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T_1 APPROACH SPACES

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Abstract. In this paper, we characterize both T_1 and local T_1 limit (resp. gauge) approach spaces as well as show how these concepts are related to each other. Finally, we compare these T_1 and the usual T_1 approach spaces.

1. Introduction

It is well-known that the category **Met** of metric spaces and non-expensive maps fails to have infinite products and coproducts. To solve this problem, in 1989, Robert Lowen [17] introduced approach spaces, a generalization of metric and topology, based upon a distance function between points and sets. Approach spaces can be defined in several equivalent ways such as in terms of limit, gauge and distance [18, 22] which correspond to limit points of filter, extended pseudo quasi-metrics determining coarser topologies and closure operators in topology respectively. Approach spaces have several applicative roots in all field of mathematics including probability theory [12], domain theory [13], group theory [19] and vector spaces [21].

In 1991, Baran [2] introduced local T_1 separation property in order to define the notion of strong closedness [2] in set-based topological category which forms closure operators in sense of Dikranjan and Giuli [14, 15] in some well known topological categories **Conv** (category of convergence spaces and continuous maps) [6, 18, 23], **Prord** (category of preordered sets and order preserving maps) [7, 15] and **SUConv** (category of semiuniform convergence spaces and uniformly continuous maps) [9, 24]. Furthermore, Baran [2] generalized T_1 axiom of topology to topological category which is used to define regular, completely regular and normal objects [4, 5] in topological categories.

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The aim of paper is

(i) to characterize T_1 limit (resp. gauge) approach spaces and show how these are related to each other.
(ii) to give characterization of local T_1 limit (resp. gauge) approach spaces and examine how these are related to each other and their relationship with T_1 axiom.
(iii) to compare these results with usual T_1 defined in [16, 20, 22] and examine their relationship.

2. Preliminaries

Let X and J be sets, $F(X)$ be the set of all filters on X and $\sigma : J \rightarrow F(X)$ be a map. Let $\mathcal{A}$ be collection of subsets of X, $2^{(I)}$ be set of finite subsets of X and 2^X be the power set of X. The stack of $\mathcal{A}$ is defined by $[\mathcal{A}] = \{B \subseteq X | \exists A \in \mathcal{A} : A \subseteq B\}$ and diagonal filter of σ is defined as for all $\alpha \in F(J)$, $\sum \sigma(\alpha) = \bigvee_{F\in\alpha} \bigcap_{j\in F} \sigma(j)$. The indicator map $\theta_A : X \rightarrow [0, \infty]$ of a subset $A \subset X$ is a map which equals 0 on A and ∞ outside A, i.e.,

$$\theta_A(x) = \begin{cases} 0, & x \in A \\ \infty, & x \notin A \end{cases}$$

Definition 1. *(cf.* [18, 22]*) A map $\lambda : F(X) \longrightarrow [0, \infty]^X$ is called a limit on X if it fulfills the following properties:*

(i) $\forall x \in X : \lambda[x](x) = 0$,
(ii) $\forall \alpha, \beta \in F(X) : \alpha \subset \beta \Rightarrow \lambda\beta \leq \lambda\alpha$,
(iii) *For any non-empty family* $(\alpha_i)_{i\in I}$ *of filters on* X : $\lambda(\bigcap_{i\in I} \alpha_i) = \sup_{i\in I} \lambda(\alpha_i)$,
(iv) *For any* $\alpha \in F(X)$ *and any selection of filters* $(\sigma(x))_{x\in X}$*:*
$\lambda\Sigma\sigma(\alpha) \leq \lambda(\alpha) + \sup_{x\in X} \lambda\sigma(x)(x)$.

The pair (X, λ) is called a limit-approach space.

Recall [18], that an extended pseudo-quasi metric on a set X is a map $d : X \times X \rightarrow [0, \infty]$ satisfies for all $x \in X$, $d(x, x) = 0$ and for all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

Definition 2. *(cf.* [18, 22]*) Let X be a set and let $pqMet^\infty(X)$ be the set of all extended pseudo-quasi metrics on X, $\mathfrak{D} \subseteq pqMet^\infty(X)$ and $d \in pqMet^\infty(X)$, then*

(i) $\mathfrak{D}$ *is called* ideal *if it is closed under the formation of finite suprema and if it is closed under the operation of taking smaller function.*
(ii) $\mathfrak{D}$ *dominates* d *if* $\forall x \in X, \epsilon > 0$ *and* $\omega < \infty$ *there exists* $e \in \mathfrak{D}$ *such that* $d(x, .) \wedge \omega \leq e(x, .) + \epsilon$ *and if* $\mathfrak{D}$ *dominates* d*, then* $\mathfrak{D}$ *is called saturated.*

If $\mathfrak{D}$ is an ideal in $pqMet^\infty(X)$ and saturated, then $\mathfrak{D}$ is called gauge. The pair $(X, \mathfrak{D})$ is called a gauge-approach space.

Definition 3. *(cf. [18, 22]) A map $\delta : X \times 2^X \to [0, \infty]$ is called distance on X if δ satisfies the followings:*

(i) $\forall A \subseteq X$ *and* $\forall x \in A$, $\delta(x, A) = 0$
(ii) $\forall x \in X$ *and* $\emptyset$, *the empty set,* $\delta(x, \emptyset) = \infty$
(iii) $\forall x \in X, \forall A, B \subseteq X$, $\delta(x, A \cup B) = \min(\delta(x, A), \delta(x, B))$
(iv) $\forall x \in X, \forall A \subseteq X, \forall \epsilon \in [0, \infty]$, $\delta(x, A) \leq \delta(x, A^{(\epsilon)}) + \epsilon$, *where* $A^{(\epsilon)} = \{x \in X | \delta(x, A) \leq \epsilon\}$.

The pair (X, δ) is called a distance-approach space.

Note that limits, gauges and distances are equivalent concepts [18, 22], and we will denote an approach space by $(X, \mathfrak{G})$.

Definition 4. *(cf. [18, 22]) Let $(X, \mathfrak{G})$ and $(X', \mathfrak{G}')$ be approach spaces. If the map $f : (X, \mathfrak{G}) \longrightarrow (X', \mathfrak{G}')$ satisfies one of the following equivalent, then f is called a contraction map.*

(i) $\forall \alpha \in F(X) : \lambda'(f(\alpha)) \leq \lambda\alpha$.
(ii) $\forall d' \in \mathfrak{D}' : d' \circ (f \times f) \in \mathfrak{D}$.
(iii) $\forall x \in X$ *and* $A \subseteq X$, $\delta'(f(x), f(A)) \leq \delta(x, A)$.

The category whose objects are approach spaces and morphisms are contraction maps is denoted by **App** and it is a topological category over **Set** [18, 22].

Lemma 5. *(cf. [18, 22]) Let $(X_i, \mathfrak{G}_i)$ be the collection of approach spaces and $f_i : X \to (X_i, \mathfrak{G}_i)$ be a source in **App**.*

(i) *The initial limit-approach structure on X is given by $\lambda\alpha = \sup_{i \in I} \lambda_i(f_i(\alpha)) \circ f_i$, where $f_i(\alpha)$ is a filter generated by $\{f_i(A_i), i \in I\}$, i.e., $f_i(\alpha) = \{A_i \subset X_i : \exists B \in \alpha$ such that $f_i(B) \subset A_i\}$.*
(ii) *The initial gauge-approach base on X is defined by*
$$\mathcal{H} = \{\sup_{i \in K} d_i \circ (f_i \times f_i) : K \in 2^{(I)}, \forall i \in K, d_i \in \mathcal{H}_i\},$$
where for any $i \in I, \mathcal{H}_i$ is a basis for gauge in X_i.
(iii) *The discrete limit-approach structure λ on X is given by*
$$\lambda\alpha = \begin{cases} \theta_{\{x\}}, & \alpha = [x] \\ \infty, & \alpha \neq [x] \end{cases}$$
for all $\alpha \in F(X)$ and $x \in X$, where $\theta_{\{x\}}$ is an indicator of $\{x\}$.
(iv) *The discrete gauge-approach structure $\mathfrak{D}$ on X is $\mathfrak{D} = pqMet^{\infty}(X)$ (all extended pseudo-quasi metric spaces on X).*

3. Local T_1 Approach Spaces

Let X be a set and $p \in X$. Let $X \vee_p X$ be the wedge at p [2], i.e., two disjoint copies of X identified at p. A point x in $X \vee_p X$ will be denoted by $x_1(x_2)$ if x is in the first (resp. the second) component of $X \vee_p X$. Note that $p_1 = p_2$.

