INTERSECTION GRAPHS OF CO-IDEALS OF SEMIRINGS

S. EBRAHIMI ATANI, S. DOLATI PISHHESARI, M. KHORAMDEL, AND Z. E. SARVANDI

ABSTRACT. Let R be a semiring with identity. In this paper, we introduce the intersection graph of co-ideals, denoted by $G(R)$. The vertices of $G(R)$ are non-trivial co-ideals of R, and two distinct vertices I and J are adjacent if and only if $I \cap J \neq \{1\}$. The basic properties and possible structures of this graph are studied and the interplay between the algebraic properties of R and the graph-theoretic structure of $G(R)$ is investigated.

1. INTRODUCTION

Semirings which are regarded as a generalization of rings have been found useful in solving problems in different disciplines of applied mathematics and information sciences because semirings provides an algebraic framework for modeling. Recently, a lot of study of algebraic structures has been explored via the graph theoretic approach. A basic question about this representation is, what graphs can represent algebraic structures? Attempts to answer this question involve looking at graph properties such as the chromatic number and maximal clique size to find rules about possible graph structures. In 1988, Istvan Beck [2] proposed the study of commutative rings by representing them as graphs, called zero divisor graph. These zero divisor graphs marked the beginning of an approach to studying commutative rings with graphs. Similarly, there is several graphs assigned to rings and semirings [1, 7, 8, 9, 10, 11]. One of the most important graphs which have been studied is the intersection graph. Bosak [4] in 1964 defined the intersection graph of semigroups. In 1969, Csákany and Pollák [14] studied the graph of subgroups of a finite group. In 2009, the intersection graph of ideals of a ring was considered by Chakrabarty, Ghosh, Mukherjee and Sen [5]. In this paper, we introduce intersection graphs of commutative semirings with respect to co-ideals. The *intersection graph* of co-ideals of a semiring R, denoted by $G(R)$, is a graph with $\mathcal{I}^*(R) = \{I : \{1\} \neq I$ is a proper co-ideal of $R\}$ as vertices and two distinct vertices are adjacent if and only if $I \cap J \neq \{1\}$.

In section 2, it is proved that if R is a semiring, then $G(R)$ is empty if and only if $\mathcal{I}^*(R) = Max(R) = \{M_1, M_2\}$ or $R \cong \mathcal{B}$ or $|\mathcal{I}^*(R)| = 1$. Also, if $G(R)$ is connected then $diam(G(R)) \leq 2$ and $gr(G(R)) \in \{3, \infty\}$. In Theorem 12, 13, we find the relation between clique number and the number of co-ideals of R, we show that $\omega(G(R))$ is finite if and only if the set of all co-ideals of R is finite and $\omega(G(R)) = 2^{|Max(R)|-1} - 1$, when $J(R) = \{1\}$ and $Max(R)$ is finite. In Theorem 15, we characterize the co-ideals of R, when $G(R)$ has an end vertex and Corollary 17 shows that if $G(R)$ is complete r-partite, then either $|\mathcal{I}^*(R)| = r$ or $r + 1$. In the last theorem of this section, we find independence number of $G(R)$ by using minimal prime co-ideals.

In section 3, the study of planar property of $G(R)$ breaks into some cases depending on the number of maximal co-ideals and simple co-ideals of R. In this section we prove that if $G(R)$ is planar, then $|\mathcal{I}^*(R)| \leq 7$.

In order to make this paper easier to follow, we recall various notions which will be used in the sequel. For a graph Γ by $E(\Gamma)$ and $V(\Gamma)$ we denote the set of all edges and vertices, respectively. A graph G is called connected if for any vertices x and y of G there is a path between x and y. Otherwise, G is called disconnected. The distance between two distinct vertices a and b, denoted by $d(a, b)$, is the length of the shortest path connecting them (if such a path does not exist, then $d(a, b) = \infty$, also $d(a, a) = 0$). The diameter of graph Γ, denoted by $\text{diam}(\Gamma)$, is equal to $\sup\{d(a, b) : a, b \in V(\Gamma)\}$. A graph is complete if it is connected with diameter less than or equal to one. A clique of a graph is its complete subgraph and the number of vertices in the largest clique of graph G, denoted by $\omega(G)$, is called the clique number of G. In a graph $G = (V, E)$, a set $S \subseteq V$ is an independent set if the subgraph induced by S is totally disconnected. The independence number $\alpha(G)$ is the maximum size of an independent set in G. Note that a graph whose vertices-set is empty is a null graph and a graph whose edge-set is empty is an empty graph.

A commutative semiring R is defined as an algebraic system $(R, +, .)$ such that $(R, +)$ and $(R, .)$ are commutative semigroups, connected by $a(b + c) = ab + ac$ for all $a, b, c \in R$, and there exist $0, 1 \in R$ such that $r + 0 = r$ and $r0 = 0r = 0$ and $r1 = 1r = r$ for each $r \in R$. In this paper all semirings considered will be assumed to be commutative semirings with a non-zero identity.

Definition 1. *Let R be a semiring.*

(1) A non-empty subset I of R is called co-ideal, if it is closed under multiplication and satisfies the condition $r + a \in I$ for all $a \in I$ and $r \in R$ (so $0 \in I$ if and only if $I = R$). A co-ideal I is called non-trivial provided that $I \neq \{1\}$ and $I \neq R$ ([6, 13]).

(2) A non-empty subset I of R is called strong co-ideal, if $1 \in I$ and I is a co-ideal ([13]).

