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Some Notes On (2,0)-Semitensor Bundle

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Abstract

We investigate some lifts of vector felds on a cross-section in the semi-tensor (pull-back) bundle tM of tensor bundle of type (2,0) by using projection (submersion) of the tangent bundle TM and we find some relation for them.

Keywords: *Cross-section, pull-back bundle, tangent bundle, semi-tensor bundle.*
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1. Introduction

Let M_n be an n-dimensional differentiable manifold of class C^∞, and let $(T(M_n), \pi_1, M_n)$ be a tangent bundle over M_n. We use the notation $(x^i) = (x^{\overline{\alpha}}, x^\alpha)$, where the indices $i, j, \ldots$ run from 1 to $2n$, the indices $\overline{\alpha}, \overline{\beta}, \ldots$ from 1 to n and the indices $\alpha, \beta, \ldots$ from $n+1$ to $2n$, x^α are coordinates in M_n, $x^{\overline{\alpha}} = y^\alpha$ are fibre coordinates of the tangent bundle $T(M_n)$.
Let now $\left(T_0^2(M_n), \widetilde{\pi}, M_n\right)$ be a tensor bundle of the type $(2,0)$ ([4], [7], [[9], p.118], [11]) over base space M_n, and let $T(M_n)$ be tangent bundle determined by a natural projection (submersion) $\pi_1 : T(M_n) \to M_n$. The semi-tensor bundle (pull-back [5],[6],[10],[12],[14],[15]) of the $(2,0)-$tensor bundle $\left(T_0^2(M_n), \widetilde{\pi}, M_n\right)$ is the bundle $\left(t_0^2(M_n), \pi_2, T(M_n)\right)$ over tangent bundle $T(M_n)$ with a total space

$$\begin{aligned} t_0^2(M_n) &= \left\{ \left(\left(x^{\overline{\alpha}}, x^\alpha\right), x^{\overline{\overline{\alpha}}} \right) \in T(M_n) \times \left(T_0^2\right)_x (M_n) : \pi_1 \left(x^{\overline{\alpha}}, x^\alpha\right) = \widetilde{\pi}\left(x^\alpha, x^{\overline{\overline{\alpha}}}\right) = (x^\alpha) \right\} \\ &\subset T(M_n) \times \left(T_0^2\right)_x (M_n) \end{aligned}$$

and with the projection map $\pi_2 : t_0^2(M_n) \to T(M_n)$ defined by $\pi_2(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}}) = \left(x^{\overline{\alpha}}, x^\alpha\right)$, where $\left(T_0^2\right)_x (M_n)$ $\left(x = \pi_1(\widetilde{x}), \widetilde{x} = \left(x^{\overline{\alpha}}, x^\alpha\right) \in T(M_n)\right)$ is the tensor space at a point x of M_n, where $x^{\overline{\overline{\alpha}}} = t^{\beta_1\beta_2}$ $\left(\overline{\overline{\alpha}}, \overline{\overline{\beta}}, \ldots = 2n+1, \ldots, 2n+n^2\right)$ are fiber coordinates of the tensor bundle $T_0^2(M_n)$.
The generalization of pull-back bundles to higher order cases is known as Pontryagin bundles [8].
If $(x^{i'}) = (x^{\overline{\alpha}'}, x^{\alpha'}, x^{\overline{\overline{\alpha}}'})$ is another system of local adapted coordinates in the semi-tensor bundle $t_0^2(M_n)$, then we have

$$\begin{cases} x^{\overline{\alpha}'} = \frac{\partial x^{\alpha'}}{\partial x^\beta} y^\beta, \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right), \\ x^{\overline{\overline{\alpha}}'} = t^{\beta_1'\beta_2'} = A^{\beta_1'}_{\alpha_1} A^{\beta_2'}_{\alpha_2} t^{\alpha_1\alpha_2}. \end{cases} \tag{1.1}$$

The Jacobian of (1.1) has components

$$\bar{A} = \left(A^{I'}_J\right) = \begin{pmatrix} A^{\alpha'}_\beta & A^{\alpha'}_{\beta\varepsilon} y^\varepsilon & 0 \\ 0 & A^{\alpha'}_\beta & 0 \\ 0 & t^{\alpha_1\alpha_2}\partial_\beta \left(A^{\beta_1'}_{\alpha_1} A^{\beta_2'}_{\alpha_2}\right) & A^{\beta_1'}_{\alpha_1} A^{\beta_2'}_{\alpha_2} \end{pmatrix}, \tag{1.2}$$

where $I = (\overline{\alpha}, \alpha, \overline{\overline{\alpha}})$, $J = (\overline{\beta}, \beta, \overline{\overline{\beta}})$, $I, J, \ldots = 1, \ldots, 2n+n^2$, $A^{\alpha'}_\beta = \frac{\partial x^{\alpha'}}{\partial x^\beta}$, $A^{\alpha'}_{\beta\varepsilon} = \frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\varepsilon}$.
It is easily verified that the condition $Det\bar{A} \neq 0$ is equivalent to the condition:

$$Det(A^{\alpha'}_\beta) \neq 0, Det(A^{\alpha'}_\beta) \neq 0, Det(A^{\beta_1'}_{\alpha_1} A^{\beta_2'}_{\alpha_2}) \neq 0.$$