Definition 6. *(cf. [2]) A map $S_p : X \vee_p X \to X^2$ is called skewed p-axis map if*

$$S_p(x_i) = \begin{cases} (x, x), & i = 1 \\ (p, x), & i = 2 \end{cases}$$

Definition 7. *(cf. [2]) A map $\nabla_p : X \vee_p X \to X$ is called folding map at p if $\nabla_p(x_i) = x$ for $i = 1, 2$.*

Recall [1, 24], that a functor $\mathcal{U} : \mathcal{E} \to \mathbf{Set}$ is called topological if $\mathcal{U}$ is concrete, consists of small fibers and each $\mathcal{U}$-source has an initial lift or equivalently, each $\mathcal{U}$-sink has a final lift and called normalized topological functor if constant objects have a unique structure.

Note that a topological functor has a left adjoint called the discrete functor [1].

Definition 8. *(cf. [2]) Let $\mathcal{U} : \mathcal{E} \to \mathbf{Set}$ be topological, X an object in $\mathcal{E}$ with $p \in \mathcal{U}(X) = B$*

If the initial lift of the $\mathcal{U}$ -source $\{S_p : B \vee_p B \to \mathcal{U}(X^2) = B^2$ and $\nabla_p : B \vee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete, where $\mathcal{D}$ is the discrete functor, then X is called T_1 at p.

Theorem 9. *A limit-approach space (X, λ) is T_1 at p if and only if for all $x \in X$ with $x \neq p$, $\lambda([x])(p) = \infty = \lambda([p])(x)$.*

Proof. Let (X, λ) be T_1 at p and $x \in X$ with $x \neq p$. Note that $[x_1], [x_2] \in F(X \vee_p X)$ and $x_1, x_2 \in X \vee_p X$.

$$\lambda_{dis}([\nabla_p x_1])(\nabla_p x_2) = \lambda_{dis}([x])(x) = 0,$$
$$\lambda([\pi_1 S_p x_1])(\pi_1 S_p x_2) = \lambda([x])(p),$$

and

$$\lambda([\pi_2 S_p x_1])(\pi_2 S_p x_2) = \lambda([x])(x) = 0,$$

where λ_{dis} is the discrete structure on X, $\pi_i : X^2 \to X$, $i = 1, 2$ are the projection maps. Since (X, λ) is T_1 at p, by Lemma 5 (i),

$$\begin{aligned} \infty &= \sup\{\lambda_{dis}([\nabla_p x_1])(\nabla_p x_2), \lambda([\pi_1 S_p x_1])(\pi_1 S_p x_2), \lambda([\pi_2 S_p x_1])(\pi_2 S_p x_2)\} \\ &= \sup\{0, \lambda([x])(p)\} = \lambda([x])(p) \end{aligned}$$

and consequently, $\lambda([x])(p) = \infty$.

Similarly,

$$\lambda_{dis}([\nabla_p x_2])(\nabla_p x_1) = \lambda_{dis}([x])(x) = 0,$$
$$\lambda([\pi_1 S_p x_2])(\pi_1 S_p x_1) = \lambda([p])(x),$$

and

$$\lambda([\pi_2 S_p x_2])(\pi_2 S_p x_1) = \lambda([x])(x) = 0.$$

Since (X, λ) is T_1 at p, by Lemma 5 (i)

$$\begin{aligned} \infty &= \sup\{\lambda_{dis}([\nabla_p x_2])(\nabla_p x_1), \lambda([\pi_1 S_p x_2])(\pi_1 S_p x_1), \lambda([\pi_2 S_p x_2])(\pi_2 S_p x_1)\} \\ &= \sup\{0, \lambda([p])(x)\} = \lambda([p])(x) \end{aligned}$$

and consequently, $\lambda([p])(x) = \infty$.

Conversely, let $\overline{\lambda}$ be an initial limit structure on $X \vee_p X$ induced by the maps $S_p : X \vee_p X \to (X^2, \lambda^2)$ and $\nabla_p : X \vee_p X \to (X, \lambda_{dis})$, where λ_{dis} is discrete limit structure on X and λ^2 is the product limit-structure on X^2 induced by $\pi_i : X^2 \to X$ the projection maps for $i = 1, 2$. Suppose $\alpha \in F(X \vee_p X)$ and $v \in X \vee_p X$ with $\nabla_p v = x$. By Lemma 5 (iii), we have to show that, for all $u \in X \vee_p X$

$$\overline{\lambda}(\alpha) = \begin{cases} \theta_{\{v\}}, & \alpha = [v] \\ \infty, & \alpha \neq [v] \end{cases}$$

where

$$\theta_{\{v\}} u = \begin{cases} 0, & v = u \\ \infty, & v \neq u \end{cases}$$

is the indicator of $\{v\}$. Note that

$$\lambda_{dis}(\nabla_p \alpha)(\nabla_p u) = \begin{cases} \theta_{\{x\}} \nabla_p u, & \nabla_p \alpha = [x] \\ \infty, & \nabla_p \alpha \neq [x] \end{cases}$$

$$= \begin{cases} 0, & \nabla_p \alpha = [x] \text{ and } \nabla_p u = x \\ \infty, & \nabla_p \alpha = [x] \text{ and } \nabla_p u \neq x \\ \infty, & \nabla_p \alpha \neq [x] \text{ and } \nabla_p u \neq x \end{cases}$$

Case 1: If $x = p$, then $\nabla_p u = x = p$ implies $u = p_1 = p_2 = v$ and $\nabla_p \alpha = [x] = [p]$ implies $\alpha = [p_i]$ for $i = 1, 2$. By Lemma 5 (i), $\overline{\lambda}(\nabla_p \alpha)(\nabla_p u) = \overline{\lambda}([p])(p) = 0$ since $\overline{\lambda}$ is a limit structure on $X \vee_p X$.

Suppose that $x \neq p$. $\nabla_p u = x$ implies $u = x_1$ or $u = x_2$ and $\nabla_p \alpha = [x]$ implies $\alpha = [x_1]$, $[x_2]$, $[\{x_1, x_2\}]$ or $\alpha \supset [\{x_1, x_2\}]$.

Firstly, we show that the case $\alpha \supset [\{x_1, x_2\}]$ with $\alpha \neq [\emptyset]$ and $\alpha \neq [\{x_1, x_2\}]$ cannot occur. To this end, if $[\emptyset] \neq \alpha \neq [\{x_1, x_2\}]$, then $\alpha \supset [\{x_1, x_2\}]$ if and only if $\alpha = [x_1]$ or $\alpha = [x_2]$. Clearly, if $\alpha = [x_1]$ or $[x_2]$, then $\alpha \supset [\{x_1, x_2\}]$. Conversely, if $\alpha \supset [\{x_1, x_2\}]$ with $[\emptyset] \neq \alpha \neq [\{x_1, x_2\}]$, then there exists $V \in \alpha$ such that $V \neq \{x_1, x_2\}$ and $V \neq \emptyset$. Since V and $W = \{x_1, x_2\}$ are in α and α is a filter, $V \cap W = \{x_1\}$ or $\{x_2\}$ is in α, i.e., $\alpha = [x_1]$ or $[x_2]$. Hence, we must have $\alpha = [x_1]$, $[x_2]$ or $[\{x_1, x_2\}]$.

