The Hadamard Products for Bi-periodic Fibonacci and Bi-periodic Lucas Generating Matrices

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Abstract

In this paper, firstly, we define the Q_q *-generating matrix* for bi-periodic Fibonacci polynomial. And we give nth power, determinant and some properties of the bi-periodic Fibonacci polynomial by considering this matrix representation. Also, we introduce the Hadamard products for bi-periodic Fibonacci Q_q^n generating matrix and bi-periodic Lucas Q_l^n generating matrix of which entries are bi-periodic Fibonacci and Lucas numbers. Then, we investigate some properties of these products.

Keywords: bi-periodic Fibonacci numbers; bi-periodic Fibonacci polynomial; bi-periodic Lucas numbers; Fibonacci Q matrix; generating matrix; matrix method.
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1. Introduction

The special sequences and their properties have been investigated in many articles and books (see, for example [1, 3, 5, 6, 8, 9], [14]-[17] and the references cited there in). The Fibonacci and Lucas numbers have attracted the attention of mathematicians because of their intrinsic theory and applications. Fibonacci $\{F_n\}_{n\in\mathbb{N}}$ and Lucas $\{L_n\}_{n\in\mathbb{N}}$ sequences was defined

$$F_{n+1} = F_n + F_{n-1}, \ L_{n+1} = L_n + L_{n-1}$$

with initial conditions $F_1 = F_2 = 1, L_1 = 2, L_2 = 1$.
Many authors have generalized Fibonacci sequence in different ways. In the one of those generalizations, in [17], we define the bi-periodic Fibonacci $\{q_n(x)\}_{n\in\mathbb{N}}$ polynomial as in the form

$$q_n(x) = \begin{cases} axq_{n-1}(x) + q_{n-2}(x), & \text{if } n \text{ is odd} \\ bxq_{n-1}(x) + q_{n-2}(x), & \text{if } n \text{ is even} \end{cases} \tag{1.1}$$

where $q_0(x) = 0$, $q_1(x) = 1$ and a, b are nonzero real numbers.
Also, the Binet formula and Cassini identity of this polynomial was given. In [5], for a and b are nonzero real numbers, the authors defined the bi-periodic Fibonacci $\{q_n\}_{n\in\mathbb{N}}$ sequence

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd} \end{cases},$$

where $q_0 = 0$, $q_1 = 1$. In [1], for a and b are nonzero real numbers, it is defined the bi-periodic Lucas $\{l_n\}_{n\in\mathbb{N}}$ sequence as

$$l_n = \begin{cases} bl_{n-1} + l_{n-2}, & \text{if } n \text{ is even} \\ al_{n-1} + l_{n-2}, & \text{if } n \text{ is odd} \end{cases},$$

where $l_0 = 2$, $l_1 = a$. The author also gave in the following relations

$$\begin{aligned} (ab+4)q_n &= l_{n+1} + l_{n-1}, \\ l_n &= q_{n+1} + q_{n-1}, \end{aligned} \tag{1.2}$$

$$a^{1-\varepsilon(n)}b^{\varepsilon(n)}q_{n+1}q_{n-1} - a^{\varepsilon(n)}b^{1-\varepsilon(n)}q_n^2 \;=\; a(-1)^n, \tag{1.3}$$

$$\left(\frac{b}{a}\right)^{1-\varepsilon(n)} l_{n+1}l_{n-1} - \left(\frac{b}{a}\right)^{\varepsilon(n)} l_n^2 \;=\; (ab+4)(-1)^{n+1}$$

where $\varepsilon(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$.

On the other hand, it have been studied the matrix representation of special sequences ([2, 4, 7, 11, 12]). In [12], Sylvester gave Fibonacci Q-matrix as

$$Q = \left[\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array}\right].$$

Then, he says that some properties of Fibonacci numbers can be founded by using this matrix. Considering this matrix, in [8], the author obtained some equalities for Q^n.

And, using this properties in [10], the authors defined the Hadamard product $Q^n \circ Q^{-n}$. Similarly, in [13], the author gave some properties for the Hadamard products of its Adjoint matrix with a square matrix.

