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Asymptotically $\mathcal{I}_{\sigma\theta}$-statistical equivalence of sequences of sets defined by a modulus functions

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Abstract

We investigate the notions of strongly asymptotically $\mathcal{I}_{\sigma\theta}$-equivalence, f-asymptotically $\mathcal{I}_{\sigma\theta}$-equivalence, strongly f-asymptotically $\mathcal{I}_{\sigma\theta}$-equivalence and asymptotically $\mathcal{I}_{\sigma\theta}$-statistical equivalence for sequences of sets. Also, we investigate some relationships among these concepts.

Keywords: asymptotic equivalence, modulus function, $\mathcal{I}$-convergence, lacunary $\mathcal{I}$-Invariant equivalence

1. INTRODUCTION

Recently, concepts of statistical convergence and ideal convergence were studied and dealt with by several authors. Fast [1] and Schoenberg [2] independently introduced statistical convergence and this concept studied by many authors. Lacunary statistical convergence was defined by Fridy and Orhan [3] using the notion of lacunary sequence $\theta = \{k_r\}$. Kostyrko et al. [4] introduced and dealt with the idea of $\mathcal{I}$-convergence. $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence were introduced by Das et al. [5].

Several authors studied some convergence types of the notion of set sequences. Nuray and Rhoades [6] defined statistical convergence of set sequences. Lacunary statistical convergence of set sequences was introduced by Ulusu and Nuray [7] and they gave some examples and investigated some properties of this notion. $\mathcal{I}$-convergence of set sequences was studied by Kişi and Nuray [8]. On $\mathcal{I}$-lacunary statistical convergence of set sequences was studied by Ulusu and Dündar [9]. Also, after these important studies, the notions of statistical convergence, ideal convergence and $\mathcal{I}$-statistical convergence of set sequences and some properties was studied and dealt with by several authors.

Several authors including Raimi [10], Schaefer [11], Mursaleen [12,13], Savaş [14,15], Mursaleen and Edely [16], Pancaroğlu and Nuray [17,18] and some authors have studied invariant convergent sequences. The notion of strong σ-convergence was defined by Savaş [16]. Savaş and Nuray [19] defined the ideas of σ-statistical convergence and lacunary σ-statistically convergence and gave some inclusion relations. Then, Pancaroğlu and Nuray [17] introduced the ideas of $\sigma\theta$-summability and the space $[V_{\sigma\theta}]_q$. Recently, Ulusu and Nuray [20] defined the notions of $\sigma\theta$-uniform density of subsets A of $\mathbb{N}$, $\mathcal{I}_{\sigma\theta}$-convergence and investigated relationships between $\mathcal{I}_{\sigma\theta}$-convergence and lacunary invariant convergence also $\mathcal{I}_{\sigma\theta}$-convergence and $[V_{\sigma\theta}]_p$-convergence.

Asymptotically equivalent and asymptotic regular matrices were peresented by Marouf [21]. Patterson and Savaş [22,23] introduced asymptotically lacunary statistically equivalent sequences and also asymptotically $\sigma\theta$-statistical equivalent sequences. Ulusu and Nuray [24] defined the ideas of some basic asymptotically equivalence for sequences of sets.

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Pancaroğlu et al. [25] studided asymptotically $\sigma\theta$-statistical equivalent sequences of sets. Ulusu and Gülle [26] introduced asymptotically $\mathcal{I}_\sigma$-equivalence of sequences of sets.

Nakano [27] introduced modulus function. Maddox [28], Pehlivan and Fisher [29], Pancaroğlu and Nuray [30,31] and several authors define some new concepts and give inclusion theorems using a modulus function f. Kumar and Sharma [32] studied $\mathcal{I}_\theta$-equivalent sequences using a modulus function f. Kişi et al. [33] introduced f-asymptotically $\mathcal{I}_\theta$-equivalent set sequences. P. Akın et al. [34] introduced f-asymptotically $\mathcal{I}$-invariant statistical equivalence of set sequences.

Now, we recall the basic concepts and some definitions and notations (See [18, 21, 24-26, 28, 29, 32, 33, 35-42]).

Two nonnegative sequences $u = (u_k)$ and $v = (v_k)$ are said to be asymptotically equivalent if

$$\lim_k \frac{u_k}{v_k} = 1$$

(denoted by $u \sim v$).