If $\alpha = [x_i]$ and $u = x_i$, $i = 1, 2$, then $\overline{\lambda}([x_i])(x_i) = 0$ since $\overline{\lambda}$ is a limit structure on $X \vee_p X$.

If $\alpha = [x_2]$ and $u = x_1$, then
$\lambda_{dis}(\nabla_p \alpha)(\nabla_p u) = \lambda_{dis}(\nabla_p [x_2])(\nabla_p x_1) = \lambda_{dis}([x])(x) = 0.$
$\lambda(\pi_1 S_p \alpha)(\pi_1 S_p u) = \lambda([\pi_1 S_p x_2])(\pi_1 S_p x_1)) = \lambda([p])(x)$ and
$\lambda(\pi_2 S_p \alpha)(\pi_2 S_p u) = \lambda([\pi_2 S_p x_2])(\pi_2 S_p x_1) = \lambda([x])(x) = 0.$

By Lemma 5 (i) and the assumption $\lambda([p])(x) = \infty$.

$$\begin{aligned}\overline{\lambda}(\alpha)(u) &= \overline{\lambda}([x_2])(x_1)\\ &= \sup\{\lambda_{dis}([\nabla_p x_2])(\nabla_p x_1), \lambda([\pi_1 S_p x_2])(\pi_1 S_p x_1), \lambda([\pi_2 S_p x_2])(\pi_2 S_p x_1)\}\\ &= \sup\{0, \lambda([p])(x)\} = \lambda([p])(x) = \infty\end{aligned}$$

If $\alpha = [\{x_1, x_2\}]$, $u = x_1$, then

$$\lambda_{dis}(\nabla_p \alpha)(\nabla_p u) = \lambda_{dis}(\nabla_p[\{x_1, x_2\}])(\nabla_p x_1) = \lambda_{dis}([x])(x) = 0.$$

$$\lambda(\pi_1 S_p \alpha)(\pi_1 S_p u) = \lambda([\{\pi_1 S_p x_1, \pi_1 S_p x_2\}])(\pi_1 S_p x_1) = \lambda([\{x, p\}])(x),$$

and

$$\lambda(\pi_2 S_p \alpha)(\pi_2 S_p u) = \lambda([\{\pi_2 S_p x_1, \pi_2 S_p x_2\}])(\pi_2 S_p x_1) = \lambda([x])(x) = 0,$$

Note that $[\{x, p\}] \subset [p]$. Since λ is a limit structure, we get $\lambda([p])(x) \leq \lambda([\{x, p\}])(x)$. Since $x \neq p$ and $\lambda([p])(x) = \infty$, by assumption, then $\lambda([\{x, p\}])(x) = \infty$, and consequently, $\overline{\lambda}(\alpha)(u) = \infty$.

If $\alpha = [x_1]$ and $u = x_2$, then
$\lambda_{dis}(\nabla_p \alpha)(\nabla_p u) = \lambda_{dis}(\nabla_p[x_1])(\nabla_p x_2) = \lambda_{dis}([x])(x) = 0.$
$\lambda(\pi_1 S_p \alpha)(\pi_1 S_p u) = \lambda([\pi_1 S_p x_1])(\pi_1 S_p x_2)) = \lambda([x])(p)$ and
$\lambda(\pi_2 S_p \alpha)(\pi_2 S_p u) = \lambda([\pi_2 S_p x_1])(\pi_2 S_p x_2) = \lambda([x])(x) = 0,$
by Lemma 5 (i)

$$\begin{aligned}\overline{\lambda}(\alpha)(u) &= \overline{\lambda}([x_1])(x_2)\\ &= \sup\{\lambda_{dis}([\nabla_p x_1])(\nabla_p x_2), \lambda([\pi_1 S_p x_1])(\pi_1 S_p x_2), \lambda([\pi_2 S_p x_1])(\pi_2 S_p x_2)\}\\ &= \sup\{0, \lambda([x])(p)\} = \lambda([x])(p) = \infty\end{aligned}$$

If $\alpha = [\{x_1, x_2\}]$, $u = x_2$, then

$$\lambda_{dis}(\nabla_p \alpha)(\nabla_p u) = \lambda_{dis}(\nabla_p[\{x_1, x_2\}])(\nabla_p x_2) = \lambda_{dis}([x])(x) = 0.$$

$$\lambda(\pi_1 S_p \alpha)(\pi_1 S_p u) = \lambda([\{\pi_1 S_p x_1, \pi_1 S_p x_2\}])(\pi_1 S_p x_2) = \lambda([\{x, p\}])(p).$$

and

$$\lambda(\pi_2 S_p \alpha)(\pi_2 S_p u) = \lambda([\{\pi_2 S_p x_1, \pi_2 S_p x_2\}])(\pi_2 S_p x_2) = \lambda([x])(x) = 0.$$

Note that $[\{x, p\}] \subset [x]$. Since λ is a limit structure, $\lambda([x])(p) \leq \lambda([\{x, p\}])(p)$ and by the assumption $\lambda([p])(x) = \infty$, then $\lambda([\{x, p\}])(x) = \infty$.

By Lemma 5 (i),

$$\begin{aligned}\overline{\lambda}(\alpha)(u) &= \overline{\lambda}([\{x_1, x_2\}])(x_2)\\ &= \sup\{\lambda_{dis}([\{\nabla_p x_1, \nabla_p x_2\}])(\nabla_p x_2), \lambda([\{\pi_1 S_p x_1, \pi_1 S_p x_2\}])(\pi_1 S_p x_2),\\ &\quad \lambda([\{\pi_2 S_p x_1, \pi_2 S_p x_2\}])(\pi_2 S_p x_2)\} = \sup\{0, \infty\} = \infty.\end{aligned}$$

Case 2: Let $p = \nabla_p u \neq x$ and $\nabla_p \alpha = [x]$. It follows that $u = p_1 = p_2$ and $\alpha = [x_1]$, $[x_2]$ or $[\{x_1, x_2\}]$.

If $\alpha = [x_1]$, $[x_2]$ or $[\{x_1, x_2\}]$ and $u = p_i$ for $i = 1, 2$, then $\lambda_{dis}(\nabla_p\alpha)(\nabla_p u) = \lambda_{dis}([x])(p) = \infty$ since λ_{dis} is a discrete limit structure and $x \neq p$. It follows that

$$\begin{aligned}\overline{\lambda}(\alpha)(u) &= \sup\{\lambda_{dis}(\nabla_p\alpha)(\nabla_p u), \lambda(\pi_1 S_p\alpha)(\pi_1 S_p u), \lambda(\pi_2 S_p\alpha)(\pi_2 S_p u)\} \\ &= \sup\{\infty, \lambda(\pi_1 S_p\alpha)(p), \lambda(\pi_2 S_p\alpha)(p)\} = \infty.\end{aligned}$$

Case 3: Suppose $\nabla_p u \neq x$ and $\nabla_p\alpha \neq [x]$, then $\lambda_{dis}(\nabla_p\alpha)(\nabla_p u) = \infty$ since λ_{dis} is a discrete limit structure. It follows that

$$\begin{aligned}\overline{\lambda}(\alpha)(u) &= \sup\{\lambda_{dis}(\nabla_p\alpha)(\nabla_p u), \lambda(\pi_1 S_p\alpha)(\pi_1 S_p u), \lambda(\pi_2 S_p\alpha)(\pi_2 S_p u)\} \\ &= \sup\{\infty, \lambda(\pi_1 S_p\alpha)(\pi_1 S_p u), \lambda(\pi_2 S_p\alpha)(\pi_2 S_p u)\} = \infty\end{aligned}$$

