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LOCAL PRE-HAUSDORFF EXTENDED PSEUDO-QUASI-SEMI METRIC SPACES

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Abstract. In this paper, we characterize local pre-Hausdorff extended pseudo-quasi-semi metric spaces and investigate the relationships between them. Finally, we show that local pre-Hausdorff extended pseudo-quasi-semi metric spaces are hereditary and productive.

1. Introduction

In 1991, Baran [4] introduced a notion of a local pre-Hausdorff object in an arbitrary topological category which reduces to a local pre-Hausdorff topological space, where a topological space (X,τ) is called a local pre-Hausdorff space, i.e., pre-Hausdorff space at $p \in X$ if for each point x of X distinct from p, the set $\{x,p\}$ is not indiscrete, then the points x and p have disjoint neighborhoods [4]. Local pre-Hausdorff objects are used to define various forms of each of local Hausdorff objects [6], local regular objects, and local normal objects [8, 9] in arbitrary topological categories. There are other uses of pre-Hausdorff objects. In 1994, Mielke [21] showed that Pre-Hausdorff objects play a role in the general theory of geometric realizations, their associated interval and corresponding homotopy structures. Also, if X is a finite set, then it is shown, in [22], that (X,τ) is a pre-Hausdorff topological space, i.e., a pre-Hausdorff space at $p \in X$ for all point x of X, if and only if τ is a Borel field or a σ-algebra, i.e., τ is closed with respect to complements and countable unions on X [25].
In general, the category of metric spaces and non-expansive maps fails to have arbitrary infinite products and coproducts. To remedy this, there are various generalizations of metric spaces by adding or omitting or weakening conditions of metric.

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In 1988, E.Lowen and R.Lowen [19] introduced the category of extended pseudo-quasi-semi metric spaces and non-expansive maps which behave well with respect to arbitrary infinite products, coproducts, and quotients. More information about various generalizations of metric spaces can be found in [2, 3, 16, 17, 19, 23].
In this paper, we characterize each of pre-Hausdorff extended pseudo-quasi-semi metric spaces at p and investigate the relationships between them. Finally, we show that each of these pre-Hausdorff extended pseudo-quasi-semi metric spaces at p is hereditary and productive.

2. Preliminaries

Recall, [1, 13, 24] that a functor $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ is said to be topological or that $\mathcal{E}$ is a topological category over $\mathcal{B}$ if $\mathcal{U}$ is concrete (i.e., faithful and amnestic (i.e., if $\mathcal{U}(f) = id$ and f is an isomorphism, then $f = id$)), has small (i.e., sets) fibers, and for which every $\mathcal{U}$-source has an initial lift or, equivalently, for which each $\mathcal{U}$-sink has a final lift. Note that a topological functor $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ is said to be normalized if constant objects, i.e., subterminals, have a unique structure and to be geometric if its left adjoint, the discrete functor D is left exact, i.e., preserves finite limits [14, 20].

An extended pseudo-quasi-semi metric space is a pair (X, d), where X is a set $d : X \times X \to [0, \infty]$ is a function fulfills the following condition $d(x, x) = 0$ for all $x \in X$ [18, 19, 23].

Moreover, if for all $x, y \in X$, $d(x, y) = d(y, x)$, then (X, d) is called an extended pseudo-semi metric space.

A map $f : (X, d) \to (Y, e)$ between extended pseudo-quasi-semi metric spaces is said to be a non-expansive if it fulfills the property $e(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$.

The construct of extended pseudo-quasi-semi metric spaces and non-expansive maps is denoted by **pqsMet**.

2.1 A source $\{f_i : (X, d) \to (X_i, d_i), i \in I\}$ in **pqsMet** is an initial lift if and only if $d = \sup_{i\in I}(d_i \circ (f_i \times f_i))$, i.e., for all $x, y \in X, d(x, y) = \sup_{i\in I}(d_i(f_i(x), f_i(y)))$ [18, 23].

2.2. Let $(X_i, d_i), i \in I\}$ be a class of extended pseudo-quasi-semi metric spaces and X be a nonempty set. A sink $\{f_i : (X_i, d_i) \to (X, d), i \in I\}$ is final in **pqsMet** if and only if for all $x, y \in X$,
$d(x, y) = \inf_{i\in I}\{(d_i(f_i(x_i), f_i(y_i))) :$ there exist $x_i, y_i \in X_i$ such that $f_i(x_i) = x$ and $f_i(y_i) = y$ } [18, 23].

