



APPROXIMATE CONTROLLABILITY OF SECOND-ORDER STOCHASTIC NON-AUTONOMOUS INTEGRODIFFERENTIAL INCLUSIONS BY RESOLVENT OPERATORS

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ABSTRACT. In this paper, we formulate a set of sufficient conditions for the approximate controllability for a class of second-order stochastic non-autonomous integrodifferential inclusions in Hilbert space. We establish the results with the help of resolvent operators and Bohnenblust-Karlin's fixed point theorem is to prove the main result. An application is given to illustrate the main result.

1. INTRODUCTION

Differential inclusions have wide applications in engineering, economics and in optimal control theory. Many authors studied the existence, controllability and stability of differential inclusions [3, 7, 8, 25, 34, 35, 36]. The idea of controllability is of immense influence in mathematical control theory, which plays a vital role in both engineering and sciences. Controllability generally means that it is possible to steer dynamical control systems from an arbitrary initial state to an arbitrary final state using the set of admissible controls. There are two basic theories of controllability can be identified as approximate controllability and exact controllability. Most of the criteria, which can be met in the literature are formulated for finite dimensional system. But in the infinite dimensional system, many unsolved problems are still exist as for as controllability is concerned. In the case of infinite dimensional system, controllability can be distinguished as approximate and exact controllability. Approximate controllability means that the system can be governed to arbitrary small neighborhood of final state whereas exact controllability allows to govern the system to arbitrary final state. In otherwords the approximate controllability gives the possibility of governing the system of states which forms a dense subspace in the

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state space. Many authors studied the existence, controllability and approximate controllability of the stochastic differential equations and inclusions in the first and second order [8, 9, 13, 19, 21, 23, 24, 25, 28, 29, 30, 31, 33, 37, 38].

Stochastic Differential Equations(SDE's) have been broadly applied in various areas of applied sciences. A common development of a deterministic model of differential equation is a structure of stochastic differential equation, where appropriate parameters are modeled as applicable stochastic process. This is because of the proof that problems in a real life situations are fundamentally modeled by stochastic systems instead deterministic systems. SDE's naturally refer to the time dynamics of the evolution of a state vector, based on the (approximate) physics of the real system, together with a driving noise process. The noise process can be assumed in several ways. Random differential and evolution systems play a crucial role in characterizing many social, physical, biological, medical and engineering problems [6, 10, 26, 27, 32] and references therein.

In the past years, the authors [1, 2, 4, 12, 16, 17], investigated the existence of abstract second order initial value problem for the non-autonomous system. In [14, 15] Grimmer.et.al, studied the analytic resolvent operators for the integral equations in Banach space and he also studied the resolvent operator for integral equations in Banach space. In [18] Henríquez.et.al studied the existence of solutions of a second order abstract functional cauchy problem with nonlocal conditions and in [20] he also studied the existence of solutions of non-autonomous abstract cauchy problem of second order integrodifferential equations by the use of resolvent operators instead of cosine family of operators. In [16] Henríquez, investigated the existence of solutions of non-autonomous second order functional differential equations with infinite delay by using Leray Schauder alternative fixed point theorem. The approximate controllability of second order stochastic non autonomous integrodifferential inclusions by resolvent operators have not studied yet. Motivated by the above facts, we establish the sufficient conditions for the approximate controllability of second order stochastic integrodifferential inclusions by resolvent operators by using Bohenblust-Karlin's fixed point theorem.

In this paper, we establish the set of sufficient conditions for the approximate controllability for a non-autonomous second order stochastic integrodifferential inclusions in Hilbert space of the form,

$$\frac{d}{dt}[x'(t)] \in A(t)x(t) + \int_0^t G(t,s)x(s)ds + Bu(t) + \Sigma(t,x(t))dw(t), \quad t \in J := [0, b], \quad (1.1)$$

$$x(0) = y, \quad x'(0) = z \quad (1.2)$$

where $A(t) : D(A(t)) \subseteq X \rightarrow X$ is a closed linear operator, $G(t,s) : D(G) \subseteq X \rightarrow X$ is a linear operator and the control function the state $x(\cdot)$ takes the values in the separable real Hilbert space X ; Further $\Sigma : J \times X \rightarrow \mathcal{L}_2^0(\mathbb{K}, X)$ is nonempty, bounded, closed and convex multivalued map and the control function $u(\cdot)$ is given

in $\mathcal{L}(J, U)$, a Hilbert space of admissible control functions with U as Hilbert space. B is a bounded linear operator from U into X .

The paper is organised as follows: In section 2, we present some basic notations and preliminaries and in section 3, we studied the approximate controllability results by resolvent operator and in section 4, an application is provide to illustrate the main results.

2. PRELIMINARIES

In this section, the basic preliminaries, definitions, lemmas, notations, multi-valued maps and some results which are needed to establish our main results are discussed.

Let $(X, \|\cdot\|_X)$ and $(\mathbb{K}, \|\cdot\|_{\mathbb{K}})$ be two real separable Hilbert spaces and for convenience, we use the same notation $\|\cdot\|$ to denote the norms in X, \mathbb{K} and $\langle \cdot, \cdot \rangle$ to denote the inner product space without any confusion. Let $\mathcal{L}(\mathbb{K}, X)$ be space of bounded linear operators from \mathbb{K} into X . Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space satisfying the usual conditions, that is the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is a right continuous increasing family and \mathcal{F}_0 contains all \mathbb{P} -null sets. Let $\{w(t) : t \geq 0\}$ be a cylindrical \mathbb{K} -valued Wiener process defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with finite trace nuclear covariance operator $Q \geq 0$ such that $Tr(Q) < \infty$. Further, we assume that there exists a complete orthonormal basis $\{e_k\}_{k \geq 1}$ in \mathbb{K} , a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k, k = 1, 2, \dots$ and sequence of independent Wiener processes. We assume that $\mathcal{F}_t = \sigma \{w(s) : 0 \leq s \leq t\}$ is the σ -algebra generated by w and $\mathcal{F}_b = \mathcal{F}$. For $\varphi \in \mathcal{L}(\mathbb{K}, X)$, define

$$\|\varphi\|_Q^2 = Tr(\varphi Q \varphi^*) = \sum_{n=1}^{\infty} \left\| \sqrt{\lambda_n} \varphi e_n \right\|^2$$

If $\|\varphi\|_Q^2 = Tr(\varphi Q \varphi^*) < \infty$, then φ is called a Q -Hilbert-Schmidt operator. Let $\mathcal{L}_Q(\mathbb{K}, X)$ denotes the space of all Q -Hilbert Schmidt operators $\varphi : \mathbb{K} \rightarrow X$. The completion $\mathcal{L}_Q(\mathbb{K}, X)$ of $\mathcal{L}(\mathbb{K}, X)$ with respect to the topology induced by the norm $\|\cdot\|_Q$, where $\|\varphi\|_Q^2 = \langle \varphi, \varphi \rangle$ is a Hilbert space with the above norm topology. The collection of all strongly measurable, square integrable, X -valued random variables denoted by $\mathcal{L}_2(\Omega, X)$ is a Banach space equipped with the norm $\|\cdot\|_{\mathcal{L}_2} = \left(E \|x(\cdot, w)\|_X^2 \right)^{\frac{1}{2}}$.