In [4], we defined the Q_l matrix as

$$Q_l = \left[\begin{array}{cc} a^2 + 2\frac{a}{b} & \frac{a^2}{b} \\ a & 2\frac{a}{b} \end{array}\right]. \tag{1.4}$$

And, we gave

$$Q_l^n = \begin{cases} \left(\frac{a}{b}\right)^n (ab+4)^{\frac{n}{2}} \left[\begin{array}{cc} q_{n+1} & q_n \\ \frac{b}{a}q_n & q_{n-1} \end{array}\right], n \text{ even} \\[3mm] \left(\frac{a}{b}\right)^n (ab+4)^{\frac{n-1}{2}} \left[\begin{array}{cc} l_{n+1} & l_n \\ \frac{b}{a}l_n & l_{n-1} \end{array}\right], n \text{ odd} \end{cases}, \qquad \det(Q_l^n) = \left(\frac{a^2}{b^2}(ab+4)\right)^n. \tag{1.5}$$

This study consists of three sections. In Section 2, we define the Q_q-generating matrix as the first time in the literature. This matrix is generalization form of well-known Q-Fibonacci matrix. By using this generalized matrix, we find some equalities for bi-periodic Fibonacci polynomial. In the third part of our work, we define the Hadamard products $Q_q^n \circ Q_q^{-n}$ and $Q_l^n \circ Q_l^{-n}$. And, we get some properties of these generalized Hadamard products.

2. The properties of Q_q-generating matrix

Definition 2.1. *Bi-periodic Fibonacci Q_q-generating matrix is defined by*

$$Q_q = \left[\begin{array}{cc} bx & \frac{b}{a} \\ 1 & 0 \end{array}\right]. \tag{2.1}$$

Theorem 2.2. *Let Q_q-generating matrix be as in equation (2.1). Then, we have*

$$Q_q^n = \left(\frac{b}{a}\right)^{\left\lfloor \frac{n}{2} \right\rfloor} \left[\begin{array}{cc} \left(\frac{b}{a}\right)^{\varepsilon(n)} q_{n+1}(x) & \frac{b}{a}q_n(x) \\ q_n(x) & \left(\frac{b}{a}\right)^{\varepsilon(n)} q_{n-1}(x) \end{array}\right] \tag{2.2}$$

where $\varepsilon(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ and $q_n(x)$ is nth bi-periodic Fibonacci polynomial.

Proof. We use mathematical induction on n, we can write

$$Q_q = \left[\begin{array}{cc} bx & \frac{b}{a} \\ 1 & 0 \end{array}\right] = \left(\frac{b}{a}\right)^{\left\lfloor \frac{1}{2} \right\rfloor} \left[\begin{array}{cc} \left(\frac{b}{a}\right)^{\varepsilon(1)} q_2(x) & \frac{b}{a}q_1(x) \\ q_1(x) & \left(\frac{b}{a}\right)^{\varepsilon(1)} q_0(x) \end{array}\right],$$

$$Q_q^2 = \left(\begin{array}{cc} b^2x^2 + \frac{b}{a} & \frac{b^2}{a}x \\ bx & \frac{b}{a} \end{array}\right) = \left(\frac{b}{a}\right)^{\left\lfloor \frac{2}{2} \right\rfloor} \left[\begin{array}{cc} \left(\frac{b}{a}\right)^{\varepsilon(2)} q_3(x) & \frac{b}{a}q_2(x) \\ q_2(x) & \left(\frac{b}{a}\right)^{\varepsilon(2)} q_1(x) \end{array}\right],$$

which show that the equation (2.2) is true for $n = 1$ and $n = 2$. Now, we suppose that equation (2.2) is true for $n = k$, that is

$$Q_q^k = \left(\frac{b}{a}\right)^{\left\lfloor \frac{k}{2} \right\rfloor} \left[\begin{array}{cc} \left(\frac{b}{a}\right)^{\varepsilon(k)} q_{k+1}(x) & \frac{b}{a}q_k(x) \\ q_k(x) & \left(\frac{b}{a}\right)^{\varepsilon(k)} q_{k-1}(x) \end{array}\right].$$