2.3. Let $\{(X_i, d_i), i \in I\}$ be a class of extended pseudo-quasi-semi metric spaces and $X = \coprod_{i\in I} X_i$. Define

$$d((k, x), (j, y)) = \begin{cases} d_k(x, y) & \text{if } k = j \\ \infty & \text{if } k \neq j \end{cases}$$

for all $(k,x),(j,y) \in X$. (X,d) is the coproduct of $\{(X_i,d_i), i \in I\}$ extended pseudo-quasi-semi metric spaces, i.e., $\{k_i : (X_i,d_i) \to (X,d), i \in I\}$ is a final lift of $\{k_i : X_i \to X, i \in I\}$, where k_i are the canonical injection maps [16].

Note that the category **pqsMet** is a Cartesian closed and hereditary topological [19].

3. Local Pre-Hausdorff Extended Pseudo-Quasi-Semi Metric Spaces

Let B be a set and $p \in B$. Let $B \vee_p B$ be the wedge at p [4], i.e., two disjoint copies of B identified at p. A point $x \in B \vee_p B$ will be denoted by x_1 (x_2) if x is in the first (respectively, the second) component of $B \vee_p B$. Note that $p_1 = p_2$. The principal p-axis map $A_p : B \vee_p B \to B^2$ is given by $A_p(x_1) = (x,p)$ and $A_p(x_2) = (p,x)$. The skewed p-axis map $S_p : B \vee_p B \to B^2$ is given by $S_p(x_1) = (x,x)$ and $S_p(x_2) = (p,x)$ [4].

Definition 1. *Let (X,τ) be a topological space and $p \in X$. (X,τ) is called pre-Hausdorff space at p, denoted by $PreT_2$ at p, if for each point x distinct from p, the set $\{x,p\}$ is not indiscrete, then the points x and p have disjoint neighborhoods* [4].

The following result is given in [7].

Theorem 2. *Let (X,τ) be a topological space and $p \in X$. The followings are equivalent:*
(1) (X,τ) is $PreT_2$ at p,
(2) The initial topology induced from $A_p : X \vee_p X \to (X^2,\tau_*)$ and $S_p : X \vee_p X \to (X^2,\tau_*)$ are the same, where τ_* is the product topology on X^2.
(3) The induced (initial) topology from $S_p : X \vee_p X \to (X^2,\tau_*)$ and the co-induced (final) topology from $i_k : (X,\tau) \to X \vee_p X, k = 1,2$ are the same, where i_1 and i_2 are the canonical quotient maps.

In view of this, Baran in [4] introduced two generalizations, denoted by $Pre\overline{T}_2$ at p and $PreT_2'$ at p, of the local pre-Hausdorff objects in an arbitrary topological category.

Definition 3. *(cf. [4]) Let $\mathcal{U} : \mathcal{E} \to$ **SET**, the category of sets and functions, be topological, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$ and $p \in \mathcal{U}(X) = B$.*
(1) If the initial lift of the $\mathcal{U}$-source $\{S_p : B \vee_p B \to \mathcal{U}(X^2) = B^2\}$ and the final lift of the $\mathcal{U}$-sink $\{i_1, i_2 : \mathcal{U}(X) = B \to B \vee_p B\}$ coincide, then X is called a $PreT_2'$ object at p.
(2) If the initial lift of the $\mathcal{U}$-sources $\{S_p : B \vee_p B \to \mathcal{U}(X^2) = B^2\}$ and $\{A_p : B \vee_p B \to \mathcal{U}(X^2) = B^2\}$ coincide, then X is called a $Pre\overline{T}_2$ object at p.

Theorem 4. *An extended pseudo-quasi-semi metric space (X,d) is $Pre\overline{T}_2$ at p if and only if the following conditions are satisfied.*

(1) For all $x \in X$ with $x \neq p$, $d(x,p) = d(p,x)$.
(2) For any two distinct points x, y of X with $x \neq p \neq y$, we have either $d(x,p) = d(p,y) \geq d(x,y), d(y,x)$ or $d(p,y) = d(x,y) = d(y,x) \geq d(x,p)$ or $d(x,p) = d(x,y) = d(y,x) \geq d(p,y)$.