Let $C(J, \mathcal{L}_2(\Omega, X))$ be the Banach space of all continuous functions from J into $\mathcal{L}_2(\Omega, X)$, satisfying $\sup_{0 \leq t \leq b} E \|x(t)\|_X^2 < \infty, \mathcal{L}_2^0(\Omega, X)$ denotes the family of all \mathcal{F}_0 -measurable, X -valued random variables.

The resolvent set of linear operators A is given by $\rho(A)$. Also, we represent by $[D(A)]$ the domain of A endowed with the graph norm $E \|A\|_A^2 = E \|x\|^2 + \sup_{0 \leq s \leq t \leq b} \|A(t)S(t, s)z\|^2, z \in Z$. Here $(Z, \|\cdot\|_Z)$ is a Hilbert space continuously

included in X and $S(t, s) \in \mathcal{L}(Z, [D])$. To get our results, we consider the abstract second order integrodifferential problem

$$x''(t) \in A(t)x(t) + \int_0^t G(t, s)x(s)ds + Bu(t), \nu \leq t \leq b, \tag{2.1}$$

$$x(\nu) = 0, \quad x'(\nu) = z \in X \tag{2.2}$$

for $0 \leq \nu \leq b$, has an associated resolvent operator of bounded linear operators $\mathcal{R}(t)$ on X which was proved in [20].

Definition 2.1. A family of bounded linear operators $\mathcal{R}(t)$ on X is called a resolvent operator of (1.1)-(1.2) if the following conditions are satisfied.

a) The operator $\mathcal{R}: \Delta = \{(t, s) : 0 \leq s \leq t \leq b\} \rightarrow \mathcal{L}(X)$ is strongly continuous, $\mathcal{R}(t, \cdot)z$ is continuously differentiable for all $z \in X$. We denote by the positive constants M and \widetilde{M} such that

$$\|\mathcal{R}(t, s)\| \leq M, \quad \left\| \frac{\partial}{\partial s} \mathcal{R}(t, s) \right\| \leq \widetilde{M}, \quad (t, s) \in \Delta. \tag{2.3}$$

b) Let $z \in D$. Since $\mathcal{R}(\cdot, \beta)z$ is a solution of the problem (1.1)-(1.2), we have

$$\frac{\partial^2}{\partial t^2} \mathcal{R}(t, \beta)z = A(t)\mathcal{R}(t, \beta)z + \int_\sigma^t G(t, \xi)R(\eta, \beta)z d\xi. \tag{2.4}$$

Now we consider the abstract second-order Cauchy problem

$$x''(t) \in A(t)x(t) + \int_0^t Q(t, s)x(s)ds, \quad t \in I = [0, b] \tag{2.5}$$

$$x(0) = y, \quad x'(0) = z \tag{2.6}$$

Definition 2.2. Under the above conditions, let $y, z \in X$ and then the mild solution $x(\cdot)$ of the problem (2.5)-(2.6) is given by

$$x(t) = -\frac{\partial \mathcal{R}(t, 0)}{\partial s} y + \mathcal{R}(t, 0)z \tag{2.7}$$

Definition 2.3. A multivalued map $G : X \rightarrow 2^X \setminus \{\emptyset\}$ is convex(closed) valued if $G(x)$ is convex(closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \bigcup_{x \in B} G(x)$ is bounded in X for any bounded set B of X i.e.,

$$\sup_{x \in B} \{ \sup \{ \|y\| : y \in G(x) \} \} < \infty.$$

Definition 2.4. G is called upper semicontinuous (u.s.c for short) on X , if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood V of x_0 such that $G(V) \subseteq N$.

Definition 2.5. The multi-valued operator G is called compact if $\overline{G(X)}$ is a compact subset of X . G is called completely continuous if $G(B)$ is relatively compact for every bounded subset B of X .

For more details on Multivalued maps, see the books of Deimling in [11], Hu and Papageorgiou in [22].

If the multivalued map G is completely continuous with nonempty values, then G is u.s.c., if and only if G has a closed graph, i.e., $x_n \rightarrow x_*$, $y_n \rightarrow y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is a $x \in X$ such that $x \in G(x)$. In the following, $BCC(X)$ denotes the set of all nonempty, bounded, closed and convex subset of X .

Definition 2.6. A multivalued map $G : J \rightarrow BCC(X)$ is said to be measurable if, for each $x \in X$, the function $v : J \rightarrow \mathbb{R}$, defined by $v(t) = d(x, Gx(t)) = \inf \{\|x - z\| : z \in G(t)\}$ belongs to $\mathcal{L}^1(J, \mathbb{R})$.

Definition 2.7. The multivalued map $\Sigma : J \times X \rightarrow BCC(X)$ is said to be \mathcal{L}^2 - Caratheodory if

- (i) $t \rightarrow \Sigma(t, x)$ is measurable for each $x \in X$,
- (ii) $x \rightarrow \Sigma(t, x)$ is upper semicontinuous for almost all $t \in J$,
- (iii) for each $r > 0$, there exists $L_r \in \mathcal{L}^1(J, \mathbb{R})$ such that

$$\|\Sigma(t, x)\|^2 = \sup \left\{ E \|\sigma\|^2 : \sigma \in \Sigma(t, x) \right\} \leq L_r(t) \text{ for almost all } t \in J \text{ and all } \|x\|^2 \leq r.$$