Hence, for all $\alpha \in F(X \vee_p X)$ and $v \in X \vee_p X$, we have

$$\overline{\lambda}(\alpha) = \begin{cases}\theta_{\{v\}}, & \alpha = [v] \\ \infty, & \alpha \neq [v]\end{cases}$$

i.e., $\overline{\lambda}$ is discrete limit structure on $X \vee_p X$ and by Definition 8, (X, λ) is T_1 at p. □

Theorem 10. *A gauge-approach space $(X, \mathfrak{D})$ is T_1 at p if and only if for all $x \in X$ with $x \neq p$, there exists $d \in \mathfrak{D}$ such that $d(x, p) = \infty = d(p, x)$.*

Proof. Let $(X, \mathfrak{D})$ be T_1 at p, $x \in X$ and $x \neq p$. Let $u = x_1$ and $v = x_2 \in X \vee_p X$. Note that

$$\begin{aligned} d(\pi_1 S_p u, \pi_1 S_p v) = d(\pi_1 S_p x_1, \pi_1 S_p x_2) = d(x, p), \\ d(\pi_2 S_p u, \pi_2 S_p v) = d(\pi_2 S_p x_1, \pi_2 S_p x_2) = d(x, x) = 0, \\ d_{dis}(\nabla_p u, \nabla_p v) = d_{dis}(\nabla_p x_1, \nabla_p x_2) = d_{dis}(x, x) = 0,\end{aligned}$$

where d_{dis} is the discrete extended pseudo-quasi metric on $X \vee_p X$ and $\pi_i : X^2 \to X$ are the projection maps and $i = 1, 2$. Since $u \neq v$ and $(X, \mathfrak{D})$ is T_1 at p, by Lemma 5 (ii),

$$\infty = \sup\{d_{dis}(\nabla_p u, \nabla_p v), d(\pi_1 S_p u, \pi_1 S_p v), d(\pi_2 S_p u, \pi_2 S_p v)\} = d(x, p)$$

and consequently, $d(x, p) = \infty$.

Similarly, if $u = x_2$ and $v = x_1 \in X \vee_p X$, then

$$\begin{aligned}\infty &= \sup\{d_{dis}(\nabla_p u, \nabla_p v), d(\pi_1 S_p u, \pi_1 S_p v), d(\pi_2 S_p u, \pi_2 S_p v)\} = \sup\{0, d(p, x)\} \\ &= d(p, x)\end{aligned}$$

and consequently, $d(p, x) = \infty$.

Conversely, let $\overline{\mathcal{H}}$ be initial gauge basis on $X \vee_p X$ induced by $S_p : X \vee_p X \to U(X^2, \mathfrak{D}^2) = X^2$ and $\nabla_p : X \vee_p X \to U(X, \mathfrak{D}_{dis}) = X$ where $\mathfrak{D}_{dis} = pqMet^{\infty}(X)$ discrete gauge-approach on X and $\mathfrak{D}^2$ is the product gauge-approach structure on X^2 induced by $\pi_i : X^2 \to X$ the projection maps for $i = 1, 2$. Suppose $\overline{d} \in \overline{\mathcal{H}}$ and $u, v \in X \vee_p X$.

If $u = v$, then $\overline{d}(u, v) = 0$.

If $u \neq v$ and $\nabla_p u \neq \nabla_p v$ implies $d_{dis}(\nabla_p u, \nabla_p v) = \infty$ since d_{dis} is a discrete structure. By Lemma 5 (ii),

$$\begin{aligned}\overline{d}(u,v) &= \sup\{d_{dis}(\nabla_p u, \nabla_p v), d(\pi_1 S_p u, \pi_1 S_p v), d(\pi_2 S_p u, \pi_2 S_p v)\}\\ &= \sup\{\infty, d(\pi_1 S_p u, \pi_1 S_p v), d(\pi_2 S_p u, \pi_2 S_p v)\} = \infty.\end{aligned}$$

Suppose $u \neq v$ and $\nabla_p u = \nabla_p v$. If $\nabla_p u = x = \nabla_p v$ for some $x \in X$ with $x \neq p$, then $u = x_1$ and $v = x_2$ or $u = x_2$ and $v = x_1$ since $u \neq v$.

If $u = x_1$ and $v = x_2$, then by Lemma 5 (ii),

$$\begin{aligned}\overline{d}(u,v) &= \overline{d}(x_1, x_2)\\ &= \sup\{d_{dis}(\nabla_p x_1, \nabla_p x_2), d(\pi_1 S_p x_1, \pi_1 S_p x_2), d(\pi_2 S_p x_1, \pi_2 S_p x_2)\}\\ &= \sup\{0, d(x,p)\} = d(x,p) = \infty\end{aligned}$$

since $x \neq p$ and $d(x,p) = \infty$.

Similarly, if $u = x_2$ and $v = x_1$, then

$$\begin{aligned}\overline{d}(u,v) &= \overline{d}(x_2, x_1)\\ &= \sup\{d_{dis}(\nabla_p x_2, \nabla_p x_1), d(\pi_1 S_p x_2, \pi_1 S_p x_1), d(\pi_2 S_p x_2, \pi_2 S_p x_1)\}\\ &= \sup\{0, d(p,x)\} = d(p,x) = \infty\end{aligned}$$

since $x \neq p$ and $d(p,x) = \infty$.

Hence, for all $u, v \in X \vee_p X$, we get

$$\overline{d}(u,v) = \begin{cases} 0, & u = v \\ \infty, & u \neq v \end{cases}$$

i.e., $\overline{d}$ is discrete extended pseudo-quasi metric on $X \vee_p X$, i.e., $\overline{\mathcal{H}} = \{\overline{d}\}$. By Definition 8, $(X, \mathfrak{D})$ is T_1 at p. □

Theorem 11. *Let $(X, \mathfrak{G})$ be approach spaces and $p \in X$. Then, following are equivalent:*

1. *$(X, \mathfrak{G})$ is T_1 at p.*
2. *For all $x \in X$ with $x \neq p$, $\lambda([x])(p) = \infty = \lambda([p])(x)$.*
3. *For all $x \in X$ with $x \neq p$, there exists $d \in \mathfrak{D}$ such that $d(x,p) = \infty = d(p,x)$.*
4. *For all $x \in X$ with $x \neq p$, $\delta(x, \{p\}) = \infty = \delta(p, \{x\})$.*

Proof. It follows from Theorems 9 and 10, and Theorem 3.1 of [10]. □

Example 12. *(i) Let $X = \{a,b,c\}$, $A \subseteq X$ and $\delta_1 : X \times 2^X \to [0,\infty]$ be a map defined as follows: For all $x \in X$, $\delta_1(x, \emptyset) = \infty$, $\delta_1(x, A) = 0$ if $x \in A$, $\delta_1(a,\{b\}) = \delta_1(b,\{a\}) = \delta_1(a,\{c\}) = \delta_1(c,\{a\}) = \infty = \delta_1(a,\{b,c\})$ and $\delta_1(b,\{c\}) = \delta_1(c,\{b\}) = \delta_1(b,\{a,c\}) = \delta_1(c,\{a,b\}) = 2$. Then, by Theorem 11, an approach space (X, δ_1) is T_1 at a but it is neither T_1 at b nor T_1 at c.*

(ii) Let $X = \{a, b, c\}$, $A \subseteq X$ and $\delta_2 : X \times 2^X \to [0, \infty]$ be a map defined as follows: For all $x \in X$, $\delta_2(x, \emptyset) = \infty$, $\delta_2(x, A) = 0$ if $x \in A$ and $\delta_2(x, A) = 1$ if $x \notin A$. Then, by Theorem 11, an approach space (X, δ_2) is not T_1 at p for all $p \in X$.