If we supposed that k is even, by using properties of the bi-periodic Fibonacci polynomial, we get

$$Q_q^{k+2} = Q_q^2 Q_q^k = \left(\frac{b}{a}\right)^{\frac{k+2}{2}} \left[\begin{array}{cc} q_{k+3}(x) & \frac{b}{a}q_{k+2}(x) \\ q_{k+2}(x) & q_{k+1}(x) \end{array}\right].$$

Similarly, for k is odd, we can write

$$Q_q^{k+2} = Q_q^2 Q_q^k = \left(\frac{b}{a}\right)^{\frac{k+1}{2}} \begin{bmatrix} \frac{b}{a}q_{k+3}(x) & \frac{b}{a}q_{k+2}(x) \\ q_{k+2}(x) & \frac{b}{a}q_{k+1}(x) \end{bmatrix}.$$

By combining this equalities, we obtain

$$Q_q^{k+2} = \left(\frac{b}{a}\right)^{\left\lfloor \frac{k+2}{2} \right\rfloor} \begin{bmatrix} \left(\frac{b}{a}\right)^{\varepsilon(k+2)} q_{k+3}(x) & \frac{b}{a}q_{k+2}(x) \\ q_{k+2}(x) & \left(\frac{b}{a}\right)^{\varepsilon(k+2)} q_{k+1}(x) \end{bmatrix}.$$

$\square$

Remark 2.3. *Let Q_q^n be as in equation (2.1). Then the following equality is true for all positive integers*

$$\det\left(Q_q^n\right) = \left(-\frac{b}{a}\right)^n. \tag{2.3}$$

Proof. By using the Cassini identity for bi-periodic Fibonacci polnomial, the desired result is obtained. $\square$

The Binet formula for bi-periodic Fibonacci polynomial, given in [17], can also be obtained by using Q_q matrix.

Theorem 2.4. *Let n be any integer. The Binet formula of bi-periodic Fibonacci polynomial is*

$$q_n(x) = \frac{a^{1-\varepsilon(n)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor} x^{n-1}} \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}$$

where $\alpha(x)$ and $\beta(x)$ are roots of $r^2 - abx^2 r - abx^2 = 0$.

Proof. Let the Q_q matrix be as in equation (2.1). Characteristic equation of Q_q-generating matrix is $\lambda^2 - bx\lambda - \frac{b}{a} = 0$. Then, eigenvalues and eigenvectors of the Q_q are

$$\lambda_1 = \frac{\alpha(x)}{ax}, \ \lambda_2 = \frac{\beta(x)}{ax}, \ u_1 = \left(\frac{b}{a}, -\frac{\beta(x)}{ax}\right), \ u_2 = \left(\frac{b}{a}, -\frac{\alpha(x)}{ax}\right).$$

The generating matrix can be diagonalized by using

$$V = U^{-1} Q_q U$$

which

$$U = \begin{bmatrix} u_1^T & u_2^T \end{bmatrix} = \begin{bmatrix} \frac{b}{a} & \frac{b}{a} \\ -\frac{\beta(x)}{ax} & -\frac{\alpha(x)}{ax} \end{bmatrix}, \ V = diag[\lambda_1, \lambda_2] = \begin{bmatrix} \frac{\alpha}{ax} & 0 \\ 0 & \frac{\beta}{ax} \end{bmatrix}.$$