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Also, $\dim t_0^2(M_n) = 2n + n^2$.
We note that cross-sections for $(2,0)-$tensor bundle and semi-tensor bundle of the type $(2,0)$ were examined in ([2],[3]). The main purpose of this paper is to study the behaviour of complete lifts of vector fields on cross-sections for $(2,0)-$semi tensor (pull-back) bundle by using projection of the tangent bundle $T(M_n)$. We denote by $\Im_q^p(T(M_n))$ and $\Im_q^p(M_n)$ the modules over $F(T(M_n))$ and $F(M_n)$ of all tensor fields of the type (p,q) on $T(M_n)$ and M_n, respectively, where $F(T(M_n))$ and $F(M_n)$ denote the rings of real-valued C^∞ $-$functions on $T(M_n)$ and M_n, respectively.

2. Vertical lifts of tensor fields and $\gamma-$operator

Let $A \in \Im_0^2(T(M_n))$. On putting

$$^{vv}A = \begin{pmatrix} ^{vv}A^{\overline{\alpha}} \\ ^{vv}A^{\alpha} \\ ^{vv}A^{\overline{\overline{\alpha}}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ A^{\alpha_1\alpha_2} \end{pmatrix}, \tag{2.1}$$

from (1.2), we easily see that with $^{vv}A' = \bar{A}\left(^{vv}A\right)$. The vector field $^{vv}A \in \Im_0^1(t_0^2(M_n))$ is called the vertical lift of $A \in \Im_0^2(T(M_n))$ to the semi-tensor bundle of the type $(2,0)$.
For any $\varphi \in \Im_1^1(M_n)$, if we take account of (1.2), we can prove that $(\gamma\varphi)' = \bar{A}(\gamma\varphi)$. Where $\gamma\varphi$ is a vector field in $\pi^{-1}(U)$ defined by

$$\gamma\varphi = (\gamma\varphi)^I = \begin{pmatrix} 0 \\ 0 \\ t^{\varepsilon\alpha_2}\varphi_\varepsilon^{\alpha_1} + t^{\alpha_1\varepsilon}\varphi_\varepsilon^{\alpha_2} \end{pmatrix}. \tag{2.2}$$

From (1.2) we easily see that the vector fields $\gamma\varphi$ defined in each $\pi^{-1}(U) \subset t_0^2(M_n)$ determine global vertical vector fields on $t_q^p(M_n)$. We call $\gamma\varphi$ the vertical-vector lift of the tensor field $\varphi \in \Im_1^1(M_n)$ to $t_0^2(M_n)$.

For any $\varphi \in \Im_1^1(T(M_n))$, if we take account of (1.2), we can prove that $(\gamma\varphi)' = \bar{A}(\gamma\varphi)$, where $\gamma\varphi$ is a vector field defined by

$$\gamma\varphi = \begin{pmatrix} y^\varepsilon\varphi_\varepsilon^\beta \\ 0 \\ 0 \end{pmatrix} \tag{2.3}$$

with respect to the coordinates $(x^{\overline{\beta}}, x^\beta, x^{\overline{\overline{\beta}}})$.

3. Complete lifts of vector fields

Let $X \in \Im_0^1(T(M_n))$, i.e. $X = X^\alpha(x^\alpha)\partial_\alpha$. The complete lift ^cX of X to tangent bundle is defined by $^cX = X^\alpha\partial_\alpha + y^\varepsilon\partial_\varepsilon X^\alpha\partial_{\overline{\alpha}}$ [[13], p.15]. On putting

$$^{cc}X = \begin{pmatrix} ^{cc}X^{\overline{\beta}} \\ ^{cc}X^{\beta} \\ ^{cc}X^{\overline{\overline{\beta}}} \end{pmatrix} = \begin{pmatrix} y^\varepsilon\partial_\varepsilon X^\beta \\ X^\beta \\ t^{\varepsilon\alpha_2}\partial_\varepsilon X^{\alpha_1} + t^{\alpha_1\varepsilon}\partial_\varepsilon X^{\alpha_2} \end{pmatrix}, \tag{3.1}$$

from (1.2), we easily see that $^{cc}X' = \bar{A}(^{cc}X)$. The vector field ^{cc}X is called the complete lift of $^cX \in \Im_0^1(T(M_n))$ to $t_0^2(M_n)$.