(3) A co-ideal I of R is called subtractive if $x, xy \in I$, then $y \in I$ ([6]).

(4) If D is an arbitrary nonempty subset of R, then the set $F(D)$ consisting of all elements of R of the form $d_1 d_2 \ldots d_n + r$ (with $d_i \in D$ for all $1 \leq i \leq n$ and $r \in R$) is a co-ideal of R containing D ([13]).

(5) A proper co-ideal P of R is called prime if $x + y \in P$, then $x \in P$ or $y \in P$ ([6]).

(6) A semiring R is called co-semidomain, if $a + b = 1$ $(a, b \in R)$, then either $a = 1$ or $b = 1$ ([6]).

(7) A semiring R is called I-semiring, if $r + 1 = 1$ for all $r \in R$ ([7]).

(8) An identity-summand graph of a commutative semiring R, denoted by $\Gamma(R)$, is a graph whose vertices are the non-identity identity-summands of R with two distinct vertices joint by an edge when the sum of the vertices is 1 ([7]).

(9) Let R be an I-semiring. R is called semisimple, if for each proper co-ideal I of R, there exists a co-ideal J of R such that $R = F(I \cup J)$ and $I \cap J = \{1\}$ ([12]).

(10) Let R be an I-semiring and I be a co-ideal of R. I is called simple (minimal) if it has no co-ideals besides the $\{1\}$ and itself. We show the set of all simple (minimal) co-ideals of R by $Min(R)$. Moreover, $Soc(R) = F(\cup_{i \in \Lambda} I_i)$, where $Min(R) = \{I_i\}_{i \in \Lambda}$ ([12]).

(11) Let R be a semiring and I, J be co-ideals of R. Then we call J is a complement of I if $I \cap J = \{1\}$ and J is maximal with respect to this property ([12]).

Proposition 2. *(i) [7, Proposition 2.5] Let R be a commutative I-semiring. Then the following statements hold:*

(1) If $a + a = 1$ for some $a \in R$, then $a = 1$;

(2) If J is a co-ideal, then J is a subtractive co-ideal of R. Moreover, if $xy \in J$, then $x, y \in J$ for every $x, y \in R$. In particular, J is subtractive;

(3) The set $(1 : x) = \{r \in R : r + x = 1\}$ is a strong co-ideal of R for every $x \in S(R)$.

(ii) [10, Proposition 2.1] Let I be a subtractive co-ideal of a semiring R. Then the following hold:

(1) If $xy \in I$, then $x, y \in I$ for all $x, y \in R$;

(2) $(I : a) = \{r \in R : r + a \in I\}$ is a subtractive co-ideal of R for all $a \in R$.

2. Intersection graph of co-ideals of a semiring

We begin with the key definition of this paper.

Definition 3. *Let R be a semiring. The intersection graph of co-ideals of R, denoted by $G(R)$, is the graph with all elements of $\mathcal{I}^*(R) = \{I : \{1\} \neq I$ is a proper co-ideal of $R\}$ as vertices and two distinct vertices I and J are adjacent if and only if $I \cap J \neq \{1\}$.*

Remark 4. *We know that R is an I-semiring if and only if $\{1\}$ is a co-ideal of R. So if there exist co-ideals I, J of R such that $I \cap J = \{1\}$, then R is an I-semiring. Hence if R is not an I-semiring, then $G(R)$ is a complete graph. Note that if R is an I-semiring, then all co-ideals of R are strong.*

In this paper, $\mathcal{B}$ denotes the boolean semiring $\{0,1\}$, which $1+1=1$.

Theorem 5. *Let R be a semiring. Then $G(R)$ is an empty graph if and only if $\mathcal{I}^*(R) = Max(R) = \{M_1, M_2\}$ or $R \cong \mathcal{B}$ or $|\mathcal{I}^*(R)| = 1$.*

Proof. If R is not an I-semiring, then $I \cap J \neq \{1\}$ for each co-ideals I, J of R. Since $G(R)$ is empty, R contains only one co-ideal and $|\mathcal{I}^*(R)| = 1$. Suppose that R is an I-semiring. We consider two cases:

Case 1: (R, M) is a local semiring. Since for each co-ideal I of R, $I \cap M \neq \{1\}$ and $G(R)$ is an empty graph, M is the only co-ideal of R. Hence by [7, Theorem 2.8] $M = \{1\}$. Now, let $1 \neq r \in R$, so $r \notin M$. Hence there exists $s \in R$ such that $r + s = 0$. Since R is an I-semiring, $r = 0$ and $R \cong \mathcal{B}$.

Case 2: Suppose that $|Max(R)| \geq 2$. Since $G(R)$ is empty, $M_i \cap M_j = \{1\}$ for each $M_i, M_j \in Max(R)$. As $R = F(M_i \cup M_j)$, M_i and M_j are simple co-ideals of R by [12, Lemma 3.5]. We show $Max(R) = \{M_i, M_j\}$. Suppose to the contrary that $M_i, M_j \neq M_k \in Max(R)$. Hence $M_k \cap M_i = M_k \cap M_j = \{1\}$. Let $a_i \in M_i$. Then $M_j, M_k \subseteq (1 : a_i)$. Thus $M_j = M_k = (1 : a_i)$, a contradiction. Therefore $Max(R) = \{M_i, M_j\}$. Moreover, $\mathcal{I}^*(R) = Max(R)$. The converse is clear. $\square$