From properties of similar matrices, for n is any integer, we obtain

$$Q_q^n = UV^n U^{-1}.$$

Thus we get

$$\begin{aligned}
Q_q^n &= \begin{bmatrix} \frac{b}{a} & \frac{b}{a} \\ -\frac{\beta(x)}{ax} & -\frac{\alpha(x)}{ax} \end{bmatrix} \begin{bmatrix} \left(\frac{\alpha(x)}{ax}\right)^n & 0 \\ 0 & \left(\frac{\beta(x)}{ax}\right)^n \end{bmatrix} \frac{a^2 x}{b(\beta(x) - \alpha(x))} \begin{bmatrix} -\frac{\alpha(x)}{ax} & -\frac{b}{a} \\ \frac{\beta(x)}{ax} & \frac{b}{a} \end{bmatrix} \\
&= \frac{1}{\alpha(x) - \beta(x)} \begin{bmatrix} \frac{\alpha^{n+1}(x) - \beta^{n+1}(x)}{a^n x^n} & \frac{b(\alpha^n(x) - \beta^n(x))}{a^n x^{n-1}} \\ \frac{\alpha^n(x) - \beta^n(x)}{a^{n-1} x^{n-1}} & \frac{b(\alpha^{n-1}(x) - \beta^{n-1}(x))}{a^{n-1} x^{n-2}} \end{bmatrix}.
\end{aligned}$$

Taking into account the Theorem (2.2), we have

$$\frac{1}{\alpha(x) - \beta(x)} \frac{\alpha^n(x) - \beta^n(x)}{a^{n-1} x^{n-1}} = \left(\frac{b}{a}\right)^{\left\lfloor \frac{n}{2} \right\rfloor} q_n(x).$$

Making the necessary arrangements, the desired result is obtained. $\square$

In the following, we give some properties of bi-periodic Fibonacci polynomial.

Theorem 2.5. *For every $m, n \in \mathbb{N}$, the following statements are valid*

$$i) \ q_{m+n}(x) = \begin{cases} q_{m+1}(x)q_n(x) + q_m(x)q_{n-1}(x), & m+n \ even \\ \left(\frac{b}{a}\right)^{\varepsilon(m)} q_{m+1}(x)q_n(x) + \left(\frac{b}{a}\right)^{\varepsilon(n)} q_m(x)q_{n-1}(x), & m+n \ odd \end{cases},$$

$$ii) \ q_{m+n}(x) = \begin{cases} q_m(x)q_{n+1}(x) + q_{m-1}(x)q_n(x), & m+n \ even \\ \left(\frac{b}{a}\right)^{\varepsilon(n)} q_m(x)q_{n+1}(x) + \left(\frac{b}{a}\right)^{\varepsilon(m)} q_{m-1}(x)q_n(x), & m+n \ odd \end{cases},$$

$$iii) \quad q_{m-n}(x) = \begin{cases} (-1)^{n+1} \{q_{m+1}(x)q_n(x) - q_m(x)q_{n+1}(x)\}, & m+n \text{ even} \\ \left(-\frac{b}{a}\right)^{\varepsilon(m)} q_{m+1}(x)q_n(x) + \left(-\frac{b}{a}\right)^{\varepsilon(n)} q_m(x)q_{n+1}(x), & m+n \text{ odd} \end{cases},$$

$$iv) \quad q_{m-n}(x) = \begin{cases} (-1)^{n+1} \{q_{m-1}(x)q_n(x) - q_m(x)q_{n-1}(x)\}, & m+n \text{ even} \\ \left(-\frac{b}{a}\right)^{\varepsilon(n)} q_m(x)q_{n-1}(x) + \left(-\frac{b}{a}\right)^{\varepsilon(m)} q_{m-1}(x)q_n(x), & m+n \text{ odd} \end{cases}.$$

Proof. By using equation (2.2), Q_q^{m+n} can be written as

$$Q_q^{m+n} = \left(\frac{b}{a}\right)^{\lfloor \frac{m+n}{2} \rfloor} \begin{bmatrix} \left(\frac{b}{a}\right)^{\varepsilon(m+n)} q_{m+n+1}(x) & \frac{b}{a}q_{m+n}(x) \\ q_{m+n}(x) & \left(\frac{b}{a}\right)^{\varepsilon(n)} q_{m+n-1}(x) \end{bmatrix}. \tag{2.4}$$