4. T_1 Approach Spaces

Let X be a nonempty set, $X^2 = X \times X$ be cartesian product of X with itself and $X^2 \vee_\triangle X^2$ be two distinct copies of X^2 identified along the diagonal [2]. A point (x, y) in $X^2 \vee_\triangle X^2$ is denoted by $(x, y)_1$ ($(x, y)_2$) if (x, y) is in the first (resp. second) component of $X^2 \vee_\triangle X^2$. Note that $(x, x)_1 = (x, x)_2$, for all $x \in X$.

Definition 13. *(cf. [2]) A map $S : X^2 \vee_\triangle X^2 \to X^3$ is called skewed axis map if*

$$S((x,y)_i) = \begin{cases} (x,y,y), & i = 1 \\ (x,x,y), & i = 2 \end{cases}$$

Definition 14. *(cf. [2]) A map $\nabla : X^2 \vee_\triangle X^2 \to X^2$ is called folding map if $\nabla((x, y)_i) = (x, y)$ for $i = 1, 2$.*

Theorem 15. *(cf. [3]) Let (X, τ) be a topological space.*

(X, τ) is T_1 if and only if the initial topology on $X^2 \vee_\triangle X^2$ induced by the maps $S : X^2 \vee_\triangle X^2 \to (X^3, \tau^*)$ and $\nabla : X^2 \vee_\triangle X^2 \to (X^2, P(X^2))$ is discrete, where τ^* is the product topology on X^3.

Definition 16. *(cf. [2]) Let $\mathcal{U} : \mathcal{E} \to$ **Set** be topological, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$.*

If the initial lift of the $\mathcal{U}$-source $\{S : B^2 \vee_\triangle B^2 \to \mathcal{U}(X^3) = B^3$ and $\nabla : B^2 \vee_\triangle B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete, then X is called a T_1 object.

Theorem 17. *A limit-approach space (X, λ) is T_1 if and only if for all $x, y \in X$ with $x \neq y$, $\lambda([x])(y) = \infty = \lambda([y])(x)$.*

Proof. Let (X, λ) be T_1, $x, y \in X$ with $x \neq y$. Suppose $[(x, y)_1]$ is a filter on $F(X^2 \vee_\triangle X^2)$ and $(x, y)_2 \in X^2 \vee_\triangle X^2$. Note that

$$\lambda_{dis}([\nabla(x,y)_1])(\nabla(x,y)_2) = \lambda_{dis}([(x,y)])(x,y) = 0,$$
$$\lambda([\pi_1 S(x,y)_1])(\pi_1 S(x,y)_2) = \lambda([x])(x) = 0,$$
$$\lambda([\pi_2 S(x,y)_1])(\pi_2 S(x,y)_2) = \lambda([y])(x).$$

and

$$\lambda([\pi_3 S(x,y)_1])(\pi_3 S(x,y)_2) = \lambda([y])(y) = 0.$$

Since (X, λ) is T_1 and $(x, y)_1 \neq (x, y)_2$, by Lemma 5 (i),

$$\begin{aligned} \infty &= \sup\{\lambda_{dis}([\nabla(x,y)_1])(\nabla(x,y)_2), \lambda([\pi_1 S(x,y)_1])(\pi_1 S(x,y)_2), \\ &\quad \lambda([\pi_2 S(x,y)_1])(\pi_2 S(x,y)_2), \lambda([\pi_3 S(x,y)_1])(\pi_3 S(x,y)_2)\} \\ &= \sup\{0, \lambda([y])(x)\} = \lambda([y])(x) \end{aligned}$$

and consequently, $\lambda([y])(x) = \infty$.

Similarly, let $[(x,y)_2] \in F(X^2 \vee_\triangle X^2)$ and $(x,y)_1 \in X^2 \vee_\triangle X^2$. Since (X,λ) is T_1 and $(x,y)_1 \neq (x,y)_2$, by Lemma 5 (i),

$$\begin{aligned} \infty &= \sup\{\lambda_{dis}([\nabla(x,y)_2])(\nabla(x,y)_1), \lambda([\pi_1 S(x,y)_2])(\pi_1 S(x,y)_1), \\ &\quad \lambda([\pi_2 S(x,y)_2])(\pi_2 S(x,y)_1), \lambda([\pi_3 S(x,y)_2])(\pi_3 S(x,y)_1)\} \\ &= \sup\{0, \lambda([x])(y)\} = \lambda([x])(y) \end{aligned}$$

and consequently, $\lambda([x])(y) = \infty$.

Conversely, let $\overline{\lambda}$ be an initial limit structure on $X^2 \vee_\triangle X^2$ induced by the maps $S : X^2 \vee_\triangle X^2 \to (X^3, \lambda^3)$ and $\nabla : X^2 \vee_\triangle X^2 \to (X^2, \lambda_{dis})$ where λ_{dis} is discrete limit structure on X^2 and λ^3 is the product limit-structure on X^3 induced by $\pi_i : X^3 \to X$ the projection maps for $i = 1,2,3$. Suppose $\alpha \in F(X^2 \vee_\triangle X^2)$ and $v \in X^2 \vee_\triangle X^2$ with $\nabla v = (x,y)$. Note that

$$\lambda_{dis}(\nabla\alpha)(\nabla u) = \begin{cases} \theta_{\{(x,y)\}} \nabla u, & \nabla\alpha = [(x,y)] \\ \infty, & \nabla\alpha \neq [(x,y)] \end{cases}$$

$$= \begin{cases} 0, & \nabla\alpha = [(x,y)] \text{ and } \nabla u = (x,y) \\ \infty, & \nabla\alpha = [(x,y)] \text{ and } \nabla u \neq (x,y) \\ \infty, & \nabla\alpha \neq [(x,y)] \text{ and } \nabla u \neq (x,y) \end{cases}$$

Case 1: If $x = y$, then $\nabla u = (x,x)$ implies $u = (x,x)_1 = (x,x)_2 = v$ and $\nabla\alpha = [(x,x)]$ implies $\alpha = [(x,x)_1] = [(x,x)_2]$. By Lemma 5 (i), $\overline{\lambda}(\nabla\alpha)(\nabla u) = \overline{\lambda}([(x,x)])(x,x) = 0$ since $\overline{\lambda}$ is a limit structure on $X^2 \vee_\triangle X^2$.

Suppose that $\nabla u = (x,y)$ for some $x,y \in X$ with $x \neq y$ implies $u = (x,y)_1$ or $u = (x,y)_2$ and $\nabla\alpha = [(x,y)]$ implies $\alpha = [(x,y)_1]$, $[(x,y)_2]$, $[\{(x,y)_1,(x,y)_2\}]$ or $\alpha \supset [\{(x,y)_1,(x,y)_2\}]$. By the same argument used in Theorem 9, $\alpha \supset [\{(x,y)_1,(x,y)_2\}]$ with $\alpha \neq [\emptyset]$ and $\alpha \neq [\{(x,y)_1,(x,y)_2\}]$ cannot occur. Hence, we must have $\alpha = [(x,y)_1]$, $[(x,y)_2]$ or $[\{(x,y)_1,(x,y)_2\}]$.

If $\alpha = [(x,y)_i]$ and $u = (x,y)_i$, $i = 1,2$, then $\overline{\lambda}([(x,y)_i])((x,y)_i) = 0$ since $\overline{\lambda}$ is a limit structure on $X^2 \vee_\triangle X^2$.