Proof. If $X \in \Im_0^1(M_n)$ and $\begin{pmatrix} ^{cc}X^{\overline{\beta}} \\ ^{cc}X^{\beta} \\ ^{cc}X^{\overline{\overline{\beta}}} \end{pmatrix}$ are components of $(^{cc}X)^J$ with respect to the coordinates $(x^{\overline{\beta}}, x^\beta, x^{\overline{\overline{\beta}}})$ on $t_0^2(M_n)$, then we have by (1.2) and (3.1):

$$\begin{aligned} (^{cc}X)^J &= A_I^J(^{cc}X)^I \\ (^{cc}X)^J &= A_{\overline{\alpha}}^J(^{cc}X)^{\overline{\alpha}} + A_\alpha^J(^{cc}X)^\alpha + A_{\overline{\overline{\alpha}}}^J(^{cc}X)^{\overline{\overline{\alpha}}}. \end{aligned}$$

Firstly, if $J = \overline{\beta}$, we have

$$\begin{aligned} (^{cc}X)^{\overline{\beta}} &= A_{\overline{\alpha}}^{\overline{\beta}}(^{cc}X)^{\overline{\alpha}} + A_\alpha^{\overline{\beta}}(^{cc}X)^\alpha + A_{\overline{\overline{\alpha}}}^{\overline{\beta}}(^{cc}X)^{\overline{\overline{\alpha}}} \\ &= A_\alpha^\beta\left(y^\varepsilon\partial_\varepsilon X^\alpha\right) + \left(A_{\beta\varepsilon}^\alpha y^\varepsilon\right)X^\alpha \\ &= y^\varepsilon A_\alpha^\beta\left(\partial_\varepsilon X^\alpha\right) + y^\varepsilon\left(\partial_\varepsilon A_\alpha^\beta\right)X^\alpha \\ &= y^\varepsilon\partial_\varepsilon\left(A_\alpha^\beta X^\alpha\right) \\ &= y^\varepsilon\partial_\varepsilon X^\beta \end{aligned}$$

by virtue of (1.2) and (3.1). Secondly, if $J=\beta$, we have

$$\begin{aligned} ({}^{cc}X)^{\beta} &= A^{\beta}_{\overline{\alpha}}({}^{cc}X)^{\overline{\alpha}}+A^{\beta}_{\alpha}({}^{cc}X)^{\alpha}+A^{\beta}_{\overline{\overline{\alpha}}}({}^{cc}X)^{\overline{\overline{\alpha}}} \\ &= A^{\beta}_{\alpha}X^{\alpha}=X^{\beta} \end{aligned}$$

by virtue of (1.2) and (3.1). Thirdly, if $J=\overline{\overline{\beta}}$, then we have

$$\begin{aligned} ({}^{cc}X)^{\overline{\overline{\beta}}} &= t^{\varepsilon\alpha_2}(\partial_{\varepsilon}X^{\alpha_1})A^{\beta_1}_{\alpha_1}+t^{\varepsilon\alpha_2}\left(\partial_{\varepsilon}A^{\beta_1}_{\alpha_1}\right)X^{\alpha_1} \\ &\quad +t^{\alpha_1\varepsilon}(\partial_{\varepsilon}X^{\alpha_2})A^{\beta_2}_{\alpha_2}+t^{\alpha_1\varepsilon}\left(\partial_{\varepsilon}A^{\beta_2}_{\alpha_2}\right)X^{\alpha_2} \\ &= \sum_{p=1}^{4}a_p, \end{aligned}$$

where

$$\begin{aligned} a_1 &= t^{\varepsilon\alpha_2}(\partial_{\varepsilon}X^{\alpha_1})A^{\beta_1}_{\alpha_1}, \\ a_2 &= t^{\varepsilon\alpha_2}\left(\partial_{\varepsilon}A^{\beta_1}_{\alpha_1}\right)X^{\alpha_1}, \\ a_3 &= t^{\alpha_1\varepsilon}(\partial_{\varepsilon}X^{\alpha_2})A^{\beta_2}_{\alpha_2}, \\ a_4 &= t^{\alpha_1\varepsilon}\left(\partial_{\varepsilon}A^{\beta_2}_{\alpha_2}\right)X^{\alpha_2}. \end{aligned}$$