Lemma 2.8 ([3], Lasota and Opial). *Let J be a compact real interval, $BCC(X)$ be the set of all nonempty, bounded, closed and convex subset of X and Σ be a multivalued map $S_{\Sigma, x} \neq \emptyset$ and let Γ be a linear combination mapping from $\mathcal{L}^2(J, X)$ to $\mathcal{C}(J, X)$. then, the operator*

$$\Gamma \circ S_{\Sigma} : \mathcal{C} \rightarrow BCC(\mathcal{C}(J, X)), \quad x \rightarrow (\Gamma \circ S_{\Sigma})(x) = \Gamma(S_{\Sigma, x}),$$

is a closed graph operator in $\mathcal{C} \times \mathcal{C}$, where $S_{\Sigma, x}$ is known as the selected operators set from Σ , is given by

$$\sigma \in S_{\Sigma, x} = \left\{ \sigma \in \mathcal{L}^2(\mathcal{L}(K, X)) : \sigma(t) \in \Sigma(t, x) \text{ for a.e } t \in J \right\}.$$

Lemma 2.9 (Bohnenblust-Karlin's [5]). *Let \mathcal{D} be a nonempty subset of X , which is bounded, closed and convex. Suppose $G : \mathcal{D} \rightarrow 2^X \setminus \{\emptyset\}$ is u.s.c with closed, convex values and such that $G(\mathcal{D}) \subseteq \mathcal{D}$ and $G(\mathcal{D})$ is compact. Then G has a fixed point.*

Definition 2.10. A continuous X - valued process x is said to be a mild solution of (1.1)-(1.2) if

- (i) $x(t)$ is \mathcal{F}_t - adapted and $\{x(t) : t \in [0, b]\}$.
- (ii) there exists $\Sigma \in \mathcal{L}^1(J, X)$ such that $\sigma(t) \in \Sigma(t, x(t))$ on $t \in J$

$$x(t) = -\frac{\partial \mathcal{R}(t, 0)}{\partial s} y + \mathcal{R}(t, 0)z + \int_0^t \mathcal{R}(t, s)Bu(s)ds + \int_0^t \mathcal{R}(t, s)\sigma(s)dw(s)ds$$

- (iii) $x_0 = y$, $x'(0) = z$. is satisfied.

Now, it is convenient to introduce two appropriate operators and basic assumptions on these operators:

$$\begin{aligned} \gamma_0^b &= \int_0^b \mathcal{R}(b, s)BB^*\mathcal{R}^*(b, s)ds : X \rightarrow X, \\ R(\alpha, \gamma_0^b) &= \alpha(\alpha I + \gamma_0^b)^{-1} : X \rightarrow X. \end{aligned}$$

where B^* denotes the adjoint of B and $\mathcal{R}^*(t)$ is the adjoint of $\mathcal{R}(t)$. It is straight forward that the operator γ_0^b is a linear bounded operator.

To study the approximate controllability of system (1.1)-(1.2), we impose the following conditions

(H₀) $\alpha R(\alpha, \gamma_0^b) = \alpha(\alpha I + \gamma_0^b)^{-1} \rightarrow 0$ as $\alpha \rightarrow 0^+$ in the strong operator topology.

It the view of [36], Hypothesis (H_0) holds if and only if the linear system

$$\begin{aligned} x''(t) &\in A(t)x(t) + \int_0^t G(t, s)x(s)ds + (Bu)(t), \quad t \in [0, b], \\ x(0) &= y, \quad x'(0) = z, \end{aligned}$$

is approximately controllable on $[0, b]$.

3. APPROXIMATE CONTROLLABILITY RESULTS

In this section, we shall formulate and prove sufficient conditions for the approximate controllability for non-autonomous second order stochastic integrodifferential inclusions of the form (1.1)-(1.2) by using Bohnenblust-Karlin's fixed point theorem.

Definition 3.1. Let $x_b(\phi, u)$ be the state value of (1.1)-(1.2) at the terminal time b corresponding to the control u and the initial value ϕ . Introduce the set

$$\mathcal{R}(b, \phi) = \{x_b(\phi; u)(0) : u(\cdot) \in \mathcal{L}(J, U)\},$$

which is called the reachable set of (1.1)-(1.2) at the time b and its closure in X is denoted by $\overline{\mathcal{R}(b, \phi)}$. The system (1.1)-(1.2) is said to be approximately controllable on J if $\overline{\mathcal{R}(b, \phi)} = X$.

In order to establish the result, we need the following hypotheses:

- (H₁)** The resolvent operator $\mathcal{R}(t, s), (t, s) \in \Delta$ is compact.
- (H₂)** The multivalued map $\Sigma : J \times X \rightarrow BCC(X)$ is an \mathcal{L}^2 caratheodory function which satisfies the following conditions:

- (i) For each $t \in J$, the function $\Sigma(t, \cdot)$ is u.s.c and for each $x \in X$, the function $\sigma(\cdot, x)$ is measurable. And for each fixed $x \in X$, the set

$$S_{\Sigma, x} = \{\sigma \in \mathcal{L}^2(\mathcal{L}(\mathbb{K}, X)) : \sigma(t) \in \Sigma(t, \psi) \text{ for a.e } t \in J\}$$

is nonempty.

- (ii) For each positive number r there exists a positive function $L_r : J \rightarrow \mathbb{R}^+$ such that

$$\sup \left\{ E \|\sigma\|^2 : \sigma(t) \in \Sigma(t, \psi) \right\} \leq L_{\sigma, r}(t) \text{ a.e } t \in J.$$

$$r \xrightarrow{\lim} \infty \inf \frac{\int_0^t L_{\sigma,r}(s)ds}{r} = \Lambda < \infty.$$

Lemma 3.2. For any $\bar{x}_b \in \mathcal{L}_2(\mathcal{F}_b, X)$, there exists $\bar{\phi} \in \mathcal{L}_2^{\mathcal{F}}(\Omega, \mathcal{L}^2(J, \mathcal{L}(\mathbb{K}, X)))$ such that $\bar{x}_b = E\bar{x}_b + \int_0^b \bar{\phi}(s)dw(s)$.

Now for any $\alpha > 0$, $\bar{x}_b \in L^2(\mathcal{F}_b, X)$ and for $\sigma \in S_{\Sigma,x}$, we define the control function

$$u_\alpha(t, x) = B^* \mathcal{R}^*(b, s)R(\alpha, \gamma_0^b)P(x(\cdot))$$

where

$$p(x(\cdot)) \left\{ E\bar{x}_b + \int_0^b \bar{\phi}(s)dw(s) - \frac{\partial \mathcal{R}(t, 0)}{\partial s}y + \mathcal{R}(t, 0)z + \int_0^t \mathcal{R}(t, s)\sigma(s)dw(s)ds \right\}.$$

Theorem 3.3. Suppose that the hypotheses $(H_1) - (H_2)$ are satisfied then the system (1.1)-(1.2) has a mild solution on J , provided that

$$4M^2\gamma b^2[1 + \frac{4}{\alpha^2}M^4M_B^4] < 1$$

and where $\|B\| = M_B$.