If $\alpha = [(x,y)_2]$ and $u = (x,y)_1$, then

$$\lambda_{dis}(\nabla\alpha)(\nabla u) = \lambda_{dis}(\nabla[(x,y)_2])(\nabla(x,y)_1) = \lambda_{dis}([(x,y)])(x,y) = 0$$

$$\lambda(\pi_1 S\alpha)(\pi_1 Su) = \lambda([\pi_1 S(x,y)_2])(\pi_1 S(x,y)_1)) = \lambda([x])(x) = 0,$$

$$\lambda(\pi_2 S\alpha)(\pi_2 Su) = \lambda([\pi_2 S(x,y)_2])(\pi_2 S(x,y)_1) = \lambda([x])(y),$$

and

$$\lambda(\pi_3 S\alpha)(\pi_3 Su) = \lambda([\pi_3 S(x,y)_2])(\pi_3 S(x,y)_1) = \lambda([y])(y) = 0.$$

By Lemma 5 (i) and by assumption

$$\begin{aligned}\overline{\lambda}(\alpha)(u) &= \overline{\lambda}([(x,y)_2])((x,y)_1)\\ &= \sup\{\lambda_{dis}([\nabla(x,y)_2])(\nabla(x,y)_1), \lambda([\pi_1 S(x,y)_2])(\pi_1 S(x,y)_1),\\ &\quad \lambda([\pi_2 S(x,y)_2])(\pi_2 S(x,y)_1), \lambda([\pi_3 S(x,y)_2])(\pi_3 S(x,y)_1)\}\\ &= \sup\{0, \lambda([x])(y)\} = \lambda([x])(y) = \infty.\end{aligned}$$

If $\alpha = [\{(x,y)_1, (x,y)_2\}]$, $u = (x,y)_1$, then

$$\lambda_{dis}(\nabla\alpha)(\nabla u) = \lambda_{dis}(\nabla[\{(x,y)_1,(x,y)_2\}])(\nabla(x,y)_1) = \lambda_{dis}([x])(x) = 0.$$

$$\lambda(\pi_1 S\alpha)(\pi_1 Su) = \lambda([\{\pi_1 S(x,y)_1, \pi_1 S(x,y)_2\}])(\pi_1 S(x,y)_1) = \lambda([x])(x) = 0,$$

$$\lambda(\pi_2 S\alpha)(\pi_2 Su) = \lambda([\{\pi_2 S(x,y)_1, \pi_2 S(x,y)_2\}])(\pi_2 S(x,y)_1) = \lambda([\{x,y\}])(y),$$

and

$$\lambda(\pi_3 S\alpha)(\pi_3 Su) = \lambda([\{\pi_3 S(x,y)_1, \pi_3 S(x,y)_2\}])(\pi_3 S(x,y)_1) = \lambda([y])(y) = 0.$$

Note that $[\{x,y\}] \subset [x]$. Since λ is a limit structure and $\lambda([x])(y) = \infty$, $\lambda([x])(y) \leq \lambda([\{x,y\}])(y)$, it follows that $\lambda([\{x,y\}])(y) = \infty$.

By Lemma 5 (i),

$$\begin{aligned}\overline{\lambda}(\alpha)(u) &= \overline{\lambda}([\{(x,y)_1,(x,y)_2\}])((x,y)_1)\\ &= \sup\{\lambda_{dis}([\{\nabla(x,y)_1, \nabla(x,y)_2\}])(\nabla(x,y)_1), \lambda([\{\pi_1 S(x,y)_1, \pi_1 S(x,y)_2\}])\\ &\quad (\pi_1 S(x,y)_1), \lambda([\{\pi_2 S(x,y)_1, \pi_2 S(x,y)_2\}])(\pi_2 S(x,y)_1), \lambda([\{\pi_3 S(x,y)_1,\\ &\quad \pi_3 S(x,y)_2\}])(\pi_3 S(x,y)_1)\} = \sup\{0,\infty\} = \infty.\end{aligned}$$

If $\alpha = [(x,y)_1]$ and $u = (x,y)_2$, then

$$\lambda_{dis}(\nabla\alpha)(\nabla u) = \lambda_{dis}(\nabla[(x,y)_1])(\nabla(x,y)_2) = \lambda_{dis}([(x,y)])(x,y) = 0$$

$$\lambda(\pi_1 S\alpha)(\pi_1 Su) = \lambda([\pi_1 S(x,y)_1])(\pi_1 S(x,y)_2)) = \lambda([x])(x) = 0,$$

$$\lambda(\pi_2 S\alpha)(\pi_2 Su) = \lambda([\pi_2 S(x,y)_1])(\pi_2 S(x,y)_2) = \lambda([y])(x),$$

and

$$\lambda(\pi_3 S\alpha)(\pi_3 Su) = \lambda([\pi_3 S(x,y)_1])(\pi_3 S(x,y)_2) = \lambda([y])(y) = 0.$$

By Lemma 5 (i) and the assumption $\lambda[y](x) = \infty$,

$$\begin{aligned}\overline{\lambda}(\alpha)(u) &= \overline{\lambda}([(x,y)_1])((x,y)_2)\\ &= \sup\{\lambda_{dis}([\nabla(x,y)_1])(\nabla(x,y)_2), \lambda([\pi_1 S(x,y)_1])(\pi_1 S(x,y)_2),\\ &\quad \lambda([\pi_2 S(x,y)_1])(\pi_2 S(x,y)_2), \lambda([\pi_3 S(x,y)_1])(\pi_3 S(x,y)_2)\}\\ &= \sup\{0, \lambda([y])(x)\} = \lambda([y])(x) = \infty\end{aligned}$$

If $\alpha = [\{(x,y)_1, (x,y)_2\}]$, $u = (x,y)_2$, then

$$\lambda_{dis}(\nabla\alpha)(\nabla u) = \lambda_{dis}(\nabla[\{(x,y)_1,(x,y)_2\}])(\nabla(x,y)_2) = \lambda_{dis}([x])(x) = 0.$$

$$\lambda(\pi_1 S\alpha)(\pi_1 Su) = \lambda([\{\pi_1 S(x,y)_1, \pi_1 S(x,y)_2\}])(\pi_1 S(x,y)_2) = \lambda([x])(x) = 0,$$

$$\lambda(\pi_2 S\alpha)(\pi_2 Su) = \lambda([\{\pi_2 S(x,y)_1, \pi_2 S(x,y)_2\}])(\pi_2 S(x,y)_2) = \lambda([\{x,y\}])(x),$$

and

$$\lambda(\pi_3 S\alpha)(\pi_3 Su) = \lambda([\{\pi_3 S(x,y)_1, \pi_3 S(x,y)_2\}])(\pi_3 S(x,y)_2) = \lambda([y])(y) = 0,$$

Note that $[\{x,y\}] \subset [y]$. Since λ is a limit structure, we get $\lambda([y])(x) \leq \lambda([\{x,y\}])(x)$. By the assumption $\lambda([y])(x) = \infty$, it follows that $\lambda([\{x,y\}])(x) = \infty$.

By Lemma 5 (i),

$$\begin{aligned} \overline{\lambda}(\alpha)(u) &= \overline{\lambda}([\{(x,y)_1, (x,y)_2\}])((x,y)_1) \\ &= \sup\{\lambda_{dis}([\{\nabla(x,y)_1, \nabla(x,y)_2\}])(\nabla(x,y)_2), \lambda([\{\pi_1 S(x,y)_1, \pi_1 S(x,y)_2\}]) \\ &\quad (\pi_1 S(x,y)_2), \lambda([\{\pi_2 S(x,y)_1, \pi_2 S(x,y)_2\}])(\pi_2 S(x,y)_2), \lambda([\{\pi_3 S(x,y)_1, \\ &\quad \pi_3 S(x,y)_2\}])(\pi_3 S(x,y)_2)\} = \sup\{0, \infty\} = \infty. \end{aligned}$$

Case 2: Let $(z,z) = \nabla u \neq (x,y)$ for some $z \in X$ and $\nabla\alpha = [(x,y)]$. It follows that $u = (z,z)_1 = (z,z)_2$ and $\alpha = [(x,y)_1]$, $[(x,y)_2]$ or $[\{(x,y)_1, (x,y)_2\}]$.