For case of odd m and n, we can write

$$Q_q^m Q_q^n = \left(\frac{b}{a}\right)^{\frac{m+n}{2}-1} \begin{bmatrix} \left(\frac{b}{a}\right)^2 q_{m+1}(x)q_{n+1}(x) + \frac{b}{a}q_m(x)q_n(x) & \left(\frac{b}{a}\right)^2 \{q_{m+1}(x)q_n(x) + q_m(x)q_{n-1}(x)\} \\ \frac{b}{a}\{q_m(x)q_{n+1}(x) + q_{m-1}(x)q_n(x)\} & \frac{b}{a}q_m(x)q_n(x) + \left(\frac{b}{a}\right)^2 q_{m-1}(x)q_{n-1}(x) \end{bmatrix}. \tag{2.5}$$

If we compare the 1*st* row and 2*nd* column entries of the matrices (2.4) and (2.5), we get

$$q_{m+n}(x) = q_{m+1}(x)q_n(x) + q_m(x)q_{n-1}(x).$$

On the other hand, comparing the 2*nd* row and 1*st* column, we obtain

$$q_{m+n}(x) = q_m(x)q_{n+1}(x) + q_{m-1}(x)q_n(x).$$

Similarly, for the case of even m and n, we have

$$q_{m+n}(x) = q_{m+1}(x)q_n(x) + q_m(x)q_{n-1}(x),$$
$$q_{m+n}(x) = q_m(x)q_{n+1}(x) + q_{m-1}(x)q_n(x).$$

And, for the case of odd m and even n (or case of even m and odd n), we have

$$q_{m+n}(x) = \frac{b}{a}q_{m+1}(x)q_n(x) + q_m(x)q_{n-1}(x),$$

$$q_{m+n}(x) = q_m(x)q_{n+1}(x) + \frac{b}{a}q_{m-1}(x)q_n(x).$$

Thus, the proof of *i*) and *ii*) is obtained.

Now, we give the proof of *iii*) and *iv*). By calculating inverse of the matrix Q_q^n in (2.2), we conclude

$$Q_q^{-n} = \begin{cases} \left(\frac{b}{a}\right)^{-\frac{n}{2}} \begin{bmatrix} q_{n-1}(x) & -\frac{b}{a}q_n(x) \\ -q_n(x) & q_{n+1}(x) \end{bmatrix}, & n \text{ even} \\ \left(\frac{b}{a}\right)^{-\frac{n+1}{2}} \begin{bmatrix} -\frac{b}{a}q_{n-1}(x) & \frac{b}{a}q_n(x) \\ q_n(x) & -\frac{b}{a}q_{n+1}(x) \end{bmatrix}, & n \text{ odd} \end{cases}. \tag{2.6}$$

Benefitting from the equality $Q_q^{m-n} = Q_q^m Q_q^{-n}$ and by comparing the entries, the desired result can be obtained. That is, for the case of odd m and n, we get

$$q_{m-n}(x) = q_{m+1}(x)q_n(x) - q_m(x)q_{n-1}(x),$$
$$q_{m-n}(x) = q_{m-1}(x)q_n(x) - q_m(x)q_{n-1}(x).$$

Similarly, for the case of even m and n, we obtain

$$q_{m-n}(x) = q_m(x)q_{n+1}(x) - q_{m+1}(x)q_n(x),$$
$$q_{m-n}(x) = q_m(x)q_{n-1}(x) - q_{m-1}(x)q_n(x).$$

And, for the case of odd m and even n (or case of even m and odd n), we have

$$q_{m-n}(x) = -\frac{b}{a}q_{m+1}(x)q_n(x) + q_m(x)q_{n+1}(x),$$

$$q_{m-n}(x) = q_m(x)q_{n-1}(x) - \frac{b}{a}q_{m-1}(x)q_n(x).$$

Thus, we have the desired expressions. □

3. On the Hadamard Products $Q_q^n \circ Q_q^{-n}$ and $Q_l^n \circ Q_l^{-n}$ for Bi-periodic Fibonacci and Lucas Generating Matrices

We will accept $x = 1$ in Q_q^n matrix throughout this section. Then, we can give the following Theorem for the generating matrices of bi-periodic Fibonacci and Lucas numbers.