Proof. The main aim in this section is to find the conditions for solvability of the system (1.1)-(1.2) for $\alpha > 0$. We show that, using the control $u_\alpha(x, t)$, the operator $\phi : \mathcal{C} \rightarrow 2^{\mathcal{C}}$, defined by

$$\phi(x) = \left\{ \varphi \in \mathcal{C} : \varphi(t) = -\frac{\partial \mathcal{R}(t, 0)}{\partial s}y + \mathcal{R}(t, 0)z + \int_0^t \mathcal{R}(t, s)Bu_\alpha(s)ds + \int_0^t \mathcal{R}(t, s)\sigma(s)dw(s)ds, \sigma \in S_{\sigma,x} \right\}$$

has a fixed point x , which is a mild solution of the system (1.1)-(1.2).

We now show that ϕ satisfies all the conditions of Lemma 2.9. To simplify the result, we subdivide the proof into five steps.

Step 1: ϕ is convex for each $x \in \mathcal{C}$. In fact, if φ_1, φ_2 belongs to $\phi(x)$, then there exist $\sigma_1, \sigma_2 \in S_{\Sigma,x}$ such that for each $t \in J$, we have

$$\begin{aligned} \varphi_i(t) &= \frac{\partial \mathcal{R}(t, 0)}{\partial s}y + \mathcal{R}(t, 0)z + \int_0^t \mathcal{R}(t, s)\sigma_i(s)dw(s)ds \\ &+ \int_0^t \mathcal{R}(t, s)BB^*\mathcal{R}^*(b, s)R(\alpha, \gamma_0^b) \\ &\times \left\{ E\bar{x}_b + \int_0^b \bar{\phi}(s)dw(s) - \frac{\partial \mathcal{R}(t, 0)}{\partial s}y + \mathcal{R}(t, 0)z + \int_0^t \mathcal{R}(t, s)\sigma_i(s)dw(s)ds \right\} (s)ds, \\ & i = 1, 2. \end{aligned}$$

Let $\lambda \in [0, 1]$. Then for each $t \in J$, we get

$$\begin{aligned}
 (\lambda\varphi_1 + (1 - \lambda)\varphi_2)(t) &= \frac{\partial\mathcal{R}(t, 0)}{\partial s}y + \mathcal{R}(t, 0)z \\
 &\quad + \int_0^t \mathcal{R}(t, s)[\lambda\sigma_1(s) + (1 - \lambda)\sigma_2(s)]dw(s)ds \\
 &\quad + \int_0^t \mathcal{R}(t, s)BB^*\mathcal{R}^*(b, s)R(\alpha, \gamma_0^b) \left\{ E\bar{x}_b + \int_0^b \bar{\phi}(s)dw(s) - \frac{\partial\mathcal{R}(t, 0)}{\partial s}y \right. \\
 &\quad \left. + \mathcal{R}(t, 0)z + \int_0^t \mathcal{R}(t, s)[\lambda\sigma_1(s) + (1 - \lambda)\sigma_2(s)]dw(s)ds \right\} (s)ds
 \end{aligned}$$

It is easy to see that $S_{\Sigma, x}$ is convex since Σ has convex values. So $\lambda\sigma_1(s) + (1 - \lambda)\sigma_2(s) \in S_{\Sigma, x}$. Thus, $\lambda\varphi_1 + (1 - \lambda)\varphi_2 \in \phi(x)$.

Step 2: For $r > 0$, let $B_r = \{x \in \mathcal{C} : \|x\|_{\mathcal{C}}^2 \in r\}$. Certainly, B_r is a bounded, closed and convex set of \mathcal{C} . We claim that there exists a positive number r such that $\phi(B_r) \subseteq B_r$.

If this is not true, then for each positive number r , there exists a function $x^r \in B_r$, but $\phi(x^r) \not\subseteq B_r$, i.e., $E\|\phi(x^r)\|_{\mathcal{C}}^2 > r$

$$\begin{aligned}
 &r < \|E\phi(x^r)(t)\|^2 \\
 &\leq 4 \left\{ E \left\| \frac{\partial\mathcal{R}(t, 0)}{\partial s}y \right\|^2 + E\|\mathcal{R}(t, 0)z\|^2 + E \left\| \int_0^t \mathcal{R}(t, s)Bu_{\alpha}^r(s, x)ds \right\|^2 \right. \\
 &\quad \left. + E \left\| \int_0^t \mathcal{R}(t, s)\sigma^r(s)dw(s)ds \right\|^2 \right\} \\
 &\leq 4 \left\{ \widetilde{M}^2 E\|y\|^2 + M^2 E\|z\|^2 + M^2 b^2 \int_0^t L_{\sigma, r}(s)dw(s) + \frac{4}{\alpha^2} M^4 M_B^4 b^2 \right. \\
 &\quad \left. \times \left[2E\|\bar{x}_b\|^2 + 2E \left\| \int_0^b \bar{\phi}(s)dw(s)ds \right\|^2 + \widetilde{M}E\|y\|^2 + M^2 E\|z\|^2 + M^2 \int_0^t L_{\sigma, r}(s)dw(s) \right] \right\}
 \end{aligned}$$

dividing both sides of the above inequality by r and taking $r \rightarrow \infty$ we have

$$4M^2\gamma b^2 \left[1 + \frac{4}{\alpha^2} M^4 M_B^4 \right] \geq 1$$

which is a contradiction to our assumption. Hence, for some positive number $r > 0$ and some $\sigma \in S_{\Sigma, x}$, $\phi(B_r) \subseteq B_r$.