Example 6. *Let $X = \{a, b\}$. Then $R = (P(X), \cup, \cap)$ is a semiring with $\mathcal{I}^*(R) = Max(R) = \{\{X, \{a\}\}, \{X, \{b\}\}\}$. It can be easily seen that $G(R)$ is an empty graph.*

Theorem 7. *Let R be a semiring such that $G(R)$ is not empty. Then $G(R)$ is connected and $diam(G(R)) \leq 2$.*

Proof. Let I, J be two non adjacent vertices of $G(R)$. Then $I \cap J = \{1\}$, which implies that R is an I-semiring. Let $I \subseteq M_1$, $J \subseteq M_2$, for some maximal co-ideals M_1, M_2 of R. If $I \cap M_2 \neq \{1\}$ or $J \cap M_1 \neq \{1\}$, then $d(I, J) = 2$. Suppose that $I \cap M_2 = \{1\}$, $J \cap M_1 = \{1\}$. Hence I, J are simple co-ideals of R, by [12, Lemma 3.5]. We show $R \neq F(I \cup J)$. Suppose that $R = F(I \cup J)$. Then by [12, Lemma 3.2], $M_1 = F(I \cup (J \cap M_1)) = F(I) = I$ and $M_2 = F(J \cup (I \cap M_2)) = F(J) = J$. Since $G(R)$ is not empty, $M_1 \cap M_2 = I \cap J \neq \{1\}$, by the proof of Theorem 5, which is a contradiction. So $F(I \cup J)$ is a proper co-ideal of R and $I - F(I \cup J) - J$ is a path between I, J. So $diam(G(R)) \leq 2$. $\square$

A cycle of a graph is a path such that the start and end vertices are the same. For a graph G, it is well-known that if G contains a cycle, then $gr(G) \leq 2diam(G) + 1$.

Theorem 8. *Let R be a semiring. Then $gr(G(R)) \in \{3, \infty\}$.*

Proof. Suppose that $G(R)$ contains a cycle. We may assume that $gr(G(R)) \leq 5$. Let $I_1 - I_2 - ... - I_n - I_1$ be a cycle of minimum length in $G(R)$, where $n \in \{4, 5\}$. Since I_1, I_3 are not adjacent, $I_1 \cap I_3 = \{1\}$. If $I_1 \cap I_2 \neq I_1$ or I_2, then $I_1 - I_1 \cap I_2 - I_2 - I_1$ is a cycle in $G(R)$, which is a contradiction. So $I_1 \cap I_2 = I_1$ or I_2. Since $I_1 \cap I_3 = \{1\}$, $I_1 \cap I_2 \neq I_2$. Hence $I_1 \cap I_2 = I_1$, which gives $I_1 \subseteq I_2$. By the similar argument for $I_2 \cap I_4$, $I_2 \subseteq I_3$. Hence $I_1 \cap I_3 = I_1 \neq \{1\}$, which is a contradiction. Therefore, there must be a shorter cycle in $G(R)$ and $gr(G(R)) = 3$. $\square$

The following useful two lemma help us to prove Theorem 12 and Theorem 15.

Lemma 9. *Let R be an I-semiring and I, J and K be simple co-ideals of R. Then $F(I \cup J) \neq F(I \cup K)$.*

Proof. Suppose, on the contrary, $F(I \cup J) = F(I \cup K)$. Let $j \in J$. So there exist $i \in I$, $k \in K$ such that $j = ik + r$, for some $r \in R$. So $ik \in (J : r)$ which implies $i, k \in (J : r)$. Hence $i + r = k + r = 1$. Thus $i, k \in (1 : r)$, gives $ik + r = 1$, a contradiction. □

Lemma 10. *Let R be a semiring and M a maximal co-ideal of R. Then there is at most one co-ideal I of R such that M is not adjacent to I.*

Proof. Suppose that there are co-ideals I, J of R such that $I \cap M = J \cap M = \{1\}$. By [12, Lemma 3.5], I, J are simple co-ideals of R. Also R is an I-semiring and $R = F(I \cup M) = F(J \cup M)$. So there exist $i \in I$, $j \in J$ and $m_1, m_2 \in M$ such that $im_1 = jm_2 = 0$. So $im_1m_2 = jm_1m_2 = 0$, which gives $m_1m_2(i + j) = 0$. Since $i + j = 1$, $m_1m_2 = 0$ which is a contradiction. □

The following result is an immediate consequence of Lemma 10.

Corollary 11. *Let R be a semiring and M a maximal co-ideal of R. Then $deg(M) = |G(R)| - 1$ or $deg(M) = |G(R)| - 2$.*

Theorem 12. *Let R be a semiring. Then the following statements are equivalent:*
(i) $deg(M)$ is finite, for some maximal co-ideal M of R;
(ii) $G(R)$ is finite;
(iii) $\omega(G(R))$ is finite.