Theorem 3.1. *For any integer n, we have*

i) $Q_q^n = bQ_q^{n-1} + \frac{b}{a}Q_q^{n-2}$,
ii) $Q_l^n = \frac{b}{a(ab+4)}Q_l^{n+1} + \frac{a}{b}Q_l^{n-1}$.

Proof. We will omit the proof of (*ii*) because it is similar to the proof of (*i*).
By considering the Equation (2.2), for even n, we can write

$$
\begin{aligned}
Q_q^n &= \left(\frac{b}{a}\right)^{\frac{n}{2}}\left[\begin{array}{cc} q_{n+1} & \frac{b}{a}q_n \\ q_n & q_{n-1} \end{array}\right] = \left(\frac{b}{a}\right)^{\frac{n}{2}}\left[\begin{array}{cc} bq_n + q_{n-1} & \frac{b}{a}\left(aq_{n-1}+q_{n-2}\right) \\ aq_{n-1}+q_{n-2} & bq_{n-2}+q_{n-3} \end{array}\right] \\
&= b\left(\frac{b}{a}\right)^{\frac{n-2}{2}}\left[\begin{array}{cc} \frac{b}{a}q_n & \frac{b}{a}aq_{n-1} \\ q_{n-1} & \frac{b}{a}q_{n-2} \end{array}\right] + \frac{b}{a}\left(\frac{b}{a}\right)^{\frac{n-2}{2}}\left[\begin{array}{cc} q_{n-1} & \frac{b}{a}q_{n-2} \\ q_{n-2} & q_{n-3} \end{array}\right].
\end{aligned}
$$

Similarly, for odd n, we have

$$
\begin{aligned}
Q_q^n &= \left(\frac{b}{a}\right)^{\frac{n-1}{2}}\left[\begin{array}{cc} \frac{b}{a}q_{n+1} & \frac{b}{a}q_n \\ q_n & \frac{b}{a}q_{n-1} \end{array}\right] = \left(\frac{b}{a}\right)^{\frac{n-1}{2}}\left[\begin{array}{cc} \frac{b}{a}\left(aq_n + q_{n-1}\right) & \frac{b}{a}\left(bq_{n-1}+q_{n-2}\right) \\ bq_{n-1}+q_{n-2} & \frac{b}{a}\left(aq_{n-2}+q_{n-3}\right) \end{array}\right] \\
&= b\left(\frac{b}{a}\right)^{\frac{n-1}{2}}\left[\begin{array}{cc} q_n & \frac{b}{a}q_{n-1} \\ q_{n-1} & q_{n-2} \end{array}\right] + \frac{b}{a}\left(\frac{b}{a}\right)^{\frac{n-2}{2}}\left[\begin{array}{cc} \frac{b}{a}q_{n-1} & \frac{b}{a}q_{n-2} \\ q_{n-2} & \frac{b}{a}q_{n-3} \end{array}\right].
\end{aligned}
$$

Thus, the desired equality is obtained.   □

Now, we define the Hadamard products for bi-periodic Fibonacci and Lucas generating matrices by considering the determinants. Thus, we can write

$$
Q_q^n \circ Q_q^{-n} = \begin{cases} \left(\frac{a}{b}\right)^n \left(Q_q^n \circ adj Q_q^n\right), n \text{ even} \\ -\left(\frac{a}{b}\right)^n \left(Q_q^n \circ adj Q_q^n\right), n \text{ odd} \end{cases} \tag{3.1}
$$

and

$$
Q_l^n \circ Q_l^{-n} = \begin{cases} \left(\frac{b^2}{a^2(ab+4)}\right)^n \left(Q_l^n \circ adj Q_l^n\right), n \text{ even} \\ -\left(\frac{b^2}{a^2(ab+4)}\right)^n \left(Q_l^n \circ adj Q_l^n\right), n \text{ odd} \end{cases}. \tag{3.2}
$$

In the following theorem, we give the determinants of the Hadamard products.