On the other hand,

$$\begin{aligned} A^{\overline{\overline{\beta}}}_{\alpha}({}^{cc}X)^{\alpha} &= A^{\overline{\overline{\beta}}}_{\overline{\alpha}}({}^{cc}X)^{\overline{\alpha}}+A^{\overline{\overline{\beta}}}_{\alpha}({}^{cc}X)^{\alpha}+A^{\overline{\overline{\beta}}}_{\overline{\overline{\alpha}}}({}^{cc}X)^{\overline{\overline{\alpha}}} \\ &= X^{\alpha}t^{\alpha_1\alpha_2}\left(\partial_{\alpha}A^{\beta_1}_{\alpha_1}\right)A^{\beta_2}_{\alpha_2}+X^{\alpha}t^{\alpha_1\alpha_2}A^{\beta_1}_{\alpha_1}\partial_{\alpha}A^{\beta_2}_{\alpha_2} \\ &\quad +A^{\beta_1}_{\alpha_1}A^{\beta_2}_{\alpha_2}t^{\varepsilon\alpha_2}\partial_{\varepsilon}X^{\alpha_1}+A^{\beta_1}_{\alpha_1}A^{\beta_2}_{\alpha_2}t^{\alpha_1\varepsilon}\partial_{\varepsilon}X^{\alpha_2} \\ &= \sum_{q=1}^{4}b_q, \end{aligned}$$

where,

$$\begin{aligned} b_1 &= X^{\alpha}t^{\alpha_1\alpha_2}\left(\partial_{\alpha}A^{\beta_1}_{\alpha_1}\right)A^{\beta_2}_{\alpha_2}, \\ b_2 &= X^{\alpha}t^{\alpha_1\alpha_2}A^{\beta_1}_{\alpha_1}\partial_{\alpha}A^{\beta_2}_{\alpha_2}, \\ b_3 &= A^{\beta_1}_{\alpha_1}A^{\beta_2}_{\alpha_2}t^{\varepsilon\alpha_2}\partial_{\varepsilon}X^{\alpha_1}, \\ b_4 &= A^{\beta_1}_{\alpha_1}A^{\beta_2}_{\alpha_2}t^{\alpha_1\varepsilon}\partial_{\varepsilon}X^{\alpha_2}. \end{aligned}$$

You can check that

$a_1=b_3, a_2=b_1, a_3=b_2, a_4=b_4.$

Thus, we have (3.1). □

4. Horizontal lifts of vector fields

Let $X\in\Im^1_0(T(M_n))$, i.e. $X=X^{\alpha}\partial_{\alpha}$. If we take account of (1.2), we can prove that ${}^{HH}X'=\bar{A}\left({}^{HH}X\right)$, where ${}^{HH}X\in\Im^1_0(t^2_0(M_n))$ is a vector field defined by

$$ {}^{HH}X=\begin{pmatrix} -\Gamma^{\beta}_{\alpha}X^{\alpha} \\ X^{\beta} \\ -\Gamma^{\alpha_1}_{\sigma\varepsilon}t^{\varepsilon\alpha_2}X^{\sigma}-\Gamma^{\alpha_2}_{\sigma\varepsilon}t^{\alpha_1\varepsilon}X^{\sigma} \end{pmatrix}, \tag{4.1}$$

with respect to the coordinates $(x^{\overline{\beta}},x^{\beta},x^{\overline{\overline{\beta}}})$ on $t^2_0(M_n)$. We call ${}^{HH}X$ the horizontal lift of the vector field X to $t^2_0(M_n)$. Where

$\Gamma^{\beta}_{\alpha}=y^{\varepsilon}\Gamma^{\beta}_{\varepsilon\,\alpha}.$

Theorem 4.1. *If $X\in\Im^1_0(T(M_n))$ then*

${}^{cc}X-{}^{HH}X=\gamma(\hat{\nabla}X)+\gamma(\nabla X),$

where the symmetric affine connection $\hat{\nabla}$ is the given by $\widehat{\Gamma}^{\alpha}_{\beta\theta}=\Gamma^{\alpha}_{\theta\beta}$.