Proof. $(i) \Rightarrow (ii)$ is clear by Corollary 11.
$(ii) \Rightarrow (iii)$ and $(ii) \Rightarrow (i)$ are clear.
$(iii) \Rightarrow (ii)$ At first we show that the number of simple co-ideals of R is finite. Suppose, on the contrary, $\{I_\alpha\}_{\alpha \in \Lambda}$ is an infinite set of simple co-ideals of R. By Lemma 9, $F(I_\alpha \cup I_\beta) \neq F(I_\alpha \cup I_\eta)$, for $\alpha, \beta, \eta \in \Lambda$. Hence for minimal co-ideal I_α of R we have the infinite complete subgraph $\{F(I_\alpha \cup I_\beta)\}_{\beta \in \Lambda}$, which is a contradiction. So R contains only finite number of simple co-ideals. Since $\omega(G(R))$ is finite, each co-ideal of R contains a simple co-ideal. Now, if $G(R)$ is infinite, then there are infinite co-ideals which contain common simple co-ideal, which is a contradiction. □

Theorem 13. *Let R be a semiring. If $Max(R)$ is finite with $J(R) = \{1\}$, then $\omega(G(R)) = 2^{|Max(R)|-1} - 1$.*

Proof. Let $Max(R) = \{M_1, M_2, ..., M_n\}$, $A_i = \{M_1, ..., M_{i-1}, M_{i+1}, ..., M_n\}$ and $P(A_i)$ the power set of A_i. For each $X \in P(A_i)$, set $T_X = \cap_{T \in X} T$, then the subgraph of $G(R)$ with vertex set $\{T_X\}_{X \in P(A_i)}$ is a complete subgraph of $G(R)$. Since $|P(A_i) \setminus \{\}| = 2^{n-1} - 1$, $\omega(G(R)) \geq 2^{n-1} - 1$. We show each co-ideal of R is of the form $\cap_{i \in \Lambda} M_i$, where $\Lambda \subseteq \{1, 2, ..., n\}$. Let $K_i = \cap_{i \neq j, 1 \leq j, i \leq n} M_j$, by [12,

Lemma 3.5], $Min(R) = \{K_i\}_{1 \leq i \leq n}$. Let K be a co-ideal of R, so there exist at least two maximal co-ideals M_i, M_j of R such that $K \nsubseteq M_i, M_j$. Let $K \subseteq \cap_{i \in \Delta} M_i$, $K \nsubseteq \cup_{i \in \Delta'} M_i$, where $\Delta \subseteq \{1, 2, ..., n\}$ and $\Delta' = \{1, 2, ..., n\} \setminus \Delta$. We show $R = F(K \cup (\cap_{i \in \Delta'} M_i))$. It is clear that $R = F(K \cup M_i)$, for each $i \in \Delta'$. So for each $i \in \Delta'$, there exist $k_i \in K$, $m_i \in M_i$ such that $0 = k_i m_i$. If $\Delta = \{i_1, i_2, ..., i_t\}$, then $0 = k_{i_1} k_{i_2} ... k_{i_t} m_{i_j}$, for each $i_j \in \Delta$. So $0 = k_{i_1} k_{i_2} ... k_{i_t} (m_{i_1} + m_{i_2} + ... + m_{i_t})$. This implies $0 \in F(K \cup (\cap_{i \in \Delta'} M_i))$, as needed. By [12, Lemma 3.2], $\cap_{i \in \Delta} M_i = F(K \cup ((\cap_{i \in \Delta} M_i) \cap (\cap_{i \in \Delta'} M_i))) = K$. So $K = \cap_{i \in \Delta} M_i$ and R has $2^n - 2$ proper co-ideals. It can be easily seen that all co-ideals of R have complement. Now, let $\Sigma = \{I_1, I_2, ...\}$ be a complete subgraph of $G(R)$. We partition the co-ideals of R in parts $V_1, V_2, .., V_{2^{n-1}-1}$ such that each part contains the co-ideal I and its complement. Now if $|\Sigma| > 2^{n-1} - 1$, then at least two of the elements of Σ are in the same part, which is a contradiction. So $\omega(G(R)) = 2^{|Max(R)|-1} - 1$. $\square$

Lemma 14. *Let R be a semiring and I be a simple co-ideal of R. Then there is at most one maximal co-ideal M such that $I \cap M = \{1\}$.*

Proof. Suppose, on the contrary, M_1, M_2 are two maximal co-ideal of R such that $I \cap M_1 = I \cap M_2 = \{1\}$. So $F(I \cup M_1) = F(I \cup M_2) = R$. Hence there exist $i_1, i_2 \in I$, $m_1 \in M_1$ and $m_2 \in M_2$ such that $i_1 m_1 = i_2 m_2 = 0$. Thus $i_1 i_2 (m_1 + m_2) = 0$, which gives $R = F(I \cup (M_1 \cap M_2))$. By [12, Lemma 3.2], $M_1 = F(M_1 \cap M_2)$, which is a contradiction. $\square$

Theorem 15. *Let R be a semiring. The following statements are equivalent:*
   (i) $G(R)$ contains an end vertex;
   (ii) $\mathcal{I}^*(R) = \{I, M\}$ or $\mathcal{I}^*(R) = \{I, J, M\}$, or $\mathcal{I}^*(R) = \{I, J, M, M'\}$ where $I, J \in Min(R)$, $M, M' \in Max(R)$;