Throughout this study, we let (V, ρ) be a metric space and G, G_k and H_k $(k = 1,2,\dots)$ be non-empty closed subsets of V.

For any point $u \in V$ and any non-empty subset G of V, we define the distance from u to G by

$$d(u, G) = \inf_{g\in G} \rho(u, g).$$

Let $G_k, H_k \subseteq V$ such that $d(u, G_k) > 0$ and $d(u, H_k) > 0$, for each $u \in V$. The sequences $\{G_k\}$ and $\{H_k\}$ are asymptotically equivalent if for each $u \in V$,

$$\lim_k \frac{d(u, G_k)}{d(u, H_k)} = 1$$

(denoted by $G_k \sim H_k$).

Let $G_k, H_k \subseteq V$ such that $d(u, G_k) > 0$ and $d(u, H_k) > 0$, for each $u \in V$. The sequences $\{G_k\}$ and $\{H_k\}$ are asymptotically statistical equivalent of multiple K if for every $\varepsilon > 0$ and for each $u \in V$,

$$\lim_n \frac{1}{n} \left| \left\{ k \le n : \left| \frac{d(u, G_k)}{d(u, H_k)} - K \right| \ge \varepsilon \right\} \right| = 0$$

(denoted by $G_k \overset{WS_L}{\sim} H_k$).

Let σ be a mapping of the positive integers into itself. A continuous linear functional φ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ mean if and only if

1. $\phi(u) \ge 0$, when the sequence $u = (u_j)$ has $u_j \ge 0$ for all j,
2. $\phi(i) = 1$, where $i = (1,1,1\dots)$,
3. $\phi(u_{\sigma(j)}) = \phi(u)$ for all $u \in \ell_\infty$.

The mappings ϕ are assumed to be one-to-one and such that $\sigma^m(j) \ne j$ for all positive integers j and m, where $\sigma^m(j)$ denotes the mth iterate of the mapping σ at j. Thus ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(u) = \lim u$ for all $u \in c$. If σ is a translation mappings that is $\sigma(j) = j + 1$, the σ mean is often called a Banach limit.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout the paper, we let $\theta = \{k_r\}$ be a lacunary sequence.

A sequence $\{G_k\}$ is Wijsman $\sigma\theta$-statistically convergent to G if for every $\varepsilon > 0$ and for each $u \in V$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |d(u, G_{\sigma^k(m)}) - d(u, G)| \ge \varepsilon\}| = 0$$

uniformly in m. It is denoted by $G_k \to G([WS_{\sigma\theta}])$.

For non-empty closed subsets G_k, H_k of V define $d(u; G_k, H_k)$ as follows:

$$d(u; G_k, H_k) = \begin{cases} \dfrac{d(u, G_k)}{d(u, H_k)}, & u \notin G_k \cup H_k; \\ K \quad , & u \in G_k \cup H_k. \end{cases}$$

The sequences $\{G_k\}$ and $\{H_k\}$are Wijsman strongly asymptotically $\sigma\theta$-equivalent of multiple K if for each $u \in V$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left| d(u; G_{\sigma^k(m)}, H_{\sigma^k(m)}) - K \right| = 0$$

uniformly in m, (denoted by $G_k \overset{[WN]_{\sigma\theta}^K}{\sim} H_k$).

The sequences $\{G_k\}$ and $\{H_k\}$ are Wijsman asymptotically $\sigma\theta$-statistical equivalent of multiple K if for each $u \in V$,

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ k \in I_r : |d(u; G_{\sigma^k(m)}, H_{\sigma^k(m)}) - K| \ge \varepsilon \right\} \right| = 0$$

uniformly in m, (denoted by $G_k \overset{WS_{\sigma\theta}^K}{\sim} H_k$).

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$,

(ii) For each $E, F \in \mathcal{I}$ we have $E \cup F \in \mathcal{I}$,

(iii) For each $E \in \mathcal{I}$ and each $F \subseteq E$ we have $F \in \mathcal{I}$.

If $\mathbb{N} \notin \mathcal{J}$, $\mathcal{J}$ is called non-trivial and if $\{n\} \in \mathcal{J}$ for each $n \in \mathbb{N}$, a non-trivial ideal is called admissible ideal. Throughout this study, we let $\mathcal{J}$ be an admissible ideal.