Theorem 3.2. *For any integer n, we have*

$$
\begin{aligned}
\det(Q_q^n \circ Q_q^{-n}) &= \begin{cases} 1 + 2\frac{b}{a}q_n^2, \ n \text{ even} \\ 1 - 2q_n^2, n \text{ odd} \end{cases}, \\
\det(Q_l^n \circ Q_l^{-n}) &= \begin{cases} 1 + 2\frac{b}{a}q_n^2, \ n \text{ even} \\ 1 + 2\frac{b}{a(ab+4)}l_n^2, n \text{ odd} \end{cases}.
\end{aligned}
$$

Proof. We are doing the proof for $\det(Q_q^n \circ Q_q^{-n})$ because the proof of $\det(Q_l^n \circ Q_l^{-n})$ can be done in similar way. By using the Equation (3.1), for even n, we can write

$$
Q_q^n \circ Q_q^{-n} = \left[\begin{array}{cc} q_{n+1}q_{n-1} & -\frac{b^2}{a^2}q_n^2 \\ -q_n^2 & q_{n+1}q_{n-1} \end{array}\right].
$$

And, we have

$$
\begin{aligned}
\det(Q_q^n \circ Q_q^{-n}) &= \det\left[\begin{array}{cc} q_{n+1}q_{n-1} & -\frac{b^2}{a^2}q_n^2 \\ -q_n^2 & q_{n+1}q_{n-1} \end{array}\right] \\
&= \left(q_{n+1}q_{n-1} - \frac{b}{a}q_n^2\right)\left(q_{n+1}q_{n-1} + \frac{b}{a}q_n^2\right) \\
&= \left(q_{n+1}q_{n-1} + \frac{b}{a}q_n^2\right) = 1 + 2\frac{b}{a}q_n^2.
\end{aligned}
$$

Similarly, for odd n, we get

$$
Q_q^n \circ Q_q^{-n} = \left[\begin{array}{cc} -\frac{b}{a}q_{n+1}q_{n-1} & \frac{b}{a}q_n^2 \\ \frac{a}{b}q_n^2 & -\frac{b}{a}q_{n+1}q_{n-1} \end{array}\right].
$$

$$\det(Q_q^n \circ Q_q^{-n}) = \det\begin{bmatrix} -\frac{b}{a}q_{n+1}q_{n-1} & \frac{b}{a}q_n^2 \\ \frac{a}{b}q_n^2 & -\frac{b}{a}q_{n+1}q_{n-1} \end{bmatrix}$$

$$= \left(\frac{b}{a}q_{n+1}q_{n-1} - q_n^2\right)\left(\frac{b}{a}q_{n+1}q_{n-1} + q_n^2\right)$$

$$= (-1)^n\left(q_{n+1}q_{n-1} + \frac{b}{a}q_n^2\right) = 1 + 2\frac{b}{a}q_n^2.$$

Thus, we get the desired result. $\square$

Using the above definition and theorem, we obtain traces, eigenvalues, eigenvectors and inverses for Hadamard products as in the following.

Remark 3.3. *We have*

i) $trace(Q_q^n \circ Q_q^{-n}) = \begin{cases} 2\left(1 + \frac{b}{a}q_n^2\right), & n\ even \\ 2\left(1 - q_n^2\right), & n\ odd \end{cases}$,

$trace(Q_l^n \circ Q_l^{-n}) = \begin{cases} 2\left(1 + \frac{b}{a}q_n^2\right), & n\ even \\ 2\left(1 + \frac{b}{a(ab+4)}l_n^2\right), & n\ odd \end{cases}$.