Step 3: ϕ sends bounded sets into equicontinuous sets of \mathcal{C} . For each $x \in B_r$,

$\varphi \in \phi(x)$, there exists $\sigma \in S_{\Sigma, x}$ such that for each $t \in J$, we have

$$\varphi(t) = -\frac{\partial \mathcal{R}(t, 0)}{\partial s} y + \mathcal{R}(t, 0)z + \int_0^t \mathcal{R}(t, s)Bu_\alpha(s)ds + \int_0^t \mathcal{R}(t, s)\sigma(s)dw(s).$$

Let $0 < \epsilon < t$ and $0 < t_1 < t_2 \leq b$, then

$$\begin{aligned} E \|\varphi(t_1) - \varphi(t_2)\|^2 &= 8E \left\| -\frac{\partial \mathcal{R}(t_1, 0)}{\partial s} + \frac{\partial \mathcal{R}(t_2, 0)}{\partial s} \right\|^2 E \|y\|^2 \\ &+ 8E \|\mathcal{R}(t_1, 0) - \mathcal{R}(t_2, 0)\|^2 E \|z\|^2 \\ &+ 8E \left\| \int_0^{t_1} \mathcal{R}(t_1, s) - \mathcal{R}(t_2, s)\sigma(s)dw(s) \right\|^2 \\ &+ 8E \left\| \int_{t_1-\epsilon}^{t_1} \mathcal{R}(t_1, s) - \mathcal{R}(t_2, s)\sigma(s)dw(s) \right\|^2 \\ &+ 8E \left\| \int_{t_1}^{t_2} \mathcal{R}(t_2, s)\sigma(s)dw(s) \right\|^2 + 8E \left\| \int_0^{t_1} \mathcal{R}(t_1, s) - \mathcal{R}(t_2, s)Bu(\eta, x)d\eta \right\|^2 \\ &+ 8E \left\| \int_{t_1-\epsilon}^{t_1} \mathcal{R}(t_1, s) - \mathcal{R}(t_2, s)Bu(\eta, x)d\eta \right\|^2 + 8E \left\| \int_{t_1}^{t_2} \mathcal{R}(t_2, s)Bu(\eta, x)d\eta \right\|^2 \\ &\leq 8E \left\| -\frac{\partial \mathcal{R}(t_1, 0)}{\partial s} + \frac{\partial \mathcal{R}(t_2, 0)}{\partial s} \right\|^2 E \|y\|^2 + 8E \|\mathcal{R}(t_1, 0) - \mathcal{R}(t_2, 0)\|^2 E \|z\|^2 \\ &+ 8E \left\| \int_0^{t_1} \mathcal{R}(t_1, s) - \mathcal{R}(t_2, s)L_{\sigma, r}(s)dw(s)ds \right\|^2 \\ &+ 8E \left\| \int_{t_1-\epsilon}^{t_1} \mathcal{R}(t_1, s) - \mathcal{R}(t_2, s)L_{\sigma, r}(s)dw(s)ds \right\|^2 \\ &+ 8E \left\| \int_{t_1}^{t_2} \mathcal{R}(t_2, s)L_{\sigma, r}(s)dw(s)ds \right\|^2 \\ &+ 8M_B^2 E \left\| \int_0^{t_1} \mathcal{R}(t_1, s) - \mathcal{R}(t_2, s)u(\eta, x)d\eta \right\|^2 \\ &+ 8M_B^2 E \left\| \int_{t_1-\epsilon}^{t_1} \mathcal{R}(t_1, s) - \mathcal{R}(t_2, s)u(\eta, x)d\eta \right\|^2 + 8M_B^2 E \left\| \int_{t_1}^{t_2} \mathcal{R}(t_2, s)u(\eta, x)d\eta \right\|^2 \end{aligned}$$

The righthand side of the above inequality tends to zero independently of $x \in B_r$ as $(t_1 - t_2) \rightarrow 0$ and ϵ sufficiently small, since the compactness of the resolvent operator $\mathcal{R}(t, s)$ implies the continuity in the uniform operator topology. Thus $\phi(x^r)$ sends B_r into equicontinuous family of functions.

Step 4: The set $\prod(t) = \{\varphi(t) : \varphi \in \phi(B_r)\}$ is relatively compact in X .

Let $t \in (0, b]$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $x \in B_r$, we define

$$\varphi_\epsilon(t) = -\frac{\partial \mathcal{R}(t, 0)}{\partial s}y + \mathcal{R}(t, 0)z + \int_0^{t-\epsilon} \mathcal{R}(t, s)Bu_\alpha(\eta, x)d\eta + \int_0^{t-\epsilon} \mathcal{R}(t, s)\sigma(s)dw(s)ds$$

Since $\mathcal{R}(t, s)$ is a compact operator, the set $\prod_\epsilon(t) = \{\varphi_\epsilon(t) : \varphi_\epsilon \in \phi(B_r)\}$ is relatively compact in X for each $\epsilon, 0 < \epsilon < t$. Moreover, for each $0 < \epsilon < t$, we have

$$E \|\varphi(t) - \varphi_\epsilon(t)\|^2 \leq E \left\| \int_t^{t-\epsilon} \mathcal{R}(t, s)Bu_\alpha(\eta, x)d\eta \right\|^2 + E \left\| \int_t^{t-\epsilon} \mathcal{R}(t, s)\sigma(s)dw(s)ds \right\|^2.$$

Therefore

$$E \|\varphi(t) - \varphi_\epsilon(t)\|^2 \rightarrow 0 \text{ as } \alpha \rightarrow 0^+.$$

Hence there exists relatively compact sets arbitrarily close to the set $\prod(t) = \{\varphi(t) : \varphi \in \phi(B_r)\}$ and the set $\tilde{\prod}(t)$ is relatively compact in X for all $t \in [0, b]$. Since it is compact at $t = 0$, hence $\prod(t)$ is relatively compact in X for all $t \in [0, b]$.

Step 5: ϕ has a closed graph.

$x_n \rightarrow x_*$ as $n \rightarrow \infty$, $\varphi_n \in \phi(x_n)$ and $\varphi_n \rightarrow \varphi_*$ as $n \rightarrow \infty$. We will show that $\varphi_* \in \phi(x_*)$. Since $\varphi_n \in \psi(x_n)$, there exists a $\sigma_n \in S_{\Sigma, x_n}$ such that

$$\begin{aligned} \varphi_n(t) &= \frac{\partial \mathcal{R}(t, 0)}{\partial s}y + \mathcal{R}(t, 0)z + \int_0^t \mathcal{R}(t, s)\sigma_n(s)dw(s)ds + \int_0^t \mathcal{R}(t, s)BB^*\mathcal{R}^*(b, s) \\ &\quad \times R(\alpha, \gamma_0^b) \left\{ E\bar{x}_b + \int_0^b \bar{\phi}(s)dw(s) - \frac{\partial \mathcal{R}(b, 0)}{\partial s}y + \mathcal{R}(b, 0)z \right. \\ &\quad \left. + \int_0^t \mathcal{R}(b, \eta)\sigma_n(\eta)dw(\eta)d\eta \right\} (s)ds. \end{aligned}$$