Proof. From (2.2), (2.3), (3.1) and (4.1), we have

$$\begin{aligned}
{}^{cc}X - {}^{HH}X &= \begin{pmatrix} y^{\varepsilon}\partial_{\varepsilon}X^{\beta} \\ X^{\beta} \\ t^{\varepsilon\alpha_2}\partial_{\varepsilon}X^{\alpha_1} + t^{\alpha_1\varepsilon}\partial_{\varepsilon}X^{\alpha_2} \end{pmatrix} - \begin{pmatrix} -\Gamma^{\beta}_{\alpha}X^{\alpha} \\ X^{\beta} \\ -\Gamma^{\alpha_1}_{\sigma\varepsilon}t^{\varepsilon\alpha_2}X^{\sigma} - \Gamma^{\alpha_2}_{\sigma\varepsilon}t^{\alpha_1\varepsilon}X^{\sigma} \end{pmatrix} \\
&= \begin{pmatrix} y^{\varepsilon}\partial_{\varepsilon}X^{\beta} + y^{\varepsilon}\Gamma^{\beta}_{\varepsilon\alpha}X^{\alpha} \\ 0 \\ t^{\varepsilon\alpha_2}\partial_{\varepsilon}X^{\alpha_1} + t^{\alpha_1\varepsilon}\partial_{\varepsilon}X^{\alpha_2} + \Gamma^{\alpha_1}_{\sigma\varepsilon}t^{\varepsilon\alpha_2}X^{\sigma} + \Gamma^{\alpha_2}_{\sigma\varepsilon}t^{\alpha_1\varepsilon}X^{\sigma} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ t^{\varepsilon\alpha_2}\partial_{\varepsilon}X^{\alpha_1} + \Gamma^{\alpha_1}_{\sigma\varepsilon}t^{\varepsilon\alpha_2}X^{\sigma} + t^{\alpha_1\varepsilon}\partial_{\varepsilon}X^{\alpha_2} + \Gamma^{\alpha_2}_{\sigma\varepsilon}t^{\alpha_1\varepsilon}X^{\sigma} \end{pmatrix} \\
&\quad + \begin{pmatrix} y^{\varepsilon}\left(\partial_{\varepsilon}X^{\beta} + \Gamma^{\beta}_{\varepsilon\alpha}X^{\alpha}\right) \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ t^{\varepsilon\alpha_2}\left(\partial_{\varepsilon}X^{\alpha_1} + \Gamma^{\alpha_1}_{\sigma\varepsilon}X^{\sigma}\right) + t^{\alpha_1\varepsilon}\left(\partial_{\varepsilon}X^{\alpha_2} + \Gamma^{\alpha_2}_{\sigma\varepsilon}X^{\sigma}\right) \end{pmatrix} \\
&\quad + \begin{pmatrix} y^{\varepsilon}\left(\partial_{\varepsilon}X^{\beta} + \Gamma^{\beta}_{\varepsilon\alpha}X^{\alpha}\right) \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ t^{\varepsilon\alpha_2}\left(\partial_{\varepsilon}X^{\alpha_1} + \Gamma^{\alpha_1}_{\varepsilon\sigma}X^{\sigma}\right) + t^{\alpha_1\varepsilon}\left(\partial_{\varepsilon}X^{\alpha_2} + \Gamma^{\alpha_2}_{\varepsilon\sigma}X^{\sigma}\right) \end{pmatrix} \\
&\quad + \begin{pmatrix} y^{\varepsilon}\left(\nabla_{\varepsilon}X^{\beta}\right) \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ t^{\varepsilon\alpha_2}\left(\hat{\nabla}_{\varepsilon}\tilde{X}^{\alpha_1}\right) + t^{\alpha_1\varepsilon}\left(\hat{\nabla}_{\varepsilon}\tilde{X}^{\alpha_2}\right) \end{pmatrix} + \begin{pmatrix} y^{\varepsilon}\left(\nabla_{\varepsilon}X^{\beta}\right) \\ 0 \\ 0 \end{pmatrix} \\
&= \gamma(\hat{\nabla}X) + \gamma(\nabla X)
\end{aligned}$$

which prove Theorem 4.1. □

5. Cross-sections in the semi-tensor bundle of the type (2,0)

Let $\xi \in \Im^2_0(M_n)$ be a tensor field of the type $(2,0)$ on M_n. Then the correspondence $x \to \xi_x$, ξ_x being the value of ξ at $x \in T(M_n)$, determines a cross-section β_{ξ} of $t^2_0(M_n)$.
Thus if $\sigma_{\xi} : M_n \to T^2_0(M_n)$ is a cross-section of $\left(T^2_0(M_n), \tilde{\pi}, M_n\right)$, such that $\tilde{\pi} \circ \sigma_{\xi} = I_{(M_n)}$, an associated cross-section $\beta_{\xi} : T(M_n) \to t^2_0(M_n)$ of semi-tensor bundle $\left(t^2_0(M_n), \pi_2, T(M_n)\right)$ defined by [[1], p. 217-218], [5], [6], [[13], p. 122]:

$$\beta_{\xi}\left(x^{\overline{\alpha}}, x^{\alpha}\right) = \left(x^{\overline{\alpha}}, x^{\alpha}, \sigma_{\xi} \circ \pi_1\left(x^{\overline{\alpha}}, x^{\alpha}\right)\right) = \left(x^{\overline{\alpha}}, x^{\alpha}, \sigma_{\xi}\left(x^{\alpha}\right)\right) = \left(x^{\overline{\alpha}}, x^{\alpha}, \xi^{\alpha_1\alpha_2}\left(x^{\beta}\right)\right).$$