Let $E \subseteq \mathbb{N}$ and

$$s_r = \min_n\{|E \cap \{\sigma^m(n): m \in I_r\}|\}$$

and

$$S_r = \max_n\{|E \cap \{\sigma^m(n): m \in I_r\}|\}.$$

If the limits

$$\underline{V_\theta}(E) = \lim_{r\to\infty} \frac{s_r}{h_r} \quad \text{and} \quad \overline{V_\theta}(E) = \lim_{r\to\infty} \frac{S_r}{h_r}$$

exist, then they are called a lower lacunary σ-uniform (lower $\sigma\theta$-uniform) density and an upper lacunary σ-uniform (upper $\sigma\theta$-uniform) density of the set E, respectively. If $\underline{V_\theta}(E) = \overline{V_\theta}(E)$, then

$$V_\theta(E) = \underline{V_\theta}(E) = \overline{V_\theta}(E)$$

is called the lacunary σ-uniform density or $\sigma\theta$-uniform density of E.

Denoted by $\mathcal{J}_{\sigma\theta}$, we denote the class of all $E \subseteq \mathbb{N}$ with $V_\theta(A) = 0$.

A sequence $\{G_k\}$ is said to be Wijsman lacunary $\mathcal{J}$-invariant convergent or $\mathcal{J}_{\sigma\theta}^W$-convergent to G if for every $\varepsilon > 0$ and for each $u \in V$, the set

$$A(\varepsilon, u) = \{k: |d(u, G_k) - d(u, G)| \geq \varepsilon\}$$

belongs to $\mathcal{J}_{\sigma\theta}$, that is, $V_\theta(A(\varepsilon, u)) = 0$. It is shown by $G_k \to G(\mathcal{J}_{\sigma\theta}^W)$.

A function $f: [0, \infty) \to [0, \infty)$ is called a modulus if

1. $f(u) = 0$ if and only if $u = 0$,
2. $f(u + v) \leq f(u) + f(v)$
3. f is increasing
4. f is continuous from the right at 0.

A modulus may be unbounded (for example $f(u) = q$, $0 < q < 1$) or bounded (for example $f(u) = \frac{u}{u+1}$).

Throughout this study, we let f be a modulus function.

The sequences $\{G_k\}$ and $\{H_k\}$ are said to be Wijsman f-asymptotically $\mathcal{J}$-equivalent of multiple K if for every $\varepsilon > 0$ and for each $u \in V$,

$$\{k \in \mathbb{N}: f(|d(u; G_k, H_k) - K|) \geq \varepsilon\} \in \mathcal{J}$$

(denoted by $G_k \overset{\mathcal{J}_W(f)}{\sim} H_k$).

The sequences $\{G_k\}$ and $\{H_k\}$ are said to be strongly Wijsman f-asymptotically $\mathcal{J}_\theta$-equivalent of multiple K if for every $\varepsilon > 0$ and for each $u \in V$,

$$\left\{r \in \mathbb{N}: \frac{1}{h_r} \sum_{k \in I_r} f(|d(u; G_k, H_k) - K|) \geq \varepsilon\right\} \in \mathcal{J}$$

(denoted by $G_k \overset{N_\theta^f(\mathcal{J}_W)}{\sim} H_k$).

The sequences $\{G_k\}$ and $\{H_k\}$ are said to be strongly Wijsman asymptotically $\mathcal{J}$-invariant equivalent of multiple K if for every $\varepsilon > 0$ and for each $u \in V$,

$$\left\{n \in \mathbb{N}: \frac{1}{n} \sum_{k=1}^{n} |d(u; G_k, H_k) - K| \geq \varepsilon\right\} \in \mathcal{J}_\sigma$$

(denoted by $G_k \overset{[W_{\mathcal{J}_\sigma}^K]}{\sim} H_k$).

The sequences $\{G_k\}$ and $\{H_k\}$ are said to be Wijsman f-asymptotically $\mathcal{J}$-invariant equivalent of multiple K if for every $\varepsilon > 0$ and for each $u \in V$,

$$\{k \in \mathbb{N}: f(|d(u; G_k, H_k) - K|) \geq \varepsilon\} \in \mathcal{J}_\sigma$$

(denoted by $G_k \overset{W_{\mathcal{J}_\sigma}^K(f)}{\sim} H_k$).