We must prove that there exists $\sigma_* \in S_{\Sigma, x_*}$ such that

$$\begin{aligned} \varphi_*(t) &= \frac{\partial \mathcal{R}(t, 0)}{\partial s}y + \mathcal{R}(t, 0)z + \int_0^t \mathcal{R}(t, s)\sigma_*(s)dw(s) \\ &+ \int_0^t \mathcal{R}(t, s)BB^*\mathcal{R}^*(b, s)R(\alpha, \gamma_0^b) \\ &\quad \times \left\{ E\bar{x}_b + \int_0^b \bar{\phi}(s)dw(s) - \frac{\partial \mathcal{R}(b, 0)}{\partial s}y + \mathcal{R}(b, 0)z + \int_0^t \mathcal{R}(b, \eta)\sigma_*(\eta)dw(\eta) \right\} (s)ds \end{aligned}$$

clearly, we have

$$\left\| \left(\varphi_n + \frac{\partial \mathcal{R}(t, 0)}{\partial s}y - \mathcal{R}(t, 0)z - \int_0^t \mathcal{R}(t, s)\sigma_n(s)dw(s)ds - \int_0^t \mathcal{R}(t, s)BB^*\mathcal{R}^*(b, s) \right. \right.$$

$$\begin{aligned} & \times R(\alpha, \gamma_0^b) \left\{ E\bar{x}_b + \int_0^b \bar{\phi}(s)dw(s) - \frac{\partial \mathcal{R}(b, 0)}{\partial s}y - \mathcal{R}(b, 0)z - \int_0^t \mathcal{R}(b, \eta)\sigma_n(\eta)dw(\eta)d\eta \right\} (s)ds \\ & - \left(\varphi_* + \frac{\partial \mathcal{R}(t, 0)}{\partial s}y - \mathcal{R}(t, 0)z - \int_0^t \mathcal{R}(t, s)\sigma_*(s)dw(s)ds - \int_0^t \mathcal{R}(t, s)BB^*\mathcal{R}^*(b, s) \right. \\ & \quad \times R(\alpha, \gamma_0^b) \left\{ E\bar{x}_b + \int_0^b \bar{\phi}(s)dw(s) - \frac{\partial \mathcal{R}(b, 0)}{\partial s}y - \mathcal{R}(b, 0)z \right. \\ & \quad \left. \left. - \int_0^t \mathcal{R}(b, \eta)\sigma_*(\eta)dw(\eta)d\eta \right\} (s)ds \right) \Big\|^2 \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Consider the linear continuous operator $\psi : \mathcal{L}^1(J \times X) \rightarrow \mathcal{L}^1(J \times X)$ and $\kappa : \mathcal{L}^1(J \times X) \rightarrow \mathcal{L}^1(J \times X)$,

$$\begin{aligned} (\kappa\sigma)(t) &= \int_0^t \mathcal{R}(t, s)\sigma(s)dw(s)ds - \int_0^t \mathcal{R}(t, s)BB^*\mathcal{R}^*(b-s)R(\alpha, \gamma_0^b) \\ & \quad \left\{ \times \int_0^b \mathcal{R}(t, s)\sigma(\tau)dw(\tau)d\tau \right\} (s)ds. \end{aligned}$$

From Lemma 2.8 it follows that $\kappa \circ S_\Sigma$ is a closed graph operator. Also, from the definition of κ , we have that

$$\begin{aligned} & \left(\varphi_n + \frac{\partial \mathcal{R}(t, 0)}{\partial s}y - \mathcal{R}(t, 0)z - \int_0^t \mathcal{R}(t, s)\sigma_n(s)dw(s)ds - \int_0^t \mathcal{R}(t, s)BB^*\mathcal{R}^*(b, s)R(\alpha, \gamma_0^b) \right. \\ & \quad \times \left\{ E\bar{x}_b + \int_0^b \bar{\phi}(s)dw(s) - \frac{\partial \mathcal{R}(b, 0)}{\partial s}y - \mathcal{R}(b, 0)z \right. \\ & \quad \left. \left. - \int_0^t \mathcal{R}(b, \eta)\sigma_n(\eta)dw(\eta)d\eta \right\} (s)ds \right) \in \kappa(S_{\sigma, x_n}). \end{aligned}$$

Since $y_n \rightarrow y_*$, for some $y_* \in S_{\Sigma, y_*}$, it follows from Lemma 2.9 that

$$\begin{aligned} & \left(\varphi_*(t) - \int_0^t \mathcal{R}(t, s)BB^*\mathcal{R}^*(b, s)R(\alpha, \gamma_0^b) \left\{ E\bar{x}_b + \int_0^b \bar{\phi}(s)dw(s) - \frac{\partial \mathcal{R}(b, 0)}{\partial s}y - \mathcal{R}(b, 0)z \right. \right. \\ & \quad \left. \left. - \int_0^b \mathcal{R}(b, s)\sigma_*(s)dw(s)ds \right\} (s)ds + \int_0^t \mathcal{R}(t, s)\sigma_*(s)dw(s) \right) \end{aligned}$$

for some $\sigma_* \in S(\Sigma, y_*)$. Therefore ϕ has a closed graph.

As a consequences of **step 1 to step 5** together with the Arzela-Ascoli theorem, we conclude that ϕ is a compact multivalued map, u.s.c with convex closed values. As a consequences of Lemma 2.9, we can deduce that ϕ has a fixed point x which is a mild solution of (1.1)-(1.2). \square

Roughly speaking, by using the control function u , from any given initial point $x_0 \in X$ we can move the system with the trajectory as close as possible to any other final point $x_b \in X$.

