A CLASS OF SUBMERSIONS AND COMPATIBLE MAPS IN FINSLER GEOMETRY

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ABSTRACT. We introduce a class of submersions between two Finslerian manifolds and the class of Finsler-compatible maps which contains the previous class. Defining also the notion of stretch it follows an upper bound for the stretch of these submersions. If the support manifold for the considered Finslerian geometries is the same we introduce a new function, called conformality, as a way to measure quantitatively the difference between the given geometries.

1. INTRODUCTION

The notion of *Riemannian submersion* is a main tool in the study of relationship between two given Riemann manifolds. In the last decades a number of books was dedicated to this subject and we cite only two of them: a most recent one is [8].

A well-known generalization of Riemann geometry is the Finslerian one and the question to extend the above notion to this setting appears in the list of open problems of [2]. Partial answers are given in [1] and [3]. A more general notion of *Lagrangian adapted to a submersion* was introduced in [7]. We remark in [3] that this problem can be interesting from the point of view of Finslerian versions of Kaluza-Klein theories.

The aim of this short note is to introduce two types of maps between two given Finslerian geometries. More precisely, in the first section we generalize two types of maps from Riemannian framework, the first being a submersion and the second a compatibility condition expressed in terms of induced norms. In the Riemannian case, the first class of maps was introduced only between Euclidean spheres and is a particular case of the second class. We introduce also the notion of *stretch function* associated to the given Finslerian manifolds. Our class of submersions have a bounded stretch.

In the second section we consider again two Finsler metrics F_1, F_2 but on the same manifold M. We introduce a new function c on the tangent bundle TM called *conformality* as a quantitative measure of the difference between F_1 and F_2. The name is chosen since in the case of a conformal transformation $F_2 = \rho F_1$ we have $c = F_2$ and in the case of a Randers transformation of the Riemannian metric a we get a conformal transformation of a. We propose as an open problem the further study of this function and its possible relationship with the quasiconformality in a Finslerian setting.

2. A class of submersions in Finsler geometry and Finsler-compatible maps

Let M be a smooth m-dimensional manifold with $m \geq 2$ and $\pi : TM \to M$ its tangent bundle. Let $x = (x^i) = (x^1, ..., x^m)$ be local coordinates on M and $(x, y) = (x^i, y^i) = (x^1, ..., x^m, y^1, ..., y^m)$ the induced coordinates on TM. Denote by O the null-section of π.

Recall after [3] that a *Finsler fundamental function* on M is a map $F : TM \to \mathbb{R}_+ = [0, +\infty)$ with the following properties:

F1) F is smooth on the slit tangent bundle $T_0 M := TM \setminus O$ and continuous on O,

F2) F is positive homogeneous of degree 1: $F(x, \lambda y) = \lambda F(x, y)$ for every $\lambda > 0$,

F3) the matrix $(g_{ij}) = \left(\frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j} \right)$ is invertible and its associated quadratic form is positive definite.

The tensor field $g = \{g_{ij}(x, y); 1 \leq i, j \leq m\}$ is called *the Finsler metric* and the homogeneity of F implies:

$$F^2(x, y) = g_{ij} y^i y^j = y_i y^i \tag{2.1}$$

where $y_i = g_{ij} y^j$. The pair (M, F) is called *Finsler manifold* and the restriction of F to a fiber $T_x M$ of TM is a norm $F(x, \cdot)$. Then g yields a Riemannian metric G_S on TM called *Sasaki*.

Example 1. *(Riemannian geometry) Let $a = (a_{ij}(x))$ be a Riemannian metric on M. It is well-known that $F(x, y) = \sqrt{a_{ij}(x) y^i y^j}$ is a Finslerian structure on M with $g = a$.*

Fix now the scalar $\eta > 0$ and a smooth map $f : (M^m, F_1) \to (N^n, F_2)$ between two Finsler manifolds with $m > n$. As usually, *the vertical bundle of f* is the vectorial bundle $V(f)$ over M having as fibre $V_x := Ker \, df(x)$. We introduce:

