Konuralp Journal of Mathematics, 6 (2) (2018) 279-285

Konuralp Journal of Mathematics

Journal Homepage: *www.dergipark.gov.tr/konuralpjournalmath*
e-ISSN: 2147-625X

Coefficient Estimates for Certain Subclasses of m-fold Symmetric Bi-univalent Functions Defined by the Q-derivative Operator

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Abstract

In the present study, we introduce two new subclasses of bi-univalent functions based on the q-derivative operator in which both f and f^{-1} are m-fold symmetric analytic functions in the open unit disk. Among other results belonging to these subclasses upper coefficients bounds $|a_{m+1}|$ and $|a_{2m+1}|$ are obtained in this study. Certain special cases are also indicated.

Keywords: *Analytic functions, univalent functions, bi-univalent functions*
2010 Mathematics Subject Classification: *30C45, 30C50*

1. Introduction

Let $\mathscr{A}$ denote the family of functions analytic in the open unit disk $\mathbb{D} = \{z : z \in \mathbb{C}, |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and having the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

A function is said to be univalent if it never takes the same value twice, that is $f(z_1) = f(z_2)$ if $z_1 \neq z_2$. We also denote by $\mathscr{S}$ the subclass of functions in $\mathscr{A}$ which are univalent in $\mathbb{D}$ (see for details [7]). From the Koebe $1/4$ Theorem (for details, see [7]) every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{D})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots$$

$$= w + \sum_{k=2}^{\infty} b_k w^k. \tag{1.2}$$

Let $f \in \mathscr{A}$. The function f is said to be *bi-univalent* in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. Let Σ denote the class of bi-univalent functions in $\mathbb{D}$ given by the Taylor-Maclaurin series expansion given by (1.1). We can accept that the beginning of estimating bounds for the coefficients of classes of bi-univalent functions is the date 1967 [11]. Later the papers of Brannan and Taha [4] and Srivastava et al. [20] were picked up the interest on the coefficient bounds of bi-univalent functions.

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For detailed information about the class of Σ was given in the references [4], [11], [14], [20] and [23].
Let $m \in \mathbb{N} = \{1,2,3...\}$. A domain $\mathbb{E}$ is said to be *m-fold symmetric* if a rotation of $\mathbb{E}$ about the origin through an angle $2\pi/m$ carries $\mathbb{E}$ on itself. It follows that, a function f analytic in $\mathbb{D}$ is said to be m-fold symmetric if

$$f\left(e^{2\pi i/m}z\right) = e^{2\pi i/m} f(z).$$

In particular every f is one-fold symmetric and every odd f is two-fold symmetric. $\mathscr{S}_m$ indicate the class of m-fold symmetric univalent functions in $\mathbb{D}$.

$f \in \mathscr{S}_m$ is characterized by having a power series as following normalized form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \qquad (z \in \mathbb{D},\ m \in \mathbb{N}). \tag{1.3}$$

In [21] Srivastava et al. defined m-fold symmetric bi-univalent function analogues to the concept of m-fold symmetric univalent functions. They introduce some important results, such as each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each $(m \in \mathbb{N})$. In addition, they acquired the series expansion for f^{-1} as follows:

$$\begin{aligned} g(w) &= w - a_{m+1}w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right] w^{2m+1} \\ &= -\left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots \\ &= z + \sum_{k=1}^{\infty} A_{mk+1} z^{mk+1} \end{aligned} \tag{1.4}$$

where $f^{-1} = g$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in $\mathbb{D}$. For some examples of m-fold symmetric bi-univalent functions, see [21]. The coefficient problem for m-fold symmetric analytic bi-univalent functions is one of the favorite subjects of geometric function theory in these days, see [1], [2], [5], [8], [21], [22]. Here, the aim of this study is to determine upper coefficients bounds $|a_{m+1}|$ and $|a_{2m+1}|$ are obtained belonging these two new subclasses.