If $\alpha = [(x,y)_i]$ or $[\{(x,y)_1, (x,y)_2\}]$ for $i = 1,2$ and $u = (z,z)_1 = (z,z)_2$, then $\lambda_{dis}(\nabla\alpha)(\nabla u) = \lambda_{dis}([(x,y)])(z,z) = \infty$ since λ_{dis} is a discrete limit structure and $(x,y) \neq (z,z)$. It follows that

$$\begin{aligned} \overline{\lambda}(\alpha)(u) &= \sup\{\lambda_{dis}(\nabla\alpha)(\nabla u), \lambda(\pi_1 S\alpha)(\pi_1 Su), \lambda(\pi_2 S\alpha)(\pi_2 Su), \lambda(\pi_3 S\alpha)(\pi_3 Su)\} \\ &= \sup\{\infty, \lambda(\pi_1 S\alpha)(z,z), \lambda(\pi_2 S\alpha)(z,z), \lambda(\pi_3 S\alpha)(z,z)\} = \infty. \end{aligned}$$

Case 3: Suppose $\nabla u \neq (x,y)$ and $\nabla\alpha \neq [(x,y)]$, then $\lambda_{dis}(\nabla\alpha)(\nabla u) = \infty$ since λ_{dis} is a discrete limit structure, and consequently

$$\begin{aligned} \overline{\lambda}(\alpha)(u) &= \sup\{\lambda_{dis}(\nabla\alpha)(\nabla u), \lambda(\pi_1 S\alpha)(\pi_1 Su), \lambda(\pi_2 S\alpha)(\pi_2 Su), \lambda(\pi_3 S\alpha)(\pi_3 Su)\} \\ &= \sup\{\infty, \lambda(\pi_1 S\alpha)(\pi_1 Su), \lambda(\pi_2 S\alpha)(\pi_2 Su), \lambda(\pi_3 S\alpha)(\pi_3 Su)\} = \infty. \end{aligned}$$

Hence, for all $\alpha \in F(X^2 \vee_\triangle X^2)$ and $v \in X^2 \vee_\triangle X^2$, we have

$$\overline{\lambda}(\alpha) = \begin{cases} \theta_{\{v\}}, & \alpha = [v] \\ \infty, & \alpha \neq [v] \end{cases}$$

i.e., $\overline{\lambda}$ is discrete limit structure on $X^2 \vee_\triangle X^2$ and by Definition 16, (X,λ) is T_1. □

Theorem 18. *A gauge-approach space $(X, \mathfrak{D})$ is T_1 if and only if for each distinct points x and y in X, there exists $d \in \mathfrak{D}$ such that $d(x,y) = \infty = d(y,x)$.*

Proof. Let $(X, \mathfrak{D})$ be T_1, $x, y \in X$ with $x \neq y$. Let $u = (x,y)_1, v = (x,y)_2 \in X^2 \vee_\triangle X^2$. Note that

$$d(\pi_1 S(u), \pi_1 S(v)) = d(\pi_1 S(x,y)_1, \pi_1 S(x,y)_2) = d(x,x) = 0,$$

$$d(\pi_2 S(u), \pi_2 S(v)) = d(\pi_2 S(x,y)_1, \pi_2 S(x,y_2) = d(y,x),$$

$$d(\pi_3 S(u), \pi_3 S(v)) = d(\pi_3 S(x,y)_1, \pi_3 S(x,y_2) = d(y,y) = 0,$$

and

$$d_{dis}(\nabla u, \nabla v) = d_{dis}(\nabla(x,y)_1, \nabla(x,y)_2) = d_{dis}((x,y),(x,y)) = 0,$$

where d_{dis} is the discrete extended pseudo-quasi metric on $X^2 \vee_{\triangle} X^2$ and $\pi_i : X^3 \to X$ are the projection maps for $i = 1, 2, 3$. Since $u \neq v$ and $(X, \mathfrak{D})$ is T_1 , by Lemma 5 (ii),

$$\begin{aligned} \infty &= \sup\{d_{dis}(\nabla u, \nabla v), d(\pi_1 S(u), \pi_1 S(v)), d(\pi_2 S(u), \pi_2 S(v)), d(\pi_3 S(u), \pi_3 S(v))\} \\ &= \sup\{0, d(y, x)\} = d(y, x) \end{aligned}$$

and consequently, $d(y, x) = \infty$.

Similarly, if $u = (x, y)_2$ and $v = (x, y)_1 \in X^2 \vee_{\triangle} X^2$, then,

$$d(\pi_1 S(u), \pi_1 S(v)) = d(\pi_1 S(x, y)_2, \pi_1 S(x, y)_1) = d(x, x) = 0,$$

$$d(\pi_2 S(u), \pi_2 S(v)) = d(\pi_2 S(x, y)_2, \pi_2 S(x, y_1) = d(x, y),$$

$$d(\pi_3 S(u), \pi_3 S(v)) = d(\pi_3 S(x, y)_2, \pi_3 S(x, y_1) = d(y, y) = 0,$$

and

$$d_{dis}(\nabla u, \nabla v) = d_{dis}(\nabla (x, y)_2, \nabla (x, y)_1) = d_{dis}((x, y), (x, y)) = 0,$$

Since $u \neq v$ and $(X, \mathfrak{D})$ is T_1 , by Lemma 5 (ii),

$$\begin{aligned} \infty &= \sup\{d_{dis}(\nabla u, \nabla v), d(\pi_1 S(u), \pi_1 S(v)), d(\pi_2 S(u), \pi_2 S(v)), d(\pi_3 S(u), \pi_3 S(v))\} \\ &= \sup\{0, d(x, y)\} = d(x, y) \end{aligned}$$

and consequently, $d(x, y) = \infty$.

Conversely, let $\overline{\mathcal{H}}$ be an initial gauge basis on $X^2 \vee_{\triangle} X^2$ induced by $S : X^2 \vee_{\triangle} X^2 \to (X^3, \mathfrak{D}^3)$ and $\nabla : X^2 \vee_{\triangle} X^2 \to (X^2, \mathfrak{D}_{dis})$, where $\mathfrak{D}_{dis}$ is discrete gauge on X^2 and $\mathfrak{D}^3$ is the product structure on X^3. Suppose $\overline{d} \in \overline{\mathcal{H}}$ and $u, v \in X^2 \vee_{\triangle} X^2$.

If $u = v$, then $\overline{d}(u, v) = 0$ since $\overline{d}$ is the extended pseudo-quasi metric.

If $u \neq v$ and $\nabla u \neq \nabla v$, then $d_{dis}(\nabla u, \nabla v) = \infty$ since d_{dis} is discrete. By Lemma 5 (ii),

$$\begin{aligned} \overline{d}(u, v) &= \sup\{d_{dis}(\nabla u, \nabla v), d(\pi_1 S(u), \pi_1 S(v)), d(\pi_2 S(u), \pi_2 S(v)), d(\pi_3 S(u), \pi_3 S(v))\} \\ &= \sup\{\infty, d(\pi_1 S(u), \pi_1 S(v)), d(\pi_2 S(u), \pi S(v)), d(\pi_3 S(u), \pi_3 S(v))\} \\ &= \infty. \end{aligned}$$

Suppose $u \neq v$ and $\nabla u = (x, y) = \nabla v$ for some $x, y \in X$ with $x \neq y$, it follows that $u = (x, y)_1$ and $v = (x, y)_2$ or $u = (x, y)_2$ and $v = (x, y)_1$. Let $u = (x, y)_1$ and $v = (x, y)_2$.

$$\begin{aligned} \overline{d}(u, v) &= \overline{d}((x, y)_1, (x, y)_2) \\ &= \sup\{d_{dis}(\nabla (x, y)_1, \nabla (x, y)_2), d(\pi_1 S(x, y)_1, \pi_1 S(x, y)_2), \\ &\quad d(\pi_2 S(x, y)_1, \pi_2 S(x, y)_2), d(\pi_3 S(x, y)_1 S, \pi_3 S(x, y)_2)\} \\ &= \sup\{0, d(y, x)\} = d(y, x) = \infty \end{aligned}$$

by the assumption $d(y, x) = \infty$.