Proof. Suppose that (X,d) is $Pre\overline{T}_2$ at p and $x \in X$ with $x \neq p$.
Let $\pi_k : X^2 \to X$, $k = 1,2$ be the projection maps. Note that
$d(\pi_1 A_p(x_1), \pi_1 A_p(x_2)) = d(x,p) = d(\pi_1 S_p(x_1), \pi_1 S_p(x_2))$,
$d(\pi_2 A_p(x_1), \pi_2 A_p(x_2)) = d(p,x)\}$ and
$d(\pi_2 S_p(x_1), \pi_2 S_p(x_2)) = d(x,x) = 0$.
$sup\{d(\pi_k A_p(x_1), \pi_k A_p(x_2)) : k = 1,2\} = sup\{d(x,p), d(p,y)\}$
and $sup\{d(\pi_k S_p(x_1), \pi_k S_p(x_2)) : k = 1,2\} = d(x,p)$.
Since (X,d) is $Pre\overline{T}_2$ at p and $x_1 \neq x_2$, by 2.1 and Definition 3,
$sup\{d(x,p), d(p,y)\} = sup\{d(\pi_1 A_p(x_1), \pi_1 A_p(x_2)), d(\pi_2 A_p(x_1), \pi_2 A_p(x_2))\} =$
$sup\{d(\pi_1 S_p(x_1), \pi_1 S_p(x_2)), d(\pi_2 S_p(x_1), \pi_2 S_p(x_2))\} = d(x,p)$
and consequently, $d(x,p) = d(p,x)$.

Suppose x, y are any two distinct points of X with $x \neq p \neq y$. Since for all $x, y \in X$ with $x \neq p \neq y$, $d(x,p) = d(p,x)$ and $d(y,p) = d(p,y)$, it follows that $sup\{d(\pi_k A_p(u), \pi_k A_p(v)) : k = 1,2\} = sup\{d(x,p), d(p,y)\}$, where $u = x_i$ and $v = y_j$ or $u = y_j$ and $v = x_i$ for $i, j = 1,2$ and $i \neq j$ and
$sup\{d(\pi_k S_p(x_1), \pi_k S_p(y_2)) : k = 1,2\} = sup\{d(x,p), d(x,y)\}$,
$sup\{d(\pi_k S_p(x_2), \pi_k S_p(y_1)) : k = 1,2\} = sup\{d(p,y), d(x,y)\}$,
$sup\{d(\pi_k S_p(y_1), \pi_k S_p(x_2)) : k = 1,2\} = sup\{d(p,y), d(y,x)\}$,
and
$sup\{d(\pi_k S_p(y_2), \pi_k S_p(x_1)) : k = 1,2\} = sup\{d(x,p), d(y,x)\}$.

Since (X,d) is $Pre\overline{T}_2$ at p and $u \neq v$, where $u = x_i$ and $v = y_j$ or $u = y_j$ and $v = x_i$ for $x, y \in X$ and $i, j = 1,2, i \neq j$, by 2.1 and Definition 3,
$sup\{d(\pi_k A_p(u), \pi_k A_p(v)) : k = 1,2\} = sup\{d(\pi_k S_p(u), \pi_k S_p(v)) : k = 1,2\}$ and consequently, we have
$sup\{d(x,p), d(p,y)\} = sup\{d(x,p), d(y,x)\} = sup\{d(x,p), d(x,y)\}$
$= sup\{d(p,y), d(x,y)\} = sup\{d(p,y), d(y,x)\}$.
Suppose that $sup\{d(x,p), d(p,y)\} = d(x,p)$. $d(x,p) = sup\{d(p,y), d(x,y)\} = sup\{d(p,y), d(y,x)\}$ implies $d(x,p) = d(p,y) \geq d(x,y), d(y,x)$ or $d(x,p) = d(x,y) = d(y,x) \geq d(p,y)$. Suppose that $sup\{d(x,p), d(p,y)\} = d(p,y)$. Note that $d(p,y) \geq d(x,p)$ and $d(p,y) = sup\{d(x,p), d(x,y)\} = sup\{d(x,p), d(y,x)\}$ implies $d(p,y) = d(x,y) = d(y,x) \geq d(x,p)$.