First formulae in what we now call q-calculus were obtained by Euler in the eighteenth century. In the second half of the twentieth century there was a significant increase of activity in the area of the q-calculus. The fractional calculus operators has gained importance and popularity, mainly due to its vast potential of demonstrated applications in various fields of applied sciences, engineering. The application of q-calculus was initiated by Jackson [9].
In the field of geometric function theory, various subclasses of analytic functions have been studied from different viewpoints. The fractional q-calculus is the important tools that are used to investigate subclasses of analytic functions. Historically speaking, a firm footing of the usage of the the q-calculus in the context of geometric function theory was actually provided and the basic (or q-) hypergeometric functions were first used in geometric function theory in a book chapter by Srivastava (see, for details, [19]). In fact, the extension of the theory of univalent functions can be described by using the theory of q-calculus. Furthermore, the q-calculus operators, such as fractional q-integral and fractional q-derivative operators, are used to construct several subclasses of analytic functions (see, [13], [15], [16]). In a recent paper Purohit and Raina [18] investigated applications of fractional q-calculus operators to defined certain new classes of functions which are analytic in the open disk. Later, Mohammed and Darus [12] studied approximation and geometric properties of these q-operators in some subclasses of analytic functions in compact disk. A comprehensive study on applications of q-calculus in operator theory may be found in [3]. For the convenience, we give some basic definitions and concept details of q-calculus which are used in this paper.

For a function $f \in \mathscr{A}$ given by (1.1) and $0 < q < 1$, the q-derivative of function f is defined by (see [7], [10])

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \qquad (z \neq 0) \tag{1.5}$$

$D_q f(0) = f'(0)$ and $D_q^2 f(z) = D_q(D_q f(z))$. From (1.5), we deduce that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \tag{1.6}$$

where

$$[k]_q = \frac{1-q^k}{1-q}. \tag{1.7}$$

As $q \to 1^-$, $[k]_q \to k$, for a function $g(z) = z^k$ we get

$$D_q(z^k) = [k]_q z^{k-1},$$

$$\lim_{q\to 1^-}(D_q(z^k)) = kz^{k-1} = g'(z),$$

where g' is the ordinary derivative.

By making use of the q-derivative of a function $f \in \mathscr{A}$, we introduce two new subclasses of the function class Σ_m and obtain estimates on the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in these new subclasses of the function class Σ_m .
Firstly, in order to derive our main results, we need to following lemma.

Lemma 1.1. [17] *If $p \in \mathscr{P}$, then $|c_k| \le 2$ for each k where $\mathscr{P}$ is the family of all functions p analytic in $\mathbb{D}$ for which*

$$\Re(p(z)) > 0, \;\; p(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

for $z \in \mathbb{D}$.

2. Definition of the Class $T_{\Sigma,m}^{q,\alpha}$ and Its Coefficient Bounds

Definition 2.1. *A function f given by (1.3) is said to be in the class $T_{\Sigma,m}^{q,\alpha}$ $(0 < q < 1, 0 < \alpha \le 1, m \in \mathbb{N})$ if the following condition are satisfied*

$$f \in \Sigma_m \;\; and \;\; \left|argD_q f(z)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{D}) \tag{2.1}$$

and

$$\left|argD_q g(w)\right| < \frac{\alpha\pi}{2} \qquad (w \in \mathbb{D}) \tag{2.2}$$

where the function g is given by Eq.(1.4).