ii) *The eigenvalues of the matrix $Q_q^n \circ Q_q^{-n}$ are given as*

$$\lambda_1 = 1, \lambda_2 = 1 + 2\frac{b}{a}q_n^2,\ for\ even\ n$$

$$\lambda_1 = 1, \lambda_2 = 1 - 2q_n^2,\ for\ odd\ n$$

and the eigenvalues of the matrix $Q_l^n \circ Q_l^{-n}$ are given as

$$\lambda_3 = 1, \lambda_4 = 1 + 2\frac{b}{a}q_n^2,\ for\ even\ n$$

$$\lambda_3 = 1, \lambda_4 = 1 + 2\frac{b}{a(ab+4)}l_n^2,\ for\ odd\ n$$

iii) *The eigenvectors corresponding to the eigenvalues of the matrices $Q_q^n \circ Q_q^{-n}$ and $Q_l^n \circ Q_l^{-n}$ are*

$$y_1 = \begin{bmatrix} \frac{b}{a} \\ 1 \end{bmatrix}, y_2 = \begin{bmatrix} -\frac{b}{a} \\ 1 \end{bmatrix}\ and\ y_3 = \begin{bmatrix} 1 \\ \frac{b}{a} \end{bmatrix}, y_4 = \begin{bmatrix} 1 \\ -\frac{b}{a} \end{bmatrix}$$

respectively.

iv) *The matrices $Q_q^n \circ Q_q^{-n}$ and $Q_l^n \circ Q_l^{-n}$ are invertible and*

$$(Q_q^n \circ Q_q^{-n})^{-1} = (Q_l^n \circ Q_l^{-n})^{-1} = \begin{bmatrix} 1 - \frac{\frac{b}{a}q_n^2}{1+2\frac{b}{a}q_n^2} & \frac{\frac{b^2}{a^2}q_n^2}{1+2\frac{b}{a}q_n^2} \\ \frac{q_n^2}{1+2\frac{b}{a}q_n^2} & 1 - \frac{\frac{b}{a}q_n^2}{1+2\frac{b}{a}q_n^2} \end{bmatrix},\ for\ even\ n$$

$$(Q_q^n \circ Q_q^{-n})^{-1} = \begin{bmatrix} 1 + \frac{q_n^2}{1-2q_n^2} & \frac{-\frac{b}{a}q_n^2}{1-2q_n^2} \\ \frac{-\frac{a}{b}q_n^2}{1-2q_n^2} & 1 - \frac{q_n^2}{1-2q_n^2} \end{bmatrix},\ for\ odd\ n$$

$$(Q_l^n \circ Q_l^{-n})^{-1} = \begin{bmatrix} 1 - \frac{\frac{b}{a(ab+4)}l_n^2}{1+2\frac{b}{a(ab+4)}l_n^2} & \frac{\frac{1}{ab+4}l_n^2}{1+2\frac{b}{a(ab+4)}l_n^2} \\ \frac{\frac{b^2}{a^2(ab+4)}l_n^2}{1+2\frac{b}{a(ab+4)}l_n^2} & 1 - \frac{\frac{b}{a(ab+4)}l_n^2}{1+2\frac{b}{a(ab+4)}l_n^2} \end{bmatrix},\ for\ odd\ n$$

4. Conclusion

In Section 2, we present some properties of bi-periodic Fibonacci polynomial by using the Q_q matrix. Then, we express that well-known matrix representations are special cases of this generalized matrices.
If we choose $x = 1$, then we get generating matrix for bi-periodic Fibonacci sequence and properties of this sequence.
Thus, if we choose the different values of a and b, then we obtain generating matrices for well-known matrix sequences in the literature:

- If we replace $a = b = 1$ in Q_q, we obtain the generating matrices for Fibonacci sequence.
- If we replace $a = b = 2$ in Q_q, we obtain the generating matrices for Pell sequence.
- If we replace $a = b = k$ in Q_q, we obtain the generating matrices for k-Fibonacci sequence.

And, in Section 3, we define the Hadamard products for the generating matrices of bi-periodic Fibonacci and Lucas sequence and give some properties of these new matrices. By taking into account these generalized matrices, it also can be obtained properties of some special matrices. Namely, for different values of a and b, we can rewrite some properties for the well-known matrices in the literature such as Fibonacci Q.

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