Proof. $(i) \Rightarrow (ii)$ Suppose that I is an end vertex of $G(R)$. We consider two cases:
   Case 1: I is maximal. So I is adjacent to all maximal co-ideals of R. This implies $|Max(R)| \leq 2$. If $|Max(R)| = 2$, then $I \cap M \neq \{1\}$, for some maximal co-ideal M of R, by the proof of Theorem 5. Hence I is adjacent to M and $M \cap I$, which is a contradiction. So $Max(R) = \{I\}$. It can be easily seen that $|\mathcal{I}^*(R)| = 2$, because $deg(I) = 1$.
   Case 2: I is not maximal. We consider two cases:
   `Case 2-1:` (R, M) is local. Hence M is the only co-ideal of R such that $I \cap M \neq \{1\}$. Since $deg(I) = 1$, I is simple. It can be easily seen that $|Min(R)| \leq 2$, because if $I, J, K \in Min(R)$, then I is adjacent to $F(I \cup J)$ and $F(I \cup K)$ which is a contradiction. If $|Min(R)| = 1$, then we show that $\mathcal{I}^*(R) = \{I, M\}$. Suppose that K is a co-ideal of R. If $I \subseteq K$, then $K = I$ or $K = M$, because $deg(I) = 1$. If $I \nsubseteq K$, then $Min(R) = \{I\}$ implies $L \subseteq K$, for some co-ideal L of R. Hence $F(I \cup L) = M$, which gives $K = F(L \cup (I \cap K)) = L$, a contradiction. If $|Min(R)| = 2$ and $J \in Min(R)$, then $M = F(I \cup J)$. Now, for each co-ideal K of R, it can be easily checked that $I \cap K \neq \{1\}$ or $J \cap K \neq \{1\}$. Since $deg(I) = 1$, $K \cap I = \{1\}$, So

$J \cap K \neq \{1\}$. Hence by [12, Lemma 3.2] $K = K \cap M = F(J \cup (K \cap I)) = J$. Thus $\mathcal{I}^*(R) = \{I, J, M\}$.

Case 2-2: R is not local. Hence $|Max(R)| = 2$ by Lemma 14. Let $Max(R) = \{M_1, M_2\}$ and $I \subseteq M_1$. It is clear that $I \cap M_2 = \{1\}$. It can be easily seen that $F(I \cup K) \neq R$, for each non-maximal co-ideal K of R. Since $deg(I) = 1$, $F(I \cup K) = M_1$, for each non-maximal co-ideal K. So by [12, Lemma 3.2], $M_1 \cap M_2 = K$. Thus $\mathcal{I}^*(R) = \{I, M_1 \cap M_2, M_1, M_2\}$.

$(ii) \Rightarrow (i)$ It can be easily checked that if $\mathcal{I}^*(R) = \{I, M\}$ or $\mathcal{I}^*(R) = \{I, J, M\}$, then $G(R)$ has an end vertex. Suppose $\mathcal{I}^*(R) = \{I, J, M, M'\}$ where $I, J \in Min(R)$, $M, M' \in Max(R)$. Since $G(R)$ is not empty, $M \cap M' \neq \{1\}$. Let $M \cap M' = I$ and $J \subseteq M$. So $J \not\subset M'$. Hence $J \cap M' = \{1\}$ and J is the end vertex of $G(R)$. $\qquad \square$

Theorem 16. *Let R be a semiring. If $G(R)$ is a complete r-partite graph, then at most one part has more than two vertex.*

Proof. If R is not an I-semiring, then $G(R)$ is complete and there is nothing to prove. Suppose that R is an I-semiring and $Min(R) = \{K_i\}_{i \in \Lambda}$. Since $K_i \cap K_j = \{1\}$, all simple co-ideals of R are in the same part, say V_1. We show $|Min(R)| \leq 2$. Let $K_i, K_j, K_\alpha \in Min(R)$. By the similar proof in Lemma 9, $F(K_i \cup K_j) \cap K_\alpha = \{1\}$, but $G(R)$ is complete r-partite implies $F(K_i \cup K_j) \cap K_i = \{1\}$, which is a contradiction. Hence R contains at most two simple co-ideals. We show that other parts contain only one co-ideal. Let I be a co-ideal of R which is not simple. Since $G(R)$ is complete r-partite, I contains simple co-ideal, say K_1. If there exists a simple co-ideal K_2 such that $K_2 \not\subseteq I$, then $I \cap K_2 = \{1\}$ implies $I \in V_1$, which is a contradiction. Hence all non simple co-ideals contain $Soc(R)$. So for each co-ideal I, J which are not simple $I \cap J \neq \{1\}$. Hence the only part which has more than one vertex is V_1. $\qquad \square$

As an immediate consequence of the proof of Theorem 16, we have the following corollary.

Corollary 17. *Let R be a semiring. If $G(R)$ is a complete r-partite graph, then $\mathcal{I}^*(R)$ is finite and $|\mathcal{I}^*(R)| = r$ or $r + 1$.*

Proof. Is clear by the proof of Theorem 16. $\qquad \square$

Lemma 18. *Let R be an I-semiring, $a, b \in R$ and $a \neq b$. If $a + b = 1$, then $F(\{a\}) \cap F(\{b\}) = \{1\}$ and $F(\{a\}) \neq F(\{b\})$.*

Proof. Since $a \in (1 : b)$, $a^n \in (1 : b)$, for each $n \in \mathbb{N}$. So $b \in (1 : a^n)$, which implies $b^m \in (1 : a^n)$, for each $m \in \mathbb{N}$. Hence $a^n + b^m = 1$, which gives $F(\{a\}) + F(\{b\}) = 1$. So $F(\{a\}) \cap F(\{b\}) = \{1\}$, by [12, Proposition 3.2]. $\qquad \square$

Proposition 19. *Let R be an I-semiring. Then $G(R)$ is a complete graph if and only if R is co-semidomain.*

Proof. Is clear by Lemma 18. □

In the next example, we show that the condition "R is an I-semiring" is not superficial in Proposition 19.

Example 20. *Let $R = (\mathbb{Z}^+ \times \mathbb{Z}^+, +, \cdot)$. It can be easily seen that R is not an I-semiring. Also R is not co-semidomain, because $(1, 0) + (0, 1) = (1, 1)$ but $G(R)$ is a complete graph.*

Let R be a semiring. Then $Min_p(R)$ is the set of all minimal prime co-ideals of R.