Theorem 3.4. *Suppose that the assumptions $(H_0) - (H_2)$ hold. Assume additionally that there exists $N \in \mathcal{L}^1([0, \infty))$ such that $\sup_{x \in X} E \|\sigma(t, x)\|^2 \leq N(t)$ for a.e. $t \in J$, then the non-autonomous second order integrodifferential inclusions (1.1)-(1.2) is approximately controllable on J .*

Proof. Let $\hat{x}^\alpha(\cdot)$ be fixed point of ϕ in B_r . By Theorem 3.3 any fixed point of ϕ is a mild solution of (1.1) – (1.2) under the control

$$\hat{u}^\alpha(t) = B^* \mathcal{R}^*(b, t) R(\alpha, \gamma_0^b) p(\hat{x}^\alpha)$$

and

$$\hat{x}^\alpha(b) = x_b + \alpha R(\alpha, \gamma_0^b) p(\hat{x}^\alpha) \tag{3.1}$$

Moreover by assumption on σ and Dunford-Pettis theorem, we have that the $\{\sigma^\alpha(s)\}$ is weakly compact in $\mathcal{L}^1(J, X)$, so there is a subsequence, still denoted by $\{\sigma^\alpha(s)\}$, that converges weakly to say $\sigma(s)$ in $\mathcal{L}^1(J, X)$. Define

$$w = E \bar{x}_b + \int_0^b \bar{\phi}(s) dw(s) - \frac{\partial \mathcal{R}(t, 0)}{\partial s} y + \mathcal{R}(t, 0) z + \int_0^t \mathcal{R}(t, s) \sigma(s) dw(s) ds$$

now we have

$$\begin{aligned} E \|p(\hat{x}^\alpha) - w\|^2 &= \left\| \int_0^b \mathcal{R}(b, s) [\sigma(s, \hat{x}^\alpha(s)) - \sigma(s)] dw(s) ds \right\|^2 \\ &\leq \sup_{t \in J} \left\| \int_0^b \mathcal{R}(b, s) [\sigma(s, \hat{x}^\alpha(s)) - \sigma(s)] dw(s) ds \right\|^2 \end{aligned} \tag{3.2}$$

By using infinite dimensional version of the Ascoli-Arzela theorem, one can show that an operator $l(\cdot) \rightarrow \int_0^\cdot S(\cdot, s) l(s) ds : \mathcal{L}^1(J, X) \rightarrow \mathcal{C}$ is compact. Therefore, we obtain that $E \|p(\hat{x}^\alpha) - w\|^2 \rightarrow 0$ as $\alpha \rightarrow 0^+$. Moreover, from (3.1) we get,

$$\begin{aligned} E \|\hat{x}^\alpha(b) - w\|^2 &\leq E \|\alpha R(\alpha, \gamma_0^b)(w)\|^2 + E \|\alpha R(\alpha, \gamma_0^b)\|^2 E \|p(\hat{x}^\alpha) - w\|^2 \\ &\leq E \|\alpha R(\alpha, \gamma_0^b)(w)\|^2 + E \|p(\hat{x}^\alpha) - w\|^2. \end{aligned}$$

It follows from assumption (H_0) and the estimation (3.2) that $E \|p(\hat{x}^\alpha) - w\|^2 \rightarrow 0$ as $\alpha \rightarrow 0^+$. This proves the approximate controllability of second order differential inclusions (1.1)-(1.2). \square

4. APPLICATION

Consider the second order Cauchy problem with control

$$\begin{aligned} \frac{\partial^2}{\partial t^2} z(t, \tau) \in & \frac{\partial^2}{\partial \tau^2} z(t, \tau) + b(t) \frac{\partial}{\partial t} z(t, \tau) \\ & + \int_0^t a(t-s) \frac{\partial}{\partial \tau} z(t, \tau) + \mu(t, \tau) + \sigma(t, z(t, \tau)) dw(t) \end{aligned} \quad (4.1)$$

for $t \in J, 0 \leq \tau \leq \pi$, subject to the initial conditions

$$z(t, 0) = z(t, \pi) = 0, \quad (4.2)$$

$$z(0, \tau) = z_0(\tau) \quad (4.3)$$

$$\frac{\partial z(0, \tau)}{\partial t} = z_1(\tau), \quad (4.4)$$

where $w(t)$ denotes a standard cylindrical process in X defined on a stochastic space $(\Omega, \mathcal{F}, \mathbb{P})$, $a, b : \mathbb{R} \rightarrow \mathbb{R}$, $\mu : J \times [0, \pi] \rightarrow [0, \pi]$ are continuous functions and $X = \mathbb{K} = \mathcal{L}^2([0, \pi])$. Define the operators $A : D(A) \subset X \rightarrow X$, by $(Az)(\xi) = \frac{d^2 z(\xi)}{d\xi^2}$, where each domain $D(A)$ is given by

$$\{z \in X, y, y' \text{ are absolutely continuous } y'' \in \mathbb{H}, y(0) = y(\pi) = 0\}.$$

It is well known that A is the infinitesimal generator of a resolvent operator $\mathcal{R}(t)$ on X . Further A and L can be written as $Az = \sum_{n=1}^{\infty} -n^2 \langle y, w_n \rangle w_n, y \in D(A)$, where $z_n(\xi) = \sqrt{\frac{2}{\pi}} \sin n\xi, n = 1, 2, 3, \dots$ is the orthogonal set of vectors of A

We take $A(t)y(\tau) = b(t)y(\tau)$ defined on X . Let $z(t)(\tau) = z(t, \tau)$ and the functions $f : J \times X \rightarrow \mathcal{L}_2^0, G(t, s) \subseteq X \rightarrow X, u : J \rightarrow U$ given by

$$\begin{aligned} \sigma(t, w)(\tau) &= f(t, w(\tau)) \\ g(t, w)y(\tau) &= a(t-s) \frac{\partial z(\tau)}{d\tau}, \\ Bu(t)(\tau) &= \mu(t, \tau), \end{aligned}$$

where $\mu : J \times [0, \pi] \rightarrow [0, \pi]$ is continuous.

Assume these functions satisfy the requirement of hypotheses. From the above choices of the functions and evolution operator $A(t)$ with $B = I$, the system (4.1)-(4.4) can be formulated as an abstract second order semilinear system (1.1)-(1.2) in X . Further, we can impose some suitable conditions on the above defined functions to verify the assumptions on Theorem 3.4, we can conclude that (4.1)-(4.4) is approximately controllable on $[0, b]$.

REFERENCES

[1] Balasubramaniam, P., Ntouyas, S.K., Controllability for neutral stochastic functional differential inclusions with infinite delay in abstract space, *Journal of Mathematical Analysis and Applications*, **324**, (2006), 161 - 176.