Conversely, suppose that the conditions hold. We need to show that (X,d) is $Pre\overline{T}_2$ at p. Let d_{A_p} and d_{S_p} be the extended pseudo-quasi-semi metric structures on $X \vee_p X$ induced by $A_p : X \vee_p X \to (X^2, d^2)$ and $S_p : X \vee_p X \to (X^2, d^2)$, respectively, where d^2 is the product extended pseudo-quasi-semi metric structure on X^2. By 2.1 and Definition 3, we need to show that for any points u and v in $X \vee_p X$, $d_{A_p}(u,v) = d_{S_p}(u,v)$.
If $u = v$, then $d_{A_p}(u,u) = 0 = d_{S_p}(u,u)$.

Suppose that $u \neq v$ and they are in the same component of the wedge $X \vee_p X$. If $u = x_k$ and $v = y_k$ for $x, y \in X$ and $k = 1, 2$, then, by 2.1, $d_{A_p}(u, v) = sup\{d(\pi_i A_p(u), \pi_i A_p(v)), i = 1, 2\} = sup\{d(x, y), d(p, p) = 0\} = d(x, y)$ and $d_{S_p}(u, v) = sup\{d(\pi_i S_p(u), \pi_i S_p(v)), i = 1, 2\} = d(x, y)$.
Thus, $d_{A_p}(u, v) = d_{S_p}(u, v)$.

Suppose $u \neq v$ and they are in the different component of the wedge $X \vee_p X$. If $u = x_1$ and $v = x_2$ for $x \in X$ with $x \neq p$, then $d_{A_p}(u, v) = sup\{d(\pi_k A_p(u), \pi_k A_p(v)), k = 1, 2\} = sup\{d(x, p), d(p, x)\}$ and
$d_{S_p}(u, v) = sup\{d(\pi_k S_p(u), \pi_k S_p(v)), k = 1, 2\} = sup\{d(x, p), d(x, x) = 0\} = d(x, p)$. Since $x \neq p$, by the assumption (1), $d(x, p) = d(p, x)$ and consequently, $d_{A_p}(u, v) = d_{S_p}(u, v)$.
If $u = x_2$ and $v = x_1$ for $x \in X$ with $x \neq p$, then
$d_{A_p}(u, v) = sup\{d(\pi_k A_p(u), \pi_k A_p(v)), k = 1, 2\}$
$= sup\{d(p, x), d(x, p)\}$ and
$d_{S_p}(u, v) = sup\{d(\pi_k S_p(u), \pi_k S_p(v)), k = 1, 2\}$
$= sup\{d(p, x), d(x, x) = 0\} = d(p, x)$. It follows from the assumption (1) that $d_{A_p}(u, v) = d_{S_p}(u, v)$.
If $u = x_i$ and $v = y_j$ or $u = y_j$ and $v = x_i$ for distinct points x, y of X with $x \neq p \neq y$ and $i, j = 1, 2, i \neq j$, then $d_{A_p}(u, v) = sup\{d(\pi_k A_p(u), \pi_k A_p(v)), k = 1, 2\} = sup\{d(x, p), d(p, y)\}$. If $u = x_1$ and $v = y_2$ (resp. $u = x_2$ and $v = y_1$ or $u = y_1$ and $v = x_2$ or $u = y_2$ and $v = x_1$), then $d_{S_p}(u, v) = sup\{d(\pi_k S_p(u), \pi_k S_p(v)), k = 1, 2\} = sup\{d(x, p), d(x, y)\}$ (resp. $d_{S_p}(u, v) = sup\{d(\pi_k S_p(u), \pi_k S_p(v)), k = 1, 2\} = sup\{d(p, y), d(x, y)\}$ or $d_{S_p}(u, v) = sup\{d(\pi_k S_p(u), \pi_k S_p(v)), k = 1, 2\} = sup\{d(y, p), d(y, x)\}$ or $d_{S_p}(u, v) = sup\{d(\pi_k S_p(u), \pi_k S_p(v)), k = 1, 2\} = sup\{d(p, x), d(y, x)\}$). By the assumption (1), $d(x, p) = d(p, x)$ and $d(y, p) = d(p, y)$ for all $x, y \in X$ with $x \neq p \neq y$. By the assumption (2), if $d(x, p) = d(p, y) \geq d(x, y), d(y, x)$, then $d_{A_p}(u, v) = sup\{d(\pi_k A_p(u), \pi_k A_p(v)), k = 1, 2\} = sup\{d(x, p), d(p, y)\} = d(x, p) = d(p, y) = sup\{d(x, p), d(x, y)\} = sup\{d(p, y), d(x, y)\}$ $= sup\{d(y, p), d(y, x)\} = sup\{d(p, x), d(y, x)\} = d_{S_p}(u, v)$.