Definition 1. *i) An Ehresmann connection or horizontal bundle for f is a distribution $H(f)$ supplementary to $V(f)$, i.e. for all $x \in M$ we have:*

$$T_x M = V_x(f) \oplus H_x(f). \tag{2.2}$$

ii) f is called η-submersion if M admits a horizontal bundle $H^\eta(f)$ of dimension n and G_S-orthogonal to $V(f)$ such that for all $x \in M$ and $v \in H_x^\eta(f)$:

$$F_2(f(x), df(x)(v)) = \eta F_1(x, v) \tag{2.3}$$

iii) f is called Finsler-compatible map if there exists a C^0-function $\lambda : N \to \mathbb{R}_+$ such that for every $x \in M$ and $Y \in T_{f(x)}N$ there exists $X \in T_xM$ with:

$$Y = df(x)(X), \quad F_2(f(x), Y) \geq \lambda(f(x))F_1(x, X). \tag{2.4}$$

Remark 1. *1) If F_1 and F_2 are the Riemann metrics of Example 1 then the notion of Finsler-compatible map is particularized to metric-compatible map introduced in [9, p. 68].*
2) Again, in the Riemannian particular case the notion of ii) from the above definition was introduced by Agnes Hsu without any name in [5, p. 195]. The support manifolds are the unit Euclidean spheres and their set is denoted $H_n^m(\eta)$. We point out that in literature there exist the notions of horizontally homothetic and horizontally conformal submersion; see for example [6].

The following result is a direct consequence of the Rank Theorem and connects the types of maps introduced in the definition 1:

Proposition 1. *If f is an η-submersion then its rank is n i.e. f is indeed a submersion. Moreover, f is a Finsler-compatible map with a constant $\lambda = \eta$.*

In the following we introduce the notion of *stretch* for a given map.

Definition 2. *Fix $x \in M$ and the smooth map $f : (M^m, F_1) \to (N^n, F_2)$ with arbitrary m and n. The stretch of f in x with respect to F_1 and F_2 is the positive scalar:*

$$\delta_{F_1, F_2, f}(x) := \sup\{F_2(x, df(x)(v)); F_1(x, v) = 1\}. \tag{2.5}$$

Remark 2. *i) If $M = N$ and f is the identity we recover the notion of [4].*
ii) Due to the homogeneity of the Finsler fundamental function we get:

$$\delta_{F_1, F_2, f}(x) := \sup\{\sqrt{(g_2)_{ab}(x, df(x)(v))\frac{\partial f^a}{\partial x^i}(x)\frac{\partial f^b}{\partial x^j}(x)v^i v^j}; (g_1)_{ij}(x, v)v^i v^j = 1\}. \tag{2.6}$$

Here $i, j = 1, ..., m$, $a, b = 1, ..., n$ and $f = (f^1, ..., f^n)$.

In particular, if f is an η-submersion then its stretch is bounded:

$$0 < \delta_{F_1, F_2, f}(x) \leq \eta \tag{2.7}$$

for all $x \in M$. For the Riemannian case this inequality is the remark 3.1.(vi) of [5, p. 197]. In fact, let $v_0 \in T_xM$ satisfying the equality case in (2.5):

$$F_1(x, v_0) = 1, \quad F_2(x, df(x)(v_0)) = \delta_{F_1, F_2, f}(x). \tag{2.8}$$

We have the G_S^1-orthogonal decomposition (2.2) and then:

$$v_0 = v_0^v + v_0^\eta, \quad v_0^v \in V_x(f), \quad v_0^\eta \in H_x^\eta(f). \tag{2.9}$$

Therefore:

$$\delta_{F_1, F_2, f}(x) = F_2(f(x), df(x)(v_0^\eta)) = \eta F_1(x, v_0^\eta) \leq \eta F_1(x, v_0) = \eta. \tag{2.10}$$

In the last inequality we use the fact that an orthogonal projection decreases the norm.

3. A conformality function between two Finslerian geometries

We return now to the general setting $f : (M, F_1) \to (N, F_2)$ and suppose now that $M = N$. Inspired by the stretch function we introduce a new way to "measure the difference" between F_1 and F_2:

Definition 3. *The conformality function* $c_{F_1, F_2, f} : TM \to \mathbb{R}_+$ *is:*

$$c_{F_1, F_2, f}(x, v) = \sup\{F_2(f(x), df(x)(w)); w \in T_x M, F_1(x, w) = F_1(x, v)\}. \quad (3.1)$$

Example 2. *Suppose that* $f = 1_M$. *Then:*

$$c_{F_1, F_2, 1_M}(x, v) = \sup\{F_2(x, w); w \in T_x M, F_1(x, w) = F_1(x, v)\} \quad (3.2)$$

and in the particular case of $F_2 = \rho F_1$ *with* $\rho \in C_+^\infty(M)$ *a smooth strictly positive function on* M *we recover* F_2. *This example explains the given name for* c.