The sequences $\{G_k\}$ and $\{H_k\}$ are said to be strongly f-asymptotically $\mathcal{J}$-invariant equivalent of multiple K if for every $\varepsilon > 0$ and for each $u \in V$,

$$\left\{n \in \mathbb{N}: \frac{1}{n} \sum_{k=1}^{n} f(|d(u; G_k, H_k) - K|) \geq \varepsilon\right\} \in \mathcal{J}_\sigma$$

(denoted by $G_k \overset{[W_{\mathcal{J}_\sigma}^K(f)]}{\sim} H_k$) .

The sequences $\{G_k\}$ and $\{H_k\}$ are said to be asymptotically $\mathcal{J}$-invariant statistical equivalent of multiple K if for every $\varepsilon > 0$, $\gamma > 0$ and for each $u \in V$,

$$\left\{n \in \mathbb{N}: \frac{1}{n} |\{k \leq n: |d(u; G_k, H_k) - K| \geq \varepsilon\}| \geq \gamma\right\} \in \mathcal{J}_\sigma$$

(denoted by $G_k \overset{W_{\mathcal{J}_\sigma}^K(S)}{\sim} H_k$).

Lemma 1 [29] Let $0 < \delta < 1$. Then, for each $u \geq \delta$ we have $f(u) \leq 2f(1)\delta^{-1}u$.

2. MAIN RESULTS

Definition 2.1 The sequences $\{G_k\}$ and $\{H_k\}$ are said to be strongly Wijsman asymptotically $\mathcal{J}_{\sigma\theta}$-equivalent of multiple K if for every $\varepsilon > 0$ and for each $u \in V$,

$$\left\{r \in \mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} |d(u;G_k,H_k) - K| \geq \varepsilon\right\} \in \mathcal{I}_{\sigma\theta}$$

(denoted by $G_k \overset{[W^K_{\mathcal{I}_{\sigma\theta}}]}{\sim} H_k$).

Definition 2.2 $\{G_k\}$ and $\{H_k\}$ are said to be Wijsman f-asymptotically $\mathcal{I}_{\sigma\theta}$-equivalent of multiple K if for every $\varepsilon > 0$ and for each $u \in V$,

$$\{k \in \mathbb{N}: f(|d(u;G_k,H_k) - K|) \geq \varepsilon\} \in \mathcal{I}_{\sigma\theta}$$

(denoted by $G_k \overset{W^K_{\mathcal{I}_{\sigma\theta}}(f)}{\sim} H_k$).

Definition 2.3 $\{G_k\}$ and $\{H_k\}$ are said to be strongly Wijsman f-asymptotically $\mathcal{I}_{\sigma\theta}$-equivalent of multiple K if for every $\varepsilon > 0$ and for each $u \in V$,

$$\left\{r \in \mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} f(|d(u;G_k,H_k) - K|) \geq \varepsilon\right\} \in \mathcal{I}_{\sigma\theta}$$

(denoted by $G_k \overset{[W^K_{\mathcal{I}_{\sigma\theta}}(f)]}{\sim} H_k$).

Theorem 2.1 For each $u \in V$, we have

$$G_k \overset{[W^K_{\mathcal{I}_{\sigma\theta}}]}{\sim} H_k \Rightarrow G_k \overset{[W^K_{\mathcal{I}_{\sigma\theta}}(f)]}{\sim} H_k.$$

Proof. Let $G_k \overset{[W^K_{\mathcal{I}_{\sigma\theta}}]}{\sim} H_k$ and $\varepsilon > 0$ be given. Select $0 < \delta < 1$ such that $f(i) < \varepsilon$ for $0 \leq i \leq \delta$. So, for each $u \in V$ and for $m = 1,2,...$, we can write