If $u = (x, y)_2$ and $v = (x, y)_1$, then

$$\begin{aligned}\overline{d}(u, v) &= \overline{d}((x, y)_2, (x, y)_1)\\ &= \sup\{d_{dis}(\nabla(x, y)_2, \nabla(x, y)_1), d(\pi_1 S(x, y)_2, \pi_1 S(x, y)_1),\\ &\quad d(\pi_2 S(x, y)_2, \pi_2 S(x, y)_1), d(\pi_3 S(x, y)_2, \pi_3 S(x, y)_1)\}\\ &= \sup\{0, d(x, y)\} = d(x, y) = \infty,\end{aligned}$$

by the assumption $d(x, y) = \infty$.

Hence, for all $u, v \in X^2 \vee_\triangle X^2$, we have

$$\overline{d}(u, v) = \begin{cases} 0, & u = v \\ \infty, & u \neq v \end{cases}$$

i.e., $\overline{d}$ is discrete extended pseudo-quasi metric on $X^2 \vee_\triangle X^2$, i.e., $\overline{\mathcal{H}} = \{\overline{d}\}$. By Definition 16, $(X, \mathfrak{D})$ is T_1. □

Remark 19. *(cf.* [18, 22]*) (i) The transition from gauge to distance is given by*

$$\delta(x, A) = \sup_{d \in \mathfrak{D}} \inf_{y \in A} d(x, y)$$

(ii) The transition from distance to gauge is determined by

$$\mathfrak{D} = \{d \in pqMet^\infty(X) | \forall A \subseteq X, \forall x \in X : \inf_{y \in A} d(x, y) \leq \delta(x, A)\}$$

Theorem 20. *Let $(X, \mathfrak{G})$ be an approach space. The following are equivalent:*

(i) *$(X, \mathfrak{G})$ is T_1.*
(ii) *For all $x, y \in X$ with $x \neq y$, $\lambda([x])(y) = \infty = \lambda([y])(x)$.*
(iii) *For all $x, y \in X$ with $x \neq y$, there exists $d \in \mathfrak{D}$ such that $d(x, y) = \infty = d(y, x)$.*
(iv) *For all $x, y \in X$ with $x \neq y$, $\delta(x, \{y\}) = \infty = \delta(y, \{x\})$.*

Proof. $(i) \Leftrightarrow (ii)$ and $(i) \Leftrightarrow (iii)$ follow from Theorem 17 and Theorem 18, respectively.

$(iii) \Rightarrow (iv)$: Suppose for all $x, y \in X$ with $x \neq y$, there exists $d \in \mathfrak{D}$ such that $d(x, y) = \infty = d(y, x)$. By Remark 19 (i), $\delta(x, \{y\}) = \sup_{e \in \mathfrak{D}} e(x, y) = \infty$, and $\delta(y, \{x\}) = \sup_{e \in \mathfrak{D}} e(y, x) = \infty$ and consequently, $\delta(x, \{y\}) = \infty = \delta(y, \{x\})$.

$(iv) \Rightarrow (iii)$: Suppose $\forall x, y \in X$ with $x \neq y$, $\delta(x, \{y\}) = \infty = \delta(y, \{x\})$. Take $A = \{y\}$, then by Remark 19 (ii), for all $e \in \mathfrak{D}$, $e(x, y) \leq \delta(x, \{y\})$ and $e(y, x) \leq \delta(y, \{x\})$. In particular, there exists $d \in \mathfrak{D}$ such that $d(x, y) = \delta(x, \{y\}) = \infty$ and $d(y, x) = \delta(y, \{x\}) = \infty$ and consequently, $d(x, y) = \infty = d(y, x)$. □

Definition 21. *(cf.* [20]*) Let $(X, \mathfrak{G})$ be an approach space.*

If topological co-reflection $(X, \tau_{\mathfrak{G}})$ is $\mathbf{T_1}$ *(we refer it to usual T_1), then an approach space $(X, \mathfrak{G})$ is called* $\mathbf{T_1}$.

Theorem 22. *Let $(X, \mathfrak{G})$ be an approach space. The following are equivalent.*

(i) $(X, \tau_{\mathfrak{G}})$ *is* $\mathbf{T_1}$.
(ii) *For all* $x, y \in X$ *with* $x \neq y$, $\lambda([x])(y) > 0$.
(iii) *For all* $x, y \in X$ *with* $x \neq y$, *there exists* $d \in \mathfrak{D}$ *such that* $d(x, y) > 0$.
(iv) *For all* $x, y \in X$ *with* $x \neq y$, $\delta(x, \{y\}) > 0$.

Proof. It is given in [16, 20, 22]. □

Example 23. *Let* $X = [0, \infty]$, $A \subset X$ *and* $\delta : X \times 2^X \longrightarrow [0, \infty]$ *be a map defined as:*

$$\delta(x, A) = \begin{cases} \infty, & A = \emptyset \\ 0, & x \in A \\ 2, & x \notin A \end{cases}$$

By Theorem 20 and Theorem 22, a distance-approach space (X, δ) *is* $\mathbf{T_1}$ *(in the usual sense) but it is not* T_1 *(in our sense).*

Remark 24. (1)

(i) *In category* **Top** *of topological spaces and continuous functions as well as in the category* **SULim** *semiuniform limit spaces and uniformly continuous maps* [24], *by Theorem 15 and by Remark 4.7(2) of* [8] *both* T_1 *(in our sense) and* $\mathbf{T_1}$ *(in the usual sense) are equivalent and they reduce to usual* T_1 *separation axiom. However, in the category* **pqsMet** *of extended pseudo-quasi-semi metric spaces and non-expensive maps, by Theorem 3.3 of* [11], *an extended pseudo-quasi-semi metric space* (X, d) *is* T_1 *iff for all distinct points* x, y *of* X, $d(x, y) = \infty$ *and by Theorem 3.4 of* [11], (X, d) *is* $\mathbf{T_1}$ *(in the usual sense, i.e.,* (X, τ_d) *is* T_1, *where* τ_d *is the topology induced from* d*) iff for all distinct points* x, y *of* X, $d(x, y) > 0$.

(ii) *By Theorem 11 and Theorem 20, an approach space* $(X, \mathfrak{G})$ *is* T_1 *if and only if* $(X, \mathfrak{G})$ *is* T_1 *at* p *for all* $p \in X$. *Moreover, by Theorem 20 and Theorem 22, if an approach space* $(X, \mathfrak{G})$ *is* T_1 *(in our sense), then* $(X, \mathfrak{G})$ *is* $\mathbf{T_1}$ *(in the usual sense) but by Example 23, reverse implication is not true.*

(iii) *By Example 12 (i), a distance-approach space* (X, δ) *is both* T_1 *at* a *and* $\mathbf{T_1}$ *(in the usual sense) but it is not* T_1 *(in our sense). Furthermore, by Example 12 (ii), a distance-approach space* (X, δ) *is* $\mathbf{T_1}$ *(in the usual sense) but it is not* T_1 *at* a. *Hence, there is no relation between* $\mathbf{T_1}$ *(in the usual sense) and local* T_1.

(iv) *By Remark 2.12 (2) of* [6], T_1 *and local* T_1 *(i.e.,* T_1 *at* p *for all* $p \in X$*) axioms could be equivalent.*

5. Conclusions

In this paper, we gave a characterization of both local T_1 and T_1 limit (resp. guage) approach spaces and determined the result that $(X, \mathfrak{G})$ approach space is T_1 at p for all $p \in X$ iff it is T_1. Moreover, it is shown that by Theorem 20 and Theorem 22, T_1 (in our sense) implies $\mathbf{T_1}$ (in the usual sense), but reverse implication, by Example 23, is not true. Furthermore, by Example 12 (i) and (ii), there is no relation between $\mathbf{T_1}$ (in the usual sense) and local T_1.

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