Example 3. *(Randers geometry) Let* (M, a) *be the Riemannian manifold of example 1 and let* $b = (b_i(x)) \in \Omega^1(M)$ *be an 1-form with* $\|b\|_a < 1$. *The function* $F : TM \to \mathbb{R}_+$, $F_R(x, y) = \sqrt{a_{ij} y^i y^j} + b_i(x) y^i$ *is a Finsler fundamental function which is called Randers. We are interested in computing* $c_{a, F_R, 1_M}$:

$$c_{a, F_R, 1_M}(x, v) = \sup\{\|w\|_a + b(w); \|w\|_a = \|v\|_a\} = \|v\|_a + \sup\{b(w); \|w\|_a = \|v\|_a\}. \quad (3.3)$$

A Cauchy-Schwarz type argument as in [4, p. 87] gives:

$$c_{a, F_R, 1_M}(x, v) = \|v\|_a + \|b(x)\|_a \|v\|_a = (1 + \|b(x)\|_a) \|v\|_a \quad (3.4)$$

and hence $c_{a, F_R, 1_M}$ *is a conformal transformation of the Riemannian metric* a.

An immediate property of the function c is given by: $c_{F_1, F_2, f}(x, 0) = 0$. Also, a straightforward computation yields the positive homogeneity:

$$\lambda \neq 0 : \quad c_{F_1, F_2, f}(x, \lambda v) = |\lambda| c_{F_1, F_2, f}(x, v). \quad (3.5)$$

For $v \in T_x M$, $v \neq 0$ we have the inequality:

$$c_{F_1, F_2, f}(x, v) \leq \delta_{F_1, F_2, f}(x) \cdot F_1(x, v) \quad (3.6)$$

since $\bar{w} := \frac{w}{F_1(x, v)}$ belongs to the indicatrix of F_1: $F_1(x, \bar{w}) = 1$ and we plug $\bar{w}$ in (2.5). We remark from Example 2 above that in the Randers geometry the relation (3.6) is an equality since the stretch of a Randers metric was computed in [4, p. 87] as: $\delta_{Randers}(x) = 1 + \|b(x)\|_a$. A first open problem is to determine conditions or geometries making equality in (3.6)

We consider a second open problem the further study of this function and its possible relationship with the quasiconformality in a Finslerian setting. Our hope is that under "mild conditions" the function c yields a (pseudo)distance:

$$d_{F_1, F_2, c}(x, y) = \inf\{\int_0^1 c_{F_1, F_2, f}(\gamma(t), \gamma'(t)) dt\} \quad (3.7)$$

where the infimum is taken over all piecewise C^1-curves $\gamma : [0, 1] \to M$ joining x to y.

References

[1] Álvarez Paiva, J. C. and Durán, C. E., Isometric submersions of Finsler manifolds, *Proc. Am. Math. Soc.* 129(8), (2001), 2409–2417.

[2] Anastasiei M., Certain generalizations of Finsler metrics, in Bao, David (Ed.) et al., Finsler geometry. Joint summer research conference, July 16-20, 1995, Seattle, WA, USA. Providence, RI: American Mathematical Society. Contemp. Math. 196, 161–169, 1996.

[3] Crasmareanu M., Fibre bundle maps and complete sprays in Finslerian setting, *J. Korean Math. Soc.* 46(3), (2009), 551–560.

[4] Crasmareanu M., New tools in Finsler geometry: stretch and Ricci solitons, *Math. Rep. (Bucur.)* 16(66)(1), (2014), 83–93.

[5] Hsu, Agnes Chi-Ling, A characterization of the Hopf map by stretch, *Math. Z.* 129, (1972), 195–206.

[6] Ou, Ye-Lin and Wilhelm, F., Horizontally homothetic submersions and nonnegative curvature, *Indiana Univ. Math. J.* 56(1), (2007), 243–261.

[7] Popescu, P., and Popescu, M., Lagrangians adapted to submersions and foliations, *Differ. Geom. Appl.* 27(2), (2009), 171–178.

[8] Şahin, B., Riemannian submersions, Riemannian maps in Hermitian geometry, and their applications, Amsterdam: Elsevier/Academic Press, 2017.

[9] Wolf, J. A., Differentiable fibre spaces and mappings compatible with Riemannian metrics, *Mich. Math. J.* 11, (1964), 65–70.

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