Remark 2.2. *We note that* $\lim_{q\to 1^-} T_{\Sigma,m}^{q,\alpha} = T_{\Sigma,m}^{\alpha}$ *and for one-fold case $T_{\Sigma,1}^{\alpha} = T_{\Sigma}^{\alpha}$ introduced by Srivastava et al. [20].*

Theorem 2.3. *Let the function f given by (1.3) be in the function class $T_{\Sigma,m}^{q,\alpha}$, $(0 < q < 1, 0 < \alpha \le 1, m \in \mathbb{N})$. Then*

$$|a_{m+1}| \le \frac{2\alpha}{\sqrt{(m+1)\alpha[1+2m]_q - (\alpha-1)[1+m]_q^2}} \tag{2.3}$$

and

$$|a_{2m+1}| \le \frac{2(m+1)\alpha^2}{[1+m]_q^2} + \frac{2\alpha}{[1+2m]_q}. \tag{2.4}$$

Proof. First of all, it follows from the conditions (2.1) and (2.2) that

$$D_q f(z) = [p(z)]^{\alpha}, \quad and \quad D_q g(w) = [q(w)]^{\alpha}, \quad (z, w \in \mathbb{D}) \tag{2.5}$$

Respectively, where $p(z)$ and $q(z)$ are in familiar Caratheodory class $\mathscr{P}$ (see for details [7]) and have the following series statement

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots \tag{2.6}$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots \tag{2.7}$$

Now, comparing the coefficients in (2.5), we have

$$[1+m]_q a_{m+1} = \alpha p_m \tag{2.8}$$

$$[1+2m]_q a_{2m+1} = \alpha p_{2m} + \frac{\alpha(\alpha-1)}{2} p_m^2 \tag{2.9}$$

$$-[1+m]_q a_{m+1} = \alpha q_m \tag{2.10}$$

$$[1+2m]_q[(m+1)a_{m+1}^2 - a_{2m+1}] = \alpha q_{2m} + \frac{\alpha(\alpha-1)}{2} q_m^2. \tag{2.11}$$

From (2.8) and (2.9), we have □

$$p_m = -q_m \tag{2.12}$$

and

$$2[1+m]_q^2 a_{m+1}^2 = \alpha^2(p_m^2+q_m^2). \tag{2.13}$$

Furthermore, from Eqs. (2.9), (2.11) and (2.13), we obtain that

$$[1+2m]_q(m+1)a_{m+1}^2 = \alpha(p_{2m}+q_{2m}) + \frac{\alpha-1}{\alpha}[1+m]_q^2 a_{m+1}^2.$$

Therefore, we get

$$a_{m+1}^2 = \frac{\alpha^2(p_{2m}+q_{2m})}{(m+1)\alpha[1+2m]_q-(\alpha-1)[1+m]_q^2}. \tag{2.14}$$

Note that, according to the Caratheodory lemma [7], $|p_m| \leq 2$ and $|q_m| \leq 2$ for $m \in \mathbb{N}$. Now taking the absolute value of (2.14) and applying the Caratheodory lemma for p_{2m} and q_{2m} we have the following inequality

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{(m+1)\alpha[1+2m]_q-(\alpha-1)[1+m]_q^2}}.$$

So, we obtain the desired estimate for $|a_{m+1}|$ given by (2.3). Next, so as to obtain solution of the coefficient bound on $|a_{2m+1}|$, we subtract (2.11) from (2.9). We thus have,

$$2[1+2m]_q a_{2m+1} - (m+1)[1+2m]_q a_{m+1}^2$$

$$= \alpha(p_{2m}-q_{2m}) + \frac{\alpha(\alpha-1)}{2}(p_m^2-q_m^2). \tag{2.15}$$

It follows from (2.13), (2.15) and observing $p_m^2 - q_m^2$, it gives that

$$a_{2m+1} = \frac{\alpha(p_{2m}-q_{2m})}{2[1+2m]_q} + \frac{(m+1)\alpha^2(p_m^2+q_m^2)}{4[1+m]_q^2}. \tag{2.16}$$

Taking the absolute value of (2.16) and applying Caratheodory lemma again for coefficients p_m, p_{2m} and q_{2m} we have

$$|a_{2m+1}| \leq \frac{2(m+1)\alpha^2}{[1+m]_q^2} + \frac{2\alpha}{[1+2m]_q}.$$

So the proof is completed.