- [2] Balasubramaniam, P., Vinayagam, D., Existence of solutions of nonlinear neutral stochastic differential inclusions in a Hilbert space, *Stoch. Anal. Appl.*, **23**, (2005), 137-151.
- [3] Balasubramaniam, P. and Tamilaagan, P., Approximate controllability of a class of fractional neutral stochastic integro-differential inclusions with infinite delay by using Mainardi's function, *Applied Mathematics and Computation*, **256**, (2015), 232-246.
- [4] Batty, C.J.K., Chill, R. and Srivastava, S., Maximal regularity for second order non-autonomous cauchy problems, *studia Mathematica*, **189**, (2008), 205-223.
- [5] Bohenblust, H.F. and Karlin, S., On a Theorem of Ville, in Contributions to the Theory of Games, *Princeton University Press, Princeton, NJ*, 155-160, 1950.
- [6] Caraballo, T., Asymptotic exponential stability of stochastic partial differential equations with delay, *Stochastics*, **33**, (1990), 27-47.
- [7] Chalishajar, D.N., Controllability of second order impulsive neutral functional differential inclusions with infinite delay, *Journal of Optimization Theory and Applications*, **154** (2), (2012), 672-684.
- [8] Chang, Y.K. and Li, W.T., Controllability of Sobolev type semilinear functional differential and integrodifferential inclusions with an unbounded delay, *Georgian Mathematical Journal*, **13** (1), (2006), 11-24.
- [9] Chang, Y.K., Controllability of impulsive functional differential systems with infinite delay in Banach space, *Chaos Solitons Fractals*, **33**, (2007), 1601-1609.
- [10] Da Prato, G. and Zabczyk, J., *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge. 1992.
- [11] Deimling, K., *Multivalued Differential Equations*, De Gruyter, Berlin. 1992.
- [12] Faraci, F and Iannizzotto, A., A multiplicity theorem for a perturbed second order non-autonomous system. *Proceedings of Edinburg Mathematical society*, **49**, (2006), 267-275.
- [13] Fattorini HO., In Second Order Linear Differential Equations in Banach Spaces, *North-Holland Mathematics Studies*, vol **108**, North-Holland, Amsterdam. 1985.
- [14] Grimmer, R., Resolvent operators for integral equations in Banach space, *Transactions of American Mathematical Society*, **273**, (1982), 333-349.
- [15] Grimmer, R. and Pritchard, A.J., Analytic Resolvent operators for integral equations in Banach space, *Journal of Differential equations*, **50**, (1983), 234-259.
- [16] Henríquez, H.R., Existence of solutions of non-autonomous second order functional differential equations with infinite delay, *Nonlinear Analysis: Theory, Method and Applications*, **74**, (2011), 3333-3352.
- [17] Henríquez, H.R., Existence of solutions of the nonautonomous abstract cauchy problem of second order. *Semigroup forum*, (2013), Doi:10.1007/s00233-013-9458-8.
- [18] Henríquez, H.R. and Hernández, E., Existence of solutions of a second order abstract functional cauchy problem with nonlocal conditions, *Annales Polonici Mathematici*, **88**(2), (2006), 141-159.
- [19] Henríquez, H.R. and Hernández, E., Approximate controllability of second order distributed implicit functional system, *Nonlinear Analysis*, **70**, (2009), 1023-1039.
- [20] Henríquez, H.R. and Pozo, J.C., Existence of solutions of abstract non-autonomous second order integrodifferential equations, *Boundary Value Problems*, **168**, (2016), 1-24.
- [21] Hernández, E and Henríquez, H.R. and dos Santos, J.P.C., Existence results for abstract partial neutral integro-differential equation with unbounded delay, *Electronic Journal of Qualitative Theory of Differential equations*, **29**, (2009), 1-23.
- [22] Hu, S and Papageorgiou, N.S., *Handbook of Multivalued Analysis(Theory)*, Kluwer Academic Publishers, Dordrecht Boston, London. 1997.
- [23] Mahmudov, N.I., Controllability of linear stochastic systems in Hilbert spaces, *Journal of Mathematical Analysis Applications*, **259**, (2001), 64-82.
- [24] Mahmudov, N.I. and Denker, A., On controllability of linear stochastic systems, *Int. J. Control*, **73**, (2000), 144-151.

- [25] Mahmudov, N.I., Vijayakumar, V. and Murugesu, R., Approximate controllability of second-order evolution differential inclusions in Hilbert spaces, *Mediterranean Journal of Mathematics*, **13**(5), (2016), 3433-3454.
- [26] Mao, X., *Stochastic Differential equations and Applications*, Horwood, Chichester. 1997.
- [27] Oksendal, B., *Stochastic Differential Equations, An Introduction with Applications*, Springer-Verlag. 2000.
- [28] Ren, Y. and Sun, D.D., Second-order neutral stochastic evolution equations with infinite delay under Caratheodory conditions, *J. Optim. Theory Appl*, **147**, (2010), 569-582.
- [29] Revathi, P., Sakthivel, R. and Ren, Y., Stochastic functional differential equations of Sobolev-type with infinite delay, *Statistics and Probability Letters*, **109**, (2016), 68-77.
- [30] Sakthivel, R., Mahmudov, N.I. and Lee, S.G., Controllability of non-linear impulsive stochastic systems, *International Journal of Control*, **82**, (2009), 801-807.
- [31] Shen, L.J. and Sun, J.T., Approximate controllability of stochastic impulsive functional systems with infinite delay, *Automatica*, **48**, (2012), 2705-2709.
- [32] Sobczyk, K. *Stochastic Differential Equations with Applications to Physics and Engineering*, Kluwer Academic, London. 1991.
- [33] Travis, C.C. and Webb, G.F., Cosine families and abstract nonlinear second order differential equations, *Acta Math. Acad. Sci. Hung*, **32**, (1978), 76-96.
- [34] Vijayakumar, V., Ravichandran, C. and Murugesu, R., Approximate controllability for a class of fractional neutral integro-differential inclusions with state-dependent delay, *Nonlinear Studies*, **20** (4), (2013), 511-530.
- [35] Vijayakumar, V., Selvakumar, A. and Murugesu, R., Controllability for a class of fractional neutral integro-differential equations with unbounded delay, *Applied Mathematics and Computation*, **232**, (2014), 303-312.
- [36] Vijayakumar, V. (2016), Approximate controllability results for analytic resolvent integrodifferential inclusions in Hilbert spaces, *International Journal of Control*, doi: 10.1080/00207179.2016.1276633.
- [37] Vijayakumar, V., Murugesu, R., Poongodi, R. and Dhanalakshmi, S. Controllability of second order impulsive nonlocal Cauchy problem via measure of noncompactness, *Mediterranean Journal of Mathematics*, **14** (1), (2017), 29-51.
- [38] Zhou, Y., Vijayakumar, V. and Murugesu, R., Controllability for fractional evolution inclusions without compactness, *Evolution Equations and Control Theory*, **4** (4), (2015), 507-524.

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