Similarly, if $d(p, y) = d(x, y) = d(y, x) \geq d(x, p$ or $d(x, p) = d(x, y) = d(y, x) \geq d(p, y$, then, by the assumptions (1) and (2), we get $d_{A_p}(u, v) = d_{S_p}(u, v)$.

Hence, for any points u and v in $X \vee_p X$, we have $d_{A_p}(u, v) = d_{S_p}(u, v)$ and by 2.1 and Definition 3, (X, d) is $Pre\overline{T}_2$ at p.

□

Theorem 5. *An extended pseudo-quasi-semi metric space (X, d) is $PreT_2'$ at p if and only if for all $x \in X$ with $x \neq p$, $d(x, p) = \infty$ and $d(p, x) = \infty$.*

Proof. Suppose that (X, d) is $PreT_2'$ at p and $x \in X$ with $x \neq p$. Let $\pi_i : X^2 \to X$, $i = 1, 2$ be the projection maps and d_1 be the final structure on $X \vee_p X$ induced from the canonical maps $i_1, i_2 : (X, d) \to X \vee_p X$. Note that $sup\{d(\pi_1 S_p(x_2), \pi_1 S_p(x_1)) = d(p, x), d(\pi_2 S_p(x_2), \pi_2 S_p(x_1)) = d(x, x) = 0\} = d(p, x)$ and since $x_1 \neq x_2$ and they are in the different component of the wedge, it follows from 2.2 and 2.3 that $d_1(x_2, x_1) = \infty$. Since (X, d) is $PreT_2'$ at p, by Definition 3, $d(p, x) = sup\{d(\pi_k S_p(x_2), \pi_k S_p(x_1)) : k = 1, 2\} = d_1(x_2, x_1) = \infty$ which shows $d(p, x) = \infty$.

Note that $sup\{d(\pi_1 S_p(x_1), \pi_1 S_p(x_2)) = d(x, p), d(\pi_2 S_p(x_1), \pi_2 S_p(x_2)) = d(x, x) = 0\} = d(x, p)$ and $d_1(x_1, x_2) = \infty$ since $x_1 \neq x_2$ and they are in the different component of the wedge. Since (X, d) is $PreT_2'$ at p, by Definition 3, $d(x, p) = sup\{d(\pi_k S_p(x_1), \pi_k S_p(x_2)) : k = 1, 2\} = d_1(x_1, x_2) = \infty$. Thus, $d(p, x) = \infty$.

Conversely, suppose that $d(x, p) = \infty$ and $d(p, x) = \infty$ for all $x \in X$ with $x \neq p$. Let d_1 and d_{S_p} be the final structure on $X \vee_p X$ induced by $i_1, i_2 : (X, d) \to X \vee_p X$ and the initial structure on $X \vee_p X$ induced by $S_p : X \vee_p X \to (X^2, d^2)$, respectively, where d^2 is the product extended pseudo-quasi-semi metric structure on X^2. We show that (X, d) is $PreT_2'$ at p, i.e., by 2.1, 2.2, and Definition 3, $d_1 = d_{S_p}$.

Let u and v be any points in $X \vee_p X$. If $u = v$, then $d_1(u, u) = 0 = d_{S_p}(u, u)$.

Suppose that $u \neq v$ and they are in the same component of the wedge. If $u = x_k$ and $v = y_k$ for $x, y \in X$ and $k = 1, 2$, then, by 2.1, $d_{S_p}(u, v) =$
$sup\{d(\pi_k S_p(u), \pi_k S_p(v)), k = 1, 2\} = sup\{d(p, p) = 0, d(x, y)\} = d(x, y)$ and $d_1(u, v) = inf\{d(x, y) : i_k(x) = x_k, i_k(y) = y_k : k = 1, 2\} = d(x, y)$. Hence, $d_{S_p}(u, v) = d(x, y) = d_1(u, v)$.

Suppose that $u \neq v$ and they are in the different component of the wedge $X \vee_p X$. If $u = x_1$ and $v = y_2$ for distinct points $x, y \in X$ with $x \neq p \neq y$, then, by 2.1 and the assumption $d(x, p) = \infty$, $d_{S_p}(u, v) = sup\{d(\pi_k S_p(u), \pi_k S_p(v)), k = 1, 2\} = sup\{d(x, p), d(x, y)\}$
$= sup\{\infty, d(x, y)\} = \infty$. By 2.2 and 2.3, $d_1(u, v) = \infty$ since $u \neq v$ and they are in the different component of the wedge. Thus, $d_{S_p}(u, v) = \infty = d_1(u, v)$.