Remark 2.4. *For one-fold case, we note that* $T_{\Sigma,1}^{q,\alpha} = H_{\Sigma}^{q,\alpha}$ *introduced by Bulut [6].*

Taking $q \to 1^-$ in Theorem 2.1, we have the class, $\lim_{q\to 1^-} T_{\Sigma,m}^{q,\alpha} = H_{\Sigma,m}^{\alpha}$ introduced by Srivastava et al. [21] and obtain the Corollary 2.1 as follows:

Corollary 2.5. [21] *Let the function* $f \in H_{\Sigma,m}^{\alpha}, (0 < \alpha \leq 1, m \in \mathbb{N})$ *be given (1.3). Then*

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{(m+1)(\alpha m+m+1)}}$$

and

$$|a_{2m+1}| \leq \frac{2\alpha^2}{m+1} + \frac{2\alpha}{2m+1}.$$

Remark 2.6. *For one-fold case, we note that* $\lim_{q\to 1^-} T_{\Sigma,1}^{q,\alpha} = H_{\Sigma}^{\alpha}$ *and we can obtain the results of Srivastava et al.[20].*

3. Definition of the Class $T^q_{\Sigma,m}(\beta)$ and Its Coefficient Bounds

Definition 3.1. *A function f given by (1.3) is said to be in the class $T^q_{\Sigma,m}(\beta)$, $(0<q<1, 0\leq\beta<1, m\in\mathbb{N})$ if the conditions given below are fulfilled:*

$$f\in\Sigma_m \text{ and } \Re\left\{D_qf(z)\right\}>\beta \qquad (z\in\mathbb{D}) \tag{3.1}$$

and

$$\Re\left\{D_qg(w)\right\}>\beta \qquad (w\in\mathbb{D}) \tag{3.2}$$

where the function g is given by Eq.(1.4).

Remark 3.2. *Note that we have the class $\lim_{q\to1^-}T^{q,\alpha}_{\Sigma,m}=T^{\alpha}_{\Sigma,m}$ and for one-fold case the class $\lim_{q\to1^-}T^q_{\Sigma,1}(\beta)=T_\Sigma(\beta)$ introduced by Srivastava et al. [20].*

Theorem 3.3. *Let the function f given by (1.3) be in the function class $T^q_{\Sigma,m}(\beta)$, $(0<q<1, 0\leq\beta<1, m\in\mathbb{N})$. Then*

$$|a_{m+1}|\leq\min\left\{\frac{2(1-\beta)}{[1+m]_q},2\sqrt{\frac{1-\beta}{[1+2m]_q(m+1)}}\right\} \tag{3.3}$$

and

$$|a_{2m+1}|\leq\frac{2(1-\beta)}{[1+2m]_q}. \tag{3.4}$$

Proof. First of all, it follows from the equations (3.1) and (3.2) that

$$D_qf(z)=[p(z)]^\alpha\ D_qg(w)=[q(w)]^\alpha, \quad (z,w\in\mathbb{D}) \tag{3.5}$$

respectively, where $p(z)$ and $q(z)$ given by (2.6) and (2.7). Now equating coefficients in (3.5), we obtain

$$[1+m]_qa_{m+1}=(1-\beta)p_m \tag{3.6}$$

$$[1+2m]_qa_{2m+1}=(1-\beta)p_{2m} \tag{3.7}$$

$$-[1+m]_qa_{m+1}=(1-\beta)q_m \tag{3.8}$$

$$[1+2m]_q[(m+1)a^2_{m+1}-a_{2m+1}]=(1-\beta)q_{2m}. \tag{3.9}$$

From Eqs. (3.6) and (3.8), we have

$$p_m=-q_m \tag{3.10}$$

and

$$2[1+m]^2_qa^2_{m+1}=(1-\beta)^2(p^2_m+q^2_m). \tag{3.11}$$

Also, from Eqs. (3.7) and (3.9), we obtain

$$[1+2m]_q(m+1)a^2_{m+1}=(1-\beta)(p_{2m}+q_{2m}). \tag{3.12}$$

Thus, applying Caratheodory lemma for (3.11) and (3.12) we obtain the coefficient estimate $|a_{m+1}|$ as follows:

