n\geq 0$ we have

$$HQ_{n+2}^{(4)}(a,b)=-HQ_n^{(4)}(a,b), \tag{3.9}$$

where $HQ_0^{(4)}(a,b)=a+bi-aj-bk$ and $HQ_1^{(4)}(a,b)=b-ai-bj+ak$.
The following theorem gives d'Ocagne's identities for fourth-order Jacobsthal quaternion.

Theorem 3.5. *If $JQ_n^{(4)}=J_n^{(4)}+iJ_{n+1}^{(4)}+jJ_{n+2}^{(4)}+kJ_{n+3}^{(4)}$ be the n-th fourth-order Jacobsthal quaternion. Then, for any integers n and m, we have*

$$JQ_m^{(4)}JQ_{n+1}^{(4)}-JQ_{m+1}^{(3)}JQ_n^{(4)}=\frac{1}{5}\left\{\begin{array}{c} 2^m\alpha UQ_n^{(4)}-2^nUQ_m^{(4)}\alpha\\ -i\left(\omega_1^{m-n}\underline{\omega_1\omega_2}-\omega_2^{m-n}\underline{\omega_2\omega_1}\right)\end{array}\right\} \tag{3.10}$$

where $\alpha=1+2i+4j+8k$, $\underline{\omega_1}=1+\omega_1 i-j-\omega_1 k$, $\underline{\omega_2}=1+\omega_2 i-j-\omega_2 k$ and $UQ_n^{(4)}=HQ_n^{(4)}(-1,-1)$.

Proof. Using the Binet formula for the fourth-order Jacobsthal quaternions and $VQ_n^{(4)}=HQ_n^{(4)}(1,-3)$ in (3.8) gives

$$\begin{aligned}&JQ_m^{(4)}JQ_{n+1}^{(4)}-JQ_{m+1}^{(4)}JQ_n^{(4)}\\ &=\frac{1}{25}\left(\begin{array}{c}\left(2^m\alpha-VQ_m^{(4)}\right)\left(2^{n+1}\alpha-VQ_{n+1}^{(4)}\right)\\ -\left(2^{m+1}\alpha-VQ_{m+1}^{(4)}\right)\left(2^n\alpha-VQ_n^{(4)}\right)\end{array}\right)\\ &=\frac{1}{25}\left(\begin{array}{c}-2^m\alpha VQ_{n+1}^{(4)}-2^{n+1}VQ_m\alpha+2^{m+1}\alpha VQ_n^{(4)}+2^nVQ_{m+1}^{(4)}\alpha\\ +VQ_m^{(4)}VQ_{n+1}^{(4)}-VQ_{m+1}^{(4)}VQ_n^{(4)}\end{array}\right)\\ &=\frac{1}{5}\left(2^m\alpha UQ_n^{(4)}-2^nUQ_m^{(4)}\alpha-i\left(\omega_1^{m-n}\underline{\omega_1\omega_2}-\omega_2^{m-n}\underline{\omega_2\omega_1}\right)\right),\end{aligned} \tag{3.11}$$

where $UQ_n^{(4)}=\frac{1}{5}\left(2VQ_n^{(4)}-VQ_{n+1}^{(4)}\right)=HQ_n^{(4)}(1,-1)$. □

Taking $m=n+1$ in this theorem and using the identity

$$-i\left(\omega_1\underline{\omega_1\omega_2}-\omega_2\underline{\omega_2\omega_1}\right)=\underline{\omega_1\omega_2}+\underline{\omega_2\omega_1}=-4(1+j),$$

we obtain Cassini's identities for fourth-order Jacobsthal quaternions.

Corollary 3.6. *For any integer $n\geq 0$, we have*

$$\left(JQ_{n+1}^{(4)}\right)^2-JQ_{n+2}^{(3)}JQ_n^{(4)}=\frac{1}{5}\left(2^n\left(2\alpha UQ_n^{(4)}-UQ_{n+1}^{(4)}\alpha\right)-4(1+j)\right). \tag{3.12}$$

We will give an example in which we check in a particular case the Cassini-like identity for fourth-order Jacobsthal quaternions.

Example 3.7. *Let* $\{JQ_s^{(4)}:\ s=0,1,2,3\}$ *be the fourth-order Jacobsthal quaternions such that* $JQ_0^{(4)}=i+j+k$, $JQ_1^{(4)}=1+i+j+3k$, $JQ_2^{(4)}=1+i+3j+7k$ *and* $JQ_3^{(4)}=1+3i+7j+13k$. *In this case,*

$$\begin{aligned}\left(JQ_1^{(4)}\right)^2-JQ_2^{(4)}JQ_0^{(4)} &= (1+i+j+3k)^2-(1+i+3j+7k)(i+j+k)\\ &= (-10+2i+2j+6k)-(-11-3i+7j-k)\\ &= 1+5i-5j+7k\\ &= \frac{1}{5}\left(\left(2\alpha UQ_0^{(4)}-UQ_1^{(4)}\alpha\right)-4(1+j)\right).\end{aligned}$$

and

$$\begin{aligned}\left(JQ_2^{(4)}\right)^2-JQ_3^{(4)}JQ_1^{(4)} &= (1+i+3j+7k)^2-(1+3i+7j+13k)(1+i+j+3k)\\ &= (-58+2i+6j+14k)-(-48+12i+12j+12k)\\ &= -10-10i-6j+2k\\ &= \frac{1}{5}\left(2\left(2\alpha UQ_1^{(4)}-UQ_2^{(4)}\alpha\right)-4(1+j)\right).\end{aligned}$$

4. Conclusions

In this work, some known identities of the sequence of Jacobsthal numbers have continued to be generalized with the use of the quaternion ring. The main motivation is based on the study of the non-commutative properties of the quaternions, and how we can solve friendly cases with sequences of recursive numbers. In particular, the ideas of finding rules of commutativity, matrix representation of quaternion sequences and their study in a wider class of rings, say in octonions or in any power associative ring.

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