If $u = x_2$ and $v = y_1$ for distinct points $x, y \in X$ with $x \neq p \neq y$, then, by 2.1, $d_{S_p}(u, v) = sup\{d(\pi_k S_p(u), \pi_k S_p(v)), k = 1, 2\} = sup\{d(p, y), d(x, y)\} = sup\{\infty, d(x, y)\} = \infty$ since $y \neq p$ and $d(p, y) = \infty$. Note, by 2.2 and 2.3, that $d_1(u, v) = \infty$ since $u \neq v$ and they are in the different component of the wedge. Thus, $d_{S_p}(u, v) = \infty = d_1(u, v)$.

Therefore, for any points u and v in $X \vee_p X$, we have $d_1(u, v) = d_{S_p}(u, v)$ and by Definition 3, (X, d) is $PreT_2'$ at p.

□

Theorem 6. *Let (X, d) be an extended pseudo-quasi-semi metric space, $A \subset X$ and $p \in A$.*
(1) If (X, d) is $Pre\overline{T}_2$ at p, then (A, d_A) is also $Pre\overline{T}_2$ at p.
(2) If (X, d) is $PreT_2'$ at p, then (A, d_A) is also $PreT_2'$ at p.

Proof. Let $f : A \hookrightarrow X$ be the inclusion map defined by $f(x) = x$ for $x \in A$ and d_A be the initial lift of $f : A \hookrightarrow (X, d)$.
(1) Suppose that (X, d) is $Pre\overline{T}_2$ at p and $x \in A$ with $x \neq p$. By 2.1 and Theorem 4, $d_A(x, p) = d(x, p) = d(p, x) = d_A(p, x)$.

Let x, y be any two distinct points of A with $x \neq p \neq y$. Since $A \subset X$ and (X, d) is $Pre\overline{T}_2$ at p, by Theorem 4, we have either $d(x,p) = d(p,y) \geq d(x,y), d(y,x)$ or $d(p,y) = d(x,y) = d(y,x) \geq d(x,p)$
or $d(x,p) = d(x,y) = d(y,x) \geq d(p,y)$.
By 2.1, $d_A(x,p) = d(x,p), d_A(p,y) = d(p,y), d_A(x,y) = d(x,y)$, and $d_A(y,x) = d(y,x)$. It follows that we have either $d_A(x,p) = d_A(p,y) \geq d_A(x,y), d_A(y,x)$ or $d_A(p,y) = d_A(x,y) = d_A(y,x) \geq d_A(x,p)$ or $d_A(x,p) = d_A(x,y) = d_A(y,x) \geq d_A(p,y)$.
Hence, by Theorem 4, (A, d_A) is $Pre\overline{T}_2$ at p.
The proof of (2) is similar to the proof of (1) by using Theorem 5.

□

Theorem 7. *Let (X_i, d_i) be extended pseudo-quasi-semi metric spaces, $X = \prod_{i\in I} X_i$ and $p = (p_1, p_2, p_3, ...)$, where $p_i \in X_i, i \in I$. The product space (X, d) is $Pre\overline{T}_2$ at p (resp. $PreT_2'$ at p) if and only if each (X_i, d_i) is $Pre\overline{T}_2$ at p_i (resp. $PreT_2'$ at p_i) .*