$$|a^2_{m+1}|\leq\frac{1-\beta}{[1+2m]_q(m+1)}\left(|p_{2m}|+|q_{2m}|\right)$$

$$|a_{m+1}|\leq2\sqrt{\frac{1-\beta}{[1+2m]_q(m+1)}}$$

which is desired coefficient bound. Next, so as to obtain bound for coefficient $|a_{2m+1}|$ by subtracting (3.9) from (3.7), we have

$$-[1+2m]_q(m+1)a^2_{m+1}+2[1+2m]_qa_{2m+1}=(1-\beta)(p_{2m}-q_{2m}) \tag{3.13}$$

or equivalently

$$a_{2m+1}=\frac{(1-\beta)(p_{2m}-q_{2m})}{2[1+2m]_q}+\frac{m+1}{2}a^2_{m+1}.$$

Upon substituting the value of a_{m+1}^2 from (3.11), we obtain

$$a_{2m+1} = \frac{(1-\beta)(p_{2m}-q_{2m})}{2[1+2m]_q} + \frac{(m+1)(1-\beta)^2(p_m^2+q_m^2)}{4[1+m]_q^2}. \tag{3.14}$$

Applying Caratheodory lemma for coefficients p_m, q_m, p_{2m} and q_{2m} we obtain

$$|a_{2m+1}| \leq \frac{2(1-\beta)}{[1+2m]_q} + \frac{2(m+1)(1-\beta)^2}{[1+m]_q^2}.$$

On the other hand, by using the equation (3.12) into (3.13), and applying Caratheodory lemma we can obtain the inequality as follows

$$|a_{2m+1}| \leq \frac{2(1-\beta)}{[1+2m]_q}$$

which is the desired bounds on coefficients $|a_{2m+1}|$ as given in Theorem 3.1.

Taking $q \to 1^-$ in Theorem 3.3, we obtain following corollary. □

Corollary 3.4. *Let the function f given by (1.3) be in the class $T_{\Sigma,m}(\beta), (0 \leq \beta < 1, m \in \mathbb{N})$, Then*

$$|a_{m+1}| \leq \begin{cases} 2\sqrt{\frac{1-\beta}{(1+2m)(1+m)}}; & 0 \leq \beta \leq \frac{m}{1+2m} \\ \frac{2(1-\beta)}{1+m}; & \frac{m}{1+2m} \leq \beta < 1. \end{cases}$$

and

$$|a_{2m+1}| \leq \frac{2(1-\beta)}{1+2m}.$$

Remark 3.5. *For one fold case, Corollary 3.1 reduces to the following corollary given by Bulut [6] for the bounds on coefficients $|a_2|$ and $|a_3|$.*

Corollary 3.6. [6] *Let the function f given by Taylor-Maclaurin series expansion (1.1) be in the class $H_{\Sigma}(\beta)$, $(0 \leq \beta < 1)$. Then*

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\beta)}{3}}; & 0 \leq \beta \leq \frac{1}{3} \\ 1-\beta; & \frac{1}{3} \leq \beta < 1 \end{cases}$$

and

$$|a_3| \leq \frac{2(1-\beta)}{3}.$$

Remark 3.7. *Corollary 3.2 given above is an improvement of the estimates for coefficients on $|a_2|$ and $|a_3|$ obtained by Srivastava et al [20].*

Corollary 3.8. [20] *Let the function f given by Taylor-Maclaurin series expansion (1.1) be in the class $H_{\Sigma}(\beta)$, $(0 \leq \beta < 1)$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3}} \quad \text{and} \quad |a_3| \leq \frac{(1-\beta)(5-3\beta)}{3}.$$

4. Acknowledgement

The present study was supported by Dicle University Scientific Research Coordination Unit. Project Number: FEN.17.027

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