Theorem 21. *Let R be an I-semiring. If $\alpha(G(R))$ is finite, then $\alpha(G(R)) = |Min_p(R)|$.*

Proof. At first, we show that $\alpha(G(R)) = \omega(\Gamma(R))$. Let $\{I_1, I_2, ..., I_n\}$ be an independent set in $G(R)$. Let $a_i \in I_i$. Then $\{a_1, a_2, ..., a_n\}$ is a vertex set of complete subgraph in $\Gamma(R)$. So $\omega(\Gamma(R)) \geq \alpha(G(R))$. Now, let $\{a_1, a_2, ...\}$ be a clique in $\Gamma(R)$. By Lemma 18, $\{F(\{a_1\}), F(\{a_2\}), ...\}$ is an independent set in $G(R)$. So $\alpha(G(R)) \geq \omega(\Gamma(R))$. Hence $\alpha(G(R)) = \omega(\Gamma(R))$. By [7, Theorem 5.4], $\omega(\Gamma(R)) = |Min_p(R)|$, so $\alpha(G(R)) = |Min_p(R)|$. □

In the following we provide an example to show that the condition "R is an I-semiring" can not be omitted in Theorem 21.

Example 22. *Let $R = (\{r_0, r_1, r_2, r_3\}, +, *)$, where*

$$a + b = \begin{cases} r_3, & \text{if } a, b \neq r_0 \\ b, & \text{if } a = r_0 \\ a, & \text{if } b = r_0. \end{cases},$$

$$a * b = \begin{cases} a, & \text{if } b = r_1 \\ b, & \text{if } a = r_1 \\ r_0, & \text{if } b = r_0 \text{ or } a = r_0 \\ r_1, & \text{if } b = a = r_2 \\ r_3, & \text{otherwise.} \end{cases}$$

for all $a, b \in R$ and $S = (\{0, 1, 2, 3, 6\}, \gcd, \text{lcm})$ (take $\gcd(0, 0) = 0$ and $\text{lcm}(0, 0) = 0$). Let $T = S \times R$. Since T is not an I-semiring, $G(T)$ is complete, hence $\alpha(G(T)) = 1$. It can be easily seen that $J_1 = \{r_1, r_2\} \times \{1, 2, 3\}$ and $J_2 = \{r_1, r_3\} \times \{1, 2, 3\}$ are the minimal prime co-ideals of T. So $\alpha(G(T)) \neq |Min_p(T)|$.

3. PLANAR PROPERTY

We do not know whether an intersection graph of co-ideals is planar. We now state our next results to investigate the planar property of $G(R)$. At first we need some proposition to gain the main theorem (Theorem 31) of this section.

Proposition 23. *Let R be a semiring. If $G(R)$ is planar, then $|Max(R)| \leq 3$.*

Proof. Suppose, on the contrary, $M_1, M_2, M_3, M_4 \in Max(R)$. By [6, Lemma 2.7], $M_1 \cap M_2 \cap M_3 \neq \{1\}$ and $M_i \cap M_j \neq \{1\}$, for each $M_i, M_j \in Max(R)$. So the vertex set $\{M_1 \cap M_2 \cap M_3, M_1 \cap M_2, M_1, M_2, M_3\}$ makes K_5 in $G(R)$, which is a contradiction. $\square$

Proposition 24. *Let R be a semiring with $|Max(R)| = 3$. If $G(R)$ is planar, then R is semisimple with $|\mathcal{I}^*(R)| = 6$.*

Proof. Suppose that $Max(R) = \{M_1, M_2, M_3\}$. If $J(R) \neq \{1\}$, then

$$\{M_1, M_2, M_3, M_1 \cap M_2, J(R)\}$$

makes K_5 in $G(R)$, which is a contradiction. Hence $J(R) = \{1\}$, which implies R is semisimple by [12, Theorem 3.7]. So there exist simple co-ideals I_i such that $R = F(I_i \cup M_i)$, for each co-ideal $M_i \in Max(R)$. Since $F(I_1 \cup I_2 \cup I_3) \not\subset M_i$ for each $M_i \in Max(R)$, $F(I_1 \cup I_2 \cup I_3) = R$. By Lemma 10 and [12, Lemma 3.2] we have $M_i = F(I_j \cup I_k)$, for $k, j \neq i$. Now let K be a co-ideal of R which is not simple and maximal. Since $G(R)$ is planar, K contains a simple co-ideal, say I_1. It is clear that if $I_2 \subseteq K$ or $I_3 \subseteq K$, then $K = M_2$ or $K = M_3$, a contradiction. So $I_2 \cap K = I_3 \cap K = \{1\}$. Hence $K \cap F(I_2 \cup I_3) = \{1\}$, which gives K is simple by [12, Lemma 3.5], a contradiction. Thus $\mathcal{I}^*(R) = \{M_1, M_2, M_3, I_1, I_2, I_3\}$. $\square$

Example 25. *Let $X = \{a, b, c\}$ and $R = (P(X), \cup, \cap)$. It can be easily seen that $G(R)$ is a planar graph and R is an I-semiring with $|\mathcal{I}^*(R)| = 6$*

Proposition 26. *Let R be a semiring with $|Max(R)| = 2$. If $G(R)$ is a planar graph, then $|Min(R)| \leq 2$.*