Proof. Suppose (X, d) is $Pre\overline{T}_2$ at p (resp. $PreT_2'$ at p). It is easy to see that for each $i \in I$, (X_i, d_i) is isomorphic to some slice in (X, d). Since (X, d) is $Pre\overline{T}_2$ at p (resp. $PreT_2'$ at p), it follows from Theorems 4-6 that for each $i \in I$, (X_i, d_i) is $Pre\overline{T}_2$ at p_i (resp. $PreT_2'$ at p_i).
Suppose that for all $i \in I$, (X_i, d_i) are $Pre\overline{T}_2$ at $p_i \in X_i$. Since **pqsMet** is a normalized topological category, by Theorem 2.6 of [5] and Theorem 3.1 of [10], the product space (X, d) is $Pre\overline{T}_2$ at p.
We show that (X, d) is $PreT_2'$ at p. Suppose that for all $i \in I$, (X_i, d_i) are $PreT_2'$ at p_i and $x = (x_1, x_2, x_3, ...) \in X$ with $x \neq p = (p_1, p_2, p_3, ...)$. It follows that there exists $j \in I$ such that $x_j \neq p_j$. Since (X_j, d_j) is $PreT_2'$ at p_j, by Theorem 5, we have $d_j(x_j, p_j) = \infty$ and $d_j(p_j, x_j) = \infty$.
If $d_j(x_j, p_j) = \infty$, then $d(x,p) = \sup\limits_{i\in I}(d_i(\pi_i(x), \pi_i(p)))$
$= \sup\ \{d_1(x_1,p_1), \{d_2(x_2,p_2), ..., d_{j-1}(x_{j-1}, p_{j-1}), \infty, d_{j+1}(x_{j+1}, p_{j+1}), ...\} = \infty$.
If $d_j(p_j, x_j) = \infty$, then $d(p,x) = \sup\limits_{i\in I}(d_i(\pi_i(p), \pi_i(x)))$
$= \sup\ \{d_1(p_1,x_1), \{d_2(p_2,x_2), ..., d_{j-1}(p_{j-1}, x_{j-1}), \infty, d_{j+1}(p_{j+1}, x_{j+1}), ...\} = \infty$.
Hence, (X, d) is $PreT_2'$ at p.

□

Example 8. *(1) The discrete extended pseudo-quasi-semi metric structure d_{dis} on X is given by*

$$d_{dis}(a,b) = \begin{cases} 0 & \text{if } a = b \\ \infty & \text{if } a \neq b \end{cases}$$

for all $a, b \in X$. By Theorems 4 and 5, (X, d_{dis}) is both $Pre\overline{T}_2$ at p and $PreT_2'$ at p for all $p \in X$.
(2) The indiscrete extended pseudo-quasi-semi metric structure d on X with $|X| \geq 2$

is given by $d(a,b) = 0$ *for all* $a, b \in X$ [18].
By Theorems 4 and 5, (X, d) *is* $Pre\overline{T}_2$ *at* p *for all* $p \in X$ *but, By Theorem 5,* (X, d) *is not* $PreT'_2$ *at* p.
(3) Let $X = \{x, y\}$ *and* $d(x, y) = 2, d(y, x) = \infty$, $d(x, x) = 0 = d(y, y)$. *By Theorems 4 and 5, the space* (X, d) *is neither* $Pre\overline{T}_2$ *nor* $PreT'_2$ *at* x *and* y.

Remark 9. *(1) For an arbitrary topological category* $\mathcal{E}$ *with* B *an object in* $\mathcal{E}$ *, the constant map at* p, $p : B \to B$ *is called a retract map if there exists a map* $r : B \to B$ *such that the composition* $rp = id$*, the identity map on* B [5]. *If* $p : B \to B$ *is a retract map, then by Theorem 2.6 of* [5] *and Theorem 3.1 of* [8], $PreT'_2$ *at* p *implies* $Pre\overline{T}_2$ *at* p *but the reverse implication is not true, in general.*

If an extended pseudo-quasi-semi metric space (X, d) *is* $PreT'_2$ *at* p*, then, by Theorems 4 and 5,* (X, d) *is* $Pre\overline{T}_2$ *at* p *but, by Example 8(2), the reverse of implication is not true.*

$Pre\overline{T}_2$ *at* p *and* $PreT'_2$ *at* p *could be equivalent. For example, for the category* **Top** *of topological spaces, by Theorem 2 as well as for the category* **Preord** *of preordered (sets with reflexive and transitive relations on them) sets and monotone (relation preserving) maps, by Theorems 6.3 and 6.4 of* [11], $Pre\overline{T}_2$ *at* p *and* $PreT'_2$ *at* p *are equivalent.*
Note, also, that all objects of a set-based arbitrary topological category may be $Pre\overline{T}_2$ *at* p*. For example, it is shown, in* [15]*, that all Cauchy spaces* [12] *are* $Pre\overline{T}_2$ *at* p.
(2) Local pre-Hausdorff objects (i.e., $Pre\overline{T_2}$ *at* p *and* $PreT'_2$ *at* p*) are used to define each of local Hausdorff objects, local regular objects, and local normal objects in arbitrary topological categories* [5, 9].

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