If the $(2,0)-$tensor field ξ has the local components $\xi^{\alpha_1\alpha_2}\left(x^{\alpha}\right)$, the cross-section $\beta_{\xi}\left(T(M_n)\right)$ of $t^2_0(M_n)$ is locally expressed by

$$\begin{cases} x^{\overline{\beta}} = y^{\beta} = V^{\beta}\left(x^{\alpha}\right), \\ x^{\beta} = x^{\beta}, \\ x^{\overline{\overline{\beta}}} = \xi^{\alpha_1\alpha_2}\left(x^{\alpha}\right), \end{cases} \tag{5.1}$$

with respect to the coordinates $x^B = (x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ in $t^2_0(M_n)$.
$x^{\overline{\alpha}} = y^{\alpha}$ being considered as parameters. Differentiating (5.1) by $x^{\overline{\alpha}} = y^{\alpha}$, we have vector fields $B_{(\overline{\theta})}$ $(\overline{\theta} = 1, \ldots, n)$ with components

$$B_{(\overline{\theta})} = \frac{\partial x^B}{\partial x^{\overline{\theta}}} = \partial_{\overline{\theta}} x^B = \begin{pmatrix} \partial_{\overline{\theta}} V^{\beta} \\ \partial_{\overline{\theta}} x^{\beta} \\ \partial_{\overline{\theta}} \xi^{\alpha_1\alpha_2} \end{pmatrix},$$

which are tangent to the cross-section $\beta_{\theta}\left(T(M_n)\right)$.
Thus $B_{(\overline{\theta})}$ have components

$$B_{(\overline{\theta})} : \left(B^B_{(\overline{\theta})}\right) = \begin{pmatrix} \delta^{\beta}_{\overline{\theta}} \\ 0 \\ 0 \end{pmatrix},$$

with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ in $t_0^2(M_n)$. Where

$$\delta_{\overline{\theta}}^{\beta} = A_{\overline{\theta}}^{\beta} = \frac{\partial x^{\beta}}{\partial x^{\overline{\theta}}}.$$

Let $X \in \Im_0^1(T(M_n))$, i.e. $X = X^{\alpha}\partial_{\alpha}$. We denote by BX the vector field with local components

$$BX : \left(B_{(\overline{\theta})}^{B} X^{\overline{\theta}}\right) = \begin{pmatrix} \delta_{\overline{\theta}}^{B} X^{\overline{\theta}} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A_{\overline{\theta}}^{B} X^{\overline{\theta}} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} X^{\beta} \\ 0 \\ 0 \end{pmatrix} \tag{5.2}$$

with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ in $t_0^2(M_n)$, which is defined globally along $\beta_{\xi}(T(M_n))$.
Differentiating (5.1) by x^{θ}, we have vector fields $C_{(\theta)}$ $(\theta = n+1, ..., 2n)$ with components

$$C_{(\theta)} = \frac{\partial x^{B}}{\partial x^{\theta}} = \partial_{\theta} x^{B} = \begin{pmatrix} \partial_{\theta} x^{\overline{\beta}} \\ \partial_{\theta} x^{\beta} \\ \partial_{\theta} \xi^{\alpha_1 \alpha_2} \end{pmatrix},$$

which are tangent to the cross-section $\beta_{\xi}(T(M_n))$.
Thus $C_{(\theta)}$ have components

$$C_{(\theta)} : \left(C_{(\theta)}^{B}\right) = \begin{pmatrix} \partial_{\theta} V^{\beta} \\ \delta_{\theta}^{\beta} \\ \partial_{\theta} \xi^{\alpha_1 \alpha_2} \end{pmatrix},$$

with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ in $t_0^2(M_n)$. Where

$$\delta_{\theta}^{\beta} = A_{\theta}^{\beta} = \frac{\partial x^{\beta}}{\partial x^{\theta}}.$$

Let $X \in \Im_0^1(T(M_n))$. Then we denote by CX the vector field with local components

$$CX : \left(C_{(\theta)}^{B} X^{\theta}\right) = \begin{pmatrix} X^{\theta} \partial_{\theta} V^{\beta} \\ X^{\beta} \\ X^{\theta} \partial_{\theta} \xi^{\alpha_1 \alpha_2} \end{pmatrix} \tag{5.3}$$

with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ in $t_0^2(M_n)$, which is defined globally along $\beta_{\xi}(T(M_n))$.
On the other hand, the fibre is locally expressed by

$$\begin{cases} x^{\overline{\beta}} = y^{\beta} = const., \\ x^{\beta} = const., \\ x^{\overline{\overline{\beta}}} = t^{\alpha_1 \alpha_2} = t^{\alpha_1 \alpha_2}, \end{cases}$$