Proof. Suppose, on the contrary, $I, J, K \in Min(R)$. So $F(I \cup J)$, $F(I \cup K)$ and $F(J \cup K)$ are proper co-ideals of R and $F(I \cup J) \neq F(I \cup K) \neq F(J \cup K)$, by Lemma 9. Without lose of generality, suppose that $F(I \cup J)$ and $F(I \cup K)$ contained in M_1. So $I, J, K \subseteq M_1$. Also by Lemma 10 we know there is at most one simple co-ideal which is not contained in M_2. Let $I, J \subseteq M_2$. Then $\{M_1, F(I \cup J), M_2, I, J, F(I \cup K)\}$ makes $K_{3,3}$ as a subgraph of $G(R)$, which is a contradiction. $\square$

Proposition 27. *Let R be a semiring with $|Max(R)| = 2$. If $G(R)$ is planar, then $|\mathcal{I}^*(R)| \leq 5$.*

Proof. Suppose, on the contrary, $|\mathcal{I}^*(R)| \geq 6$. By Proposition 26, $|Min(R)| \leq 2$. If $|Min(R)| = 1$, then $G(R)$ is planar implies $G(R)$ is a complete graph, which is a contradiction. Suppose that $|Min(R)| = 2$ and $Min(R) = \{I, J\}$. Let $Max(R) = \{M_1, M_2\}$. We consider two cases:

  Case 1: Let $M_1 \cap M_2$ be simple. Suppose that $M_1 \cap M_2 = I$. So either $J \not\subseteq M_1$ or $J \not\subseteq M_2$. Let $J \not\subseteq M_2$. Hence $F(J \cup M_2) = R$ and $F(I \cup J) = M_1$. Let K be another co-ideal of R. We show $J \not\subseteq K$. Because if $J \subseteq K$, then $K \not\subseteq M_2$. Hence $K \subseteq M_1$. Since K is not simple, $K \cap M_2 \neq \{1\}$. Also $K \cap M_2 = K \cap M_1 \cap M_2 =$

$K \cap I$. Since I is simple and $K \cap M_2 \neq \{1\}$, $K \cap I = I$, which gives $I \subseteq K$. Thus $K = M_1$, which is a contradiction. So $J \not\subseteq K$. Since K is not simple, K contains simple co-ideal I. We show $F(K \cup J) \neq R, M_1$. If $F(K \cup J) = M_1$, then $K \subseteq M_1$ implies $M_1 = F(K \cup (M_1 \cap M_2)) = F(K \cup I) = K$, a contradiction. If $F(K \cup J) = R$, then $K \not\subseteq M_1$. So $K \subseteq M_2$ and $F(K \cup M_1) = R$. Hence $M_2 = F(K \cup (M_1 \cap M_2)) = F(K \cup I) = K$, a contradiction. So $F(K \cup J) \neq R, M_1$. Hence $\{I, K, F(K \cup J), M_1, M_2\}$ makes K_5 in $G(R)$, which is a contradiction.

Case 2: $M_1 \cap M_2$ is not simple. So it contains simple co-ideal, say I. Let K be another co-ideal of R. We consider two cases:

Case 2-1: $J \subseteq M_1 \cap M_2$. Since K is not simple it contains a simple co-ideal. If $I \subseteq K$, then $\{I, K, M_1 \cap M_2, M_1, M_2\}$ makes K_5, which is a contradiction. If $J \subseteq K$, then $\{J, K, M_1 \cap M_2, M_1, M_2\}$ makes K_5, a contradiction. Thus $J \not\subseteq M_1 \cap M_2$.

Case 2-2: $J \not\subseteq M_1 \cap M_2$. So $J \not\subseteq M_1$ or $J \not\subseteq M_2$. Without lose of generality, suppose that $J \not\subseteq M_2$, hence $J \subseteq M_1$. Since $J \not\subseteq M_2$, $F(I \cup J) \neq M_1 \cap M_2$. Also $F(I \cup J) \neq M_1$, because if $F(I \cup J) = M_1$, then $M_1 \cap M_2 = F(I \cup (J \cap M_2)) = I$, a contradiction. Thus $\{I, M_1, M_2, F(I \cup J), M_1 \cap M_2\}$, makes K_5 in $G(R)$ which is a contradiction. $\square$

Proposition 28. *Let R be a local semiring. If $G(R)$ is planar with $|Min(R)| = 2$, then $|\mathcal{I}^*(R)| \leq 5$.*

Proof. Let $Min(R) = \{I, J\}$ and $M \in Max(R)$. If $F(I \cup J) = M$, then $\mathcal{I}^*(R) = \{I, J, M\}$ and there is nothing to prove. Suppose that $F(I \cup J) \neq M$. Let K, L be another co-ideals of R. If $I \subseteq K, L$, then $\{I, L, K, F(I \cup J), M\}$ makes K_5 in $G(R)$, a contradiction. Suppose that $J \not\subseteq K$, $I \not\subseteq L$. So $I \subseteq K$, $J \subseteq L$. It can be easily seen that $F(J \cup K) \neq F(I \cup L) \neq M$. Hence $\{I, K, F(I \cup L), F(I \cup J), M\}$ makes K_5, a contradiction. $\square$

Proposition 29. *Let R be a local semiring. If $G(R)$ is a planar graph with $|Min(R)| = 3$, then $|\mathcal{I}^*(R)| \leq 7$.*