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|d\left(u;G_{\sigma^k(m)},H_{\sigma^k(m)}\right) - K\right|\right) =$$

$$\frac{1}{h_r}\sum_{\substack{k\in I_r \\ \left|d\left(u;G_{\sigma^k(m)},H_{\sigma^k(m)}\right)-K\right|\leq\delta}} f\left(\left|d\left(u;G_{\sigma^k(m)},H_{\sigma^k(m)}\right) - K\right|\right)$$

$$+\frac{1}{h_r}\sum_{\substack{k\in I_r \\ \left|d\left(u;G_{\sigma^k(m)},H_{\sigma^k(m)}\right)-K\right|>\delta}} f\left(\left|d\left(u;G_{\sigma^k(m)},H_{\sigma^k(m)}\right) - K\right|\right)$$

and so, by Lemma 1, we have

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|d\left(u;G_{\sigma^k(m)},H_{\sigma^k(m)}\right) - K\right|\right)$$

$$< \varepsilon + \left(\frac{2f(1)}{\delta}\right)\frac{1}{h_r}\sum\nolimits_{k\in I_r} \left|d\left(u;G_{\sigma^k(m)},H_{\sigma^k(m)}\right) - K\right|$$

uniformly in m. Thus, for every $\gamma > 0$ and for each $u \in V$,

$$\left\{r \in \mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|d\left(u;G_{\sigma^k(m)},H_{\sigma^k(m)}\right) - K\right|\right) \geq \gamma\right\}$$

$$\subseteq \left\{r \in \mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} \left|d\left(u;G_{\sigma^k(m)},H_{\sigma^k(m)}\right) - K\right| \geq \frac{(\gamma-\varepsilon)\delta}{2f(1)}\right\}$$

uniformly in m.

Since $G_k \overset{[W^K_{\mathcal{I}_{\sigma\theta}}]}{\sim} H_k$, the second set belongs to $\mathcal{I}_{\sigma\theta}$ and thus, the first set belongs to $\mathcal{I}_{\sigma\theta}$. This proves that $G_k \overset{[W^K_{\mathcal{I}_{\sigma\theta}}(f)]}{\sim} H_k$.

Theorem 2.2 If $\lim_{i\to\infty}\frac{f(i)}{i} = \alpha > 0$, then $G_k \overset{[W^K_{\mathcal{I}_{\sigma\theta}}]}{\sim} H_k \Leftrightarrow G_k \overset{[W^K_{\mathcal{I}_{\sigma\theta}}(f)]}{\sim} H_k$.

Proof. If $\lim_{i\to\infty}\frac{f(i)}{i} = \alpha > 0$, then we have $f(i) \geq \alpha i$ for all $i \geq 0$. Assume that $G_k \overset{[W^K_{\mathcal{I}_{\sigma\theta}}(f)]}{\sim} H_k$. Since for $m = 1,2,...$ and for each $u \in V$

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|d\left(u;G_{\sigma^k(m)},H_{\sigma^k(m)}\right) - K\right|\right)$$

$$\geq \frac{1}{h_r}\sum_{k\in I_r} \alpha\left(\left|d\left(u;G_{\sigma^k(m)},H_{\sigma^k(m)}\right) - K\right|\right)$$

$$= \alpha\left(\frac{1}{h_r}\sum_{k\in I_r} \left|d\left(u;G_{\sigma^k(m)},H_{\sigma^k(m)}\right) - K\right|\right),$$

it follows that for each $\varepsilon > 0$, we have

$$\left\{r \in \mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} \left|d\left(u;G_{\sigma^k(m)},H_{\sigma^k(m)}\right) - K\right| \geq \varepsilon\right\}$$

$$\subseteq \left\{n \in \mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|d\left(u;G_{\sigma^k(m)},H_{\sigma^k(m)}\right) - K\right|\right) \geq \alpha\varepsilon\right\}$$

uniformly in m. Since $G_k \overset{[W^K_{\mathcal{I}_{\sigma\theta}}(f)]}{\sim} H_k$, it follows that second set belongs to $\mathcal{I}_{\sigma\theta}$. This proves that

$$G_k \overset{[W^K_{\mathcal{I}_{\sigma\theta}}]}{\sim} H_k \Leftrightarrow G_k \overset{[W^K_{\mathcal{I}_{\sigma\theta}}(f)]}{\sim} H_k.$$

Definition 2.4 We say that the sequences $\{G_k\}$ and $\{H_k\}$ are said to be Wijsman asymptotically lacunary $\mathcal{I}$-invariant statistical equivalent of multiple K, if for every $\varepsilon, \gamma > 0$ and for each $u \in V$

$$\left\{r \in \mathbb{N}: \frac{1}{h_r}|\{k \in I_r: |d(u;G_k,H_k) - K| \geq \varepsilon\}| \geq \gamma\right\} \in \mathcal{I}_{\sigma\theta}$$

(denoted by $G_k \overset{W^K_{\mathcal{I}_{\sigma\theta}}(S)}{\sim} H_k$).