$t^{\alpha_1 \alpha_2}$ being considered as parameters. Thus, on differentiating with respect to $x^{\overline{\overline{\beta}}} = t^{\alpha_1 \alpha_2}$, we easily see that the vector fields $E_{(\overline{\overline{\theta}})}$ $(\overline{\theta} = 2n+1, ..., 2n+n^2)$ with components

$$E_{(\overline{\overline{\theta}})} : \left(E_{(\overline{\overline{\theta}})}^{B}\right) = \partial_{\overline{\overline{\theta}}} x^{B} = \begin{pmatrix} \partial_{\overline{\overline{\theta}}} y^{\beta} \\ \partial_{\overline{\overline{\theta}}} x^{\beta} \\ \partial_{\overline{\overline{\theta}}} t^{\alpha_1 \alpha_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \delta_{\gamma_1}^{\alpha_1} \delta_{\gamma_2}^{\alpha_2} \end{pmatrix}$$

is tangent to the fibre, where δ is the Kronecker symbol.
Let ξ be a tensor field of the type $(2,0)$ with local components

$$\xi = \xi^{\gamma_1 \gamma_2} \partial_{\gamma_1} \otimes \partial_{\gamma_2}$$

on M_n.
We denote by $E\xi$ the vector field with local components

$$E\xi : \left(E_{(\overline{\overline{\theta}})}^{B} \xi^{\gamma_1 \gamma_2}\right) = \begin{pmatrix} 0 \\ 0 \\ \xi^{\alpha_1 \alpha_2} \end{pmatrix}, \tag{5.4}$$

which is tangent to the fibre.

Theorem 5.1. *Let X be a vector field on $T(M_n)$, we have along $\beta_{\xi}(T(M_n))$ the formula*

$${}^{cc}X = CX + B(L_V X) + E(-L_X \xi),$$

where $L_V X$ denotes the Lie derivative of X with respect to V, and $L_X \xi$ denotes the Lie derivative of ξ with respect to X.

Proof. Using (3.1), (5.2), (5.3) and (5.4), we have

$$\begin{aligned} CX+B(L_VX)+E(-L_X\xi) &= \begin{pmatrix} X^{\theta}\partial_{\theta}V^{\beta} \\ X^{\beta} \\ X^{\theta}\partial_{\theta}\xi^{\alpha_1\alpha_2} \end{pmatrix} + \begin{pmatrix} V^{\alpha}\partial_{\alpha}X^{\beta}-X^{\alpha}\partial_{\alpha}V^{\beta} \\ 0 \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ 0 \\ -X^{\theta}\partial_{\theta}\xi^{\alpha_1\alpha_2}+\xi^{\varepsilon\alpha_2}\partial_{\varepsilon}X^{\alpha_1}+\xi^{\alpha_1\varepsilon}\partial_{\varepsilon}X^{\alpha_2} \end{pmatrix} \\ &= \begin{pmatrix} V^{\alpha}\partial_{\alpha}X^{\beta} \\ X^{\beta} \\ \xi^{\varepsilon\alpha_2}\partial_{\varepsilon}X^{\alpha_1}+\xi^{\alpha_1\varepsilon}\partial_{\varepsilon}X^{\alpha_2} \end{pmatrix} = {}^{cc}X. \end{aligned}$$

Thus, we have Theorem 5.1. □

On the other hand, on putting $C_{(\overline{\overline{\beta}})}=E_{(\overline{\overline{\beta}})}$, we write the adapted frame of $\beta_{\xi}(T(M_n))$ as $\left\{B_{(\overline{\beta})},C_{(\beta)},C_{(\overline{\overline{\beta}})}\right\}$. The adapted frame $\left\{B_{(\overline{\beta})},C_{(\beta)},C_{(\overline{\overline{\beta}})}\right\}$ of $\beta_{\xi}(T(M_n))$ is given by the matrix

$$\widetilde{A}=\left(\widetilde{A}_B^A\right)=\begin{pmatrix} \delta_{\beta}^{\alpha} & \partial_{\beta}V^{\alpha} & 0 \\ 0 & \delta_{\beta}^{\alpha} & 0 \\ 0 & \partial_{\beta}\xi_{\alpha_1\alpha_2}^{\sigma_1\sigma_2} & \delta_{\alpha_1}^{\beta_1}\delta_{\alpha_2}^{\beta_2} \end{pmatrix}. \tag{5.5}$$

Since the matrix $\widetilde{A}$ in (5.5) is non-singular, it has the inverse. Denoting this inverse by $\left(\widetilde{A}\right)^{-1}$, we have

$$\left(\widetilde{A}\right)^{-1}=\left(\widetilde{A}_C^B\right)^{-1}=\begin{pmatrix} \delta_{\theta}^{\beta} & -\partial_{\theta}V^{\beta} & 0 \\ 0 & \delta_{\theta}^{\beta} & 0 \\ 0 & -\partial_{\theta}\xi_{\beta_1\beta_2}^{\sigma_1\sigma_2} & \delta_{\beta_1}^{\theta_1}\delta_{\beta_2}^{\theta_2} \end{pmatrix}, \tag{5.6}$$

where $\widetilde{A}\left(\widetilde{A}\right)^{-1}=(\widetilde{A}_B^A)\left(\widetilde{A}_C^B\right)^{-1}=\delta_C^A=\widetilde{I}$, where $A=(\overline{\alpha},\alpha,\overline{\overline{\alpha}})$, $B=\left(\overline{\beta},\beta,\overline{\overline{\beta}}\right)$, $C=\left(\overline{\theta},\theta,\overline{\overline{\theta}}\right)$.