Proof. Let $Min(R) = \{I, J, K\}$ and $M \in Max(R)$. If $F(I \cup J \cup K) \neq M$, then $\{F(I \cup J), F(I \cup K), F(I \cup J \cup K), M, I\}$ makes K_5 in $G(R)$ which is a contradiction. So $F(I \cup J \cup K) = M$. Now, let L be a co-ideal of R which contains simple co-ideal I. Then by [12, Lemma 3.2] $L = I$. By the similar way if L contains simple co-ideals I, J, then $L = F(I \cup J)$. Hence $\mathcal{I}^*(R) = \{I, J, K, F(I \cup J), F(I \cup K), F(J \cup K), M\}$. $\square$

Example 30. *Let $R = (\{0, 1, 2, 3, 5, 6, 10, 15, 30\}, gcd, lcm)$. Then R is a local semiring with simple co-ideals $I = \{1, 2\}$, $J = \{1, 3\}$ and $K = \{1, 5\}$. It can be easily seen that $G(R)$ is planar and $\mathcal{I}^*(R) = \{I, J, K, F(I \cup J), F(I \cup K), F(J \cup K), M\}$.*

Now we can conclude the following theorem from above arguments.

Theorem 31. *Let R be a semiring. If $G(R)$ is planar, then $|\mathcal{I}^*(R)| \leq 7$.*

Proof. Let $G(R)$ be planar. Then $|Max(R)| \geq 3$, by Proposition 23. By Proposition 24, if $|Max(R)| = 3$, then we have $|\mathcal{I}^*(R)| = 6$. Also, by Proposition 27, if $|Max(R)| = 2$, then $|\mathcal{I}^*(R)| \leq 5$ and if R is local, then $|\mathcal{I}^*(R)| \leq 7$, by Propositions 28, 29. Therefore we can conclude that $|\mathcal{I}^*(R)| \leq 7$, provided that $G(R)$ is planar. $\square$

References

[1] Anderson, D.F., Livingston, P.S., The zero-divisor graph of a commutative rings, *J. Algebra,* 217 (1999), 434– 447.

[2] Beck, I., Coloring of commutative rings, *J. Algebra* 116 (1988), 208–226.

[3] Bondy, J.A., Murty, U.S.R., Graph Theory, Graduate Texts in Mathematics, 244, Springer, New York, 2008.

[4] Bosak, J., The graphs of semigroups, in Theory of Graphs and its Applications, *Academic Press*, New York, 1964, pp. 119–125.

[5] Chakrabarty, I., Ghosh, S., Mukherjee, T. K., Sen, M. K., Intersection graphs of ideals of rings, *Discrete Math.* 309 (2009), 5381–5392.

[6] Ebrahimi Atani, S., Dolati Pish Hesari, S., and Khoramdel, M., Strong co-ideal theory in quotients of semirings, *J. of Advanced Research in Pure Math.*, 5(3) (2013), 19–32.

[7] Ebrahimi Atani, S., Dolati Pish Hesari, S., and Khoramdel, M., The identity-summand graph of commutative semirings, *J. Korean Math. Soc.*, 51 (2014), 189–202.

[8] Ebrahimi Atani, S., Dolati Pish Hesari, S., and Khoramdel, M., Total graph of a commutative semiring with respect to identity-summand elements, *J. Korean Math. Soc.* 51(3) (2014), 593– 607.

[9] Ebrahimi Atani, S., Dolati Pish Hesari, S., and Khoramdel, M., Total identity-summand graph of a commutative semiring with respect to a co-ideal, *J. Korean Math. Soc.* 52 (2015), No. 1, pp. 159–176.

[10] Ebrahimi Atani, S., Dolati Pish Hesari, S., and Khoramdel, M., A co-ideal based identity-summand graph of a commutative semiring, *Comment.Math.Univ.Carolin.* 56,3 (2015)269–285.

[11] Ebrahimi Atani, S., Dolati Pish Hesari, S., and Khoramdel, M., A graph associated to proper non-small ideals of a commutative ring, *Comment.Math.Univ.Carolin.* 58,1 (2017) 1–12.

[12] Ebrahimi Atani, S., Dolati Pish Hesari, S., and Khoramdel, M., Semisimple semirings with respect to co-ideals theory, *Asian-European Journal of Mathematics*, 11(3) (2018), (13 pages).

[13] Golan, J. S., Semirings and their Applications, *Kluwer Academic Publisher Dordrecht*, 1999.

[14] Csákány, B., Pollák, G., The graph of subgroups of a finite group, *Czechoslovak Math. J.* 19 (1969), 241–247.

[15] Wang, H., On rational series and rational language, *Theoret. Comput. Sci.* 205 (1998), 329–336.

Current address: Shahabaddin Ebrahimi Atani: Department of Mathematics, University of Guilan, P.O.Box 1914, Rasht, Iran.

E-mail address: `ebrahimi@guilan.ac.ir`

ORCID Address: `http://orcid.org/0000-0003-0568-9452`

Current address: Saboura Dolati Pishhesari: Department of Mathematics, University of Guilan, P.O.Box 1914, Rasht, Iran.

E-mail address: `Saboura_dolati@yahoo.com`

ORCID Address: `http://orcid.org/0000-0001-8830-636X`

Current address: Mehdi Khoramdel: Department of Mathematics, University of Guilan, P.O.Box 1914, Rasht, Iran.

E-mail address: `mehdikhoramdel@gmail.com`

ORCID Address: `http://orcid.org/0000-0003-0663-0356`

Current address: Zahra Ebrahimi Sarvandi: Department of Mathematics, University of Guilan, P.O.Box 1914, Rasht, Iran.

E-mail address: `zahra_2006_ebrahimi@yahoo.com`

ORCID Address: `http://orcid.org/0000-0002-7662-4587`