Theorem 2.3 For each $u \in V$, we have

$$G_k \overset{[W_{\mathcal{J}_{\sigma\theta}}^K(f)]}{\sim} H_k \Rightarrow G_k \overset{W_{\mathcal{J}_{\sigma\theta}}^K(S)}{\sim} H_k.$$

Proof. Assume that $G_k \overset{[W_{\mathcal{J}_{\sigma\theta}}^K(f)]}{\sim} H_k$ and $\varepsilon > 0$ be given. Since for each $u \in V$ and for $m = 1,2,...$

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|d\left(u; G_{\sigma^k(m)}, H_{\sigma^k(m)}\right) - K\right|\right) \geq$$

$$\frac{1}{h_r}\sum_{\substack{k\in I_r \\ \left|d\left(u;G_{\sigma^k(m)},H_{\sigma^k(m)}\right)-K\right|\geq\varepsilon}} f\left(\left|d\left(u; G_{\sigma^k(m)}, H_{\sigma^k(m)}\right) - K\right|\right)$$

$$\geq f(\varepsilon).\frac{1}{h_r}\left|\left\{k \in I_r: \left|d\left(u; G_{\sigma^k(m)}, H_{\sigma^k(m)}\right) - K\right| \geq \varepsilon\right\}\right|,$$

then for any $\gamma > 0$ and for each $u \in V$

$$\{r \in \mathbb{N}: \frac{1}{h_r}|\{k \in I_r: \left|d\left(u; G_{\sigma^k(m)}, H_{\sigma^k(m)}\right) - K\right| \geq \varepsilon\}| \geq \frac{\gamma}{f(\varepsilon)}\}$$

$$\subseteq \left\{r \in \mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|d\left(u; G_{\sigma^k(m)}, H_{\sigma^k(m)}\right) - K\right|\right) \geq \gamma\right\}$$

uniformly in m. Since $G_k \overset{[W_{\mathcal{J}_{\sigma\theta}}^K(f)]}{\sim} H_k$, the last set belongs to $\mathcal{J}_{\sigma\theta}$. So, the first set belongs to $\mathcal{J}_{\sigma\theta}$ and $G_k \overset{W_{\mathcal{J}_{\sigma\theta}}^L(S)}{\sim} H_k$.

Theorem 2.4 If f is bounded, then for each $u \in V$

$$G_k \overset{[W_{\mathcal{J}_{\sigma\theta}}^K(f)]}{\sim} H_k \Leftrightarrow G_k \overset{W_{\mathcal{J}_{\sigma\theta}}^K(S)}{\sim} H_k.$$

Proof. Let f be bounded and $G_k \overset{W_{\mathcal{J}_{\sigma\theta}}^K(S)}{\sim} H_k$. Then, there exists a $M > 0$ such that $|f(a)| \leq M$ for all $a \geq 0$. Further using the fact, for $m = 1,2,...$, we have

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|d\left(u; G_{\sigma^k(m)}, H_{\sigma^k(m)}\right) - K\right|\right) =$$

$$\frac{1}{h_r}\sum_{\substack{k\in I_r \\ \left|d\left(u;G_{\sigma^k(m)},H_{\sigma^k(m)}\right)-K\right|\geq\varepsilon}} f\left(\left|d\left(u; G_{\sigma^k(m)}, H_{\sigma^k(m)}\right) - K\right|\right)$$

$$+\frac{1}{h_r}\sum_{\substack{k\in I_r \\ \left|d\left(u;G_{\sigma^k(m)},H_{\sigma^k(m)}\right)-K\right|<\varepsilon}} f\left(\left|d\left(u; G_{\sigma^k(m)}, H_{\sigma^k(m)}\right) - K\right|\right)$$

$$\leq \frac{M}{h_r}\left|\left\{k \in I_r: \left|d\left(u; G_{\sigma^k(m)}, H_{\sigma^k(m)}\right) - K\right| \geq \varepsilon\right\}\right| + f(\varepsilon)$$

uniformly in m. This proves that $G_k \overset{[W_{\mathcal{J}_{\sigma\theta}}^K(f)]}{\sim} H_k$.

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