Proof. In fact, from (5.5) and (5.6), we easily see that

$$\begin{aligned} \widetilde{A}\left(\widetilde{A}\right)^{-1} &= (\widetilde{A}_B^A)\left(\widetilde{A}_C^B\right)^{-1}=\begin{pmatrix} \delta_{\beta}^{\alpha} & \partial_{\beta}V^{\alpha} & 0 \\ 0 & \delta_{\beta}^{\alpha} & 0 \\ 0 & \partial_{\beta}\xi_{\alpha_1\alpha_2}^{\sigma_1\sigma_2} & \delta_{\alpha_1}^{\beta_1}\delta_{\alpha_2}^{\beta_2} \end{pmatrix}\begin{pmatrix} \delta_{\theta}^{\beta} & -\partial_{\theta}V^{\beta} & 0 \\ 0 & \delta_{\theta}^{\beta} & 0 \\ 0 & -\partial_{\theta}\xi_{\beta_1\beta_2}^{\sigma_1\sigma_2} & \delta_{\beta_1}^{\theta_1}\delta_{\beta_2}^{\theta_2} \end{pmatrix} \\ &= \begin{pmatrix} \delta_{\theta}^{\alpha} & -\partial_{\theta}V^{\alpha}+\partial_{\theta}V^{\alpha} & 0 \\ 0 & \delta_{\theta}^{\alpha} & 0 \\ 0 & \partial_{\theta}\xi_{\alpha_1\alpha_2}^{\sigma_1\sigma_2}-\partial_{\theta}\xi_{\alpha_1\alpha_2}^{\sigma_1\sigma_2} & \delta_{\alpha_1}^{\theta_1}\delta_{\alpha_2}^{\theta_2} \end{pmatrix} = \begin{pmatrix} \delta_{\theta}^{\alpha} & 0 & 0 \\ 0 & \delta_{\theta}^{\alpha} & 0 \\ 0 & 0 & \delta_{\alpha}^{\theta} \end{pmatrix} = \delta_C^A=\widetilde{I}. \end{aligned}$$

□

Then we see from Theorem 5.1 that the complete lift ${}^{cc}X$ of a vector field $X\in\Im_0^1(T(M_n))$ has along $\beta_{\xi}(T(M_n))$ components of the form

$${}^{cc}X:\begin{pmatrix} L_VX \\ X \\ -L_X\xi \end{pmatrix},$$

with respect to the adapted frame $\left\{B_{(\overline{\beta})},C_{(\beta)},C_{(\overline{\overline{\beta}})}\right\}$.

Let $A\in\Im_0^2(T(M_n))$. If we take account of (2.1) and (5.5), we can easily prove that ${}^{vv}A'=\widetilde{A}\,({}^{vv}A)$, where ${}^{vv}A\in\Im_0^1(t_0^2(M_n))$ is a vector field defined by

$${}^{vv}A=\begin{pmatrix} {}^{vv}A^{\overline{\alpha}} \\ {}^{vv}A^{\alpha} \\ {}^{vv}A^{\overline{\overline{\alpha}}} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ A^{\alpha_1\alpha_2} \end{pmatrix},$$

with respect to the adapted frame $\left\{B_{(\overline{\beta})},C_{(\beta)},C_{(\overline{\overline{\beta}})}\right\}$ of $\beta_{\xi}(T(M_n))$.

Let $\varphi\in\Im_1^1(M_n)$ now. If we take account of (2.2) and (5.5), we see that $(\gamma\varphi)'=\widetilde{A}(\gamma\varphi)$. $\gamma\varphi$ is given by

$$\gamma\varphi=(\gamma\varphi)^I=\begin{pmatrix} 0 \\ 0 \\ t^{\varepsilon\alpha_2}\varphi_{\varepsilon}^{\alpha_1}+t^{\alpha_1\varepsilon}\varphi_{\varepsilon}^{\alpha_2} \end{pmatrix},$$

with respect to the adapted frame $\left\{B_{(\overline{\beta})}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)}\right\}$.

BX, CX and $E\xi$ also have components:

$$BX = \begin{pmatrix} X^{\alpha} \\ 0 \\ 0 \end{pmatrix}, CX = \begin{pmatrix} 0 \\ X^{\alpha} \\ 0 \end{pmatrix}, E\xi = \begin{pmatrix} 0 \\ 0 \\ \xi^{\alpha_1 \alpha_2} \end{pmatrix}$$

respectively, with respect to the adapted frame $\left\{B_{(\overline{\beta})}, C_{(\beta)}, C_{\left(\overline{\overline{\beta}}\right)}\right\}$ of the cross-section $\beta_{\xi}\left(T(M_n)\right)$ determined by a tensor field ξ of the type $(2,0)$ in $T(M_n)$.

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