Çankaya University Journal of Science and Engineering
Volume 15, No. 2 (2018) 054–062
Date Received: 20/06/2018
Date Accepted: 09/11/2018

n-copure submodules of modules

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Abstract: Let R be a commutative ring, M an R-module, and $n \geq 1$ an integer. In this paper, we will introduce the concept of n-copure submodules of M as a generalization of copure submodules and obtain some related results.

Keywords: Copure submodule, n-pure submodule, n-copure submodule, strong comultiplication module

1. Introduction

Throughout this paper, R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers. Further, n will denote a positive integer.

Let M be an R-module. M is said to be a *multiplication module* if for every submodule N of M, there exists an ideal I of R such that $N = IM$ [8].

Cohn [9] defined a submodule N of M a *pure submodule* if the sequence $0 \to N \otimes E \to M \otimes E$ is exact for every R-module E. Anderson and Fuller [3] called the submodule N a *pure submodule* of M if $IN = N \cap IM$ for every ideal I of R. Ribenboim [14] called N to be *pure* in M if $rM \cap N = rN$ for each $r \in R$. Although the first condition implies the second [13, p.158], and the second obviously implies the third, these definitions are not equivalent in general, see [13, p.158] for an example. The three definitions of purity given above are equivalent if M is flat. In particular, if M is a faithful multiplication module [1].

In this paper, our definition of purity will be that of Anderson and Fuller [3].

In [6], H. Ansari-Toroghy and F. Farshadifar introduced the dual notion of pure submodules (that is copure submodules) and investigated the first properties of this class of modules. A submodule N of M is said to be *copure* if $(N :_M I) = N + (0 :_M I)$ for every ideal I of R [6].

The concept of n-pure submodules of an R-module M as a generalization of pure submodules was introduced in [10]. A submodule N of an R-module M is said to be a *n-pure submodule of M* if

ISSN 2564 - 7954

$I_1I_2...I_nN = I_1N \cap I_2N \cap ...I_nN \cap (I_1I_2...I_n)M$ for all proper ideals $I_1, I_2, ...I_n$ of R. Also, an ideal I of R is said to be a *n-pure ideal of* R if I is a n-pure submodule of R.

The main purpose of this paper is to introduce the concepts of n-copure submodules of an R-module M as a generalization of copure submodules and investigate some results concerning this notion.

2. Main results

Definition 2.1. Let n be a positive integer. We say that a submodule N of an R-module M is a *n-copure submodule of* M if

$$(N :_M I_1I_2...I_n) = (N :_M I_1) + (N :_M I_2) + ... + (N :_M I_n) + (0 :_M I_1I_2...I_n)$$

for all proper ideals $I_1, I_2, ..., I_n$ of R. This can be regarded as a dual notion of the n-pure submodule of M.

Remark 2.2. Let n be a positive integer. Clearly every $(n-1)$-copure submodule of an R-module M is a n-copure submodule of M. But we see in the Example 2.3 that the converse is not true in general.

Example 2.3. Let n be a positive integer. The submodule $\bar{2}\mathbb{Z}_{2^n}$ of the $\mathbb{Z}_{2^n}$-module $\mathbb{Z}_{2^n}$ is a n-copure submodule of $\mathbb{Z}_{2^n}$ but it is not a $(n-1)$-copure submodule of $\mathbb{Z}_{2^n}$.

Example 2.4. Let $n > 1$ be an integer. Since $1/2^n \in (\mathbb{Z} :_{\mathbb{Q}} \underbrace{(2\mathbb{Z})(2\mathbb{Z})...(2\mathbb{Z})}_{n \text{ times}})$ but

$$1/2^n \notin \underbrace{(\mathbb{Z} :_{\mathbb{Q}} 2\mathbb{Z}) + (\mathbb{Z} :_{\mathbb{Q}} 2\mathbb{Z}) + ... + (\mathbb{Z} :_{\mathbb{Q}} 2\mathbb{Z})}_{n \text{ times}} + (0 :_{\mathbb{Q}} \underbrace{(2\mathbb{Z})(2\mathbb{Z})...(2\mathbb{Z})}_{n \text{ times}}).$$

The submodule $\mathbb{Z}$ of the $\mathbb{Z}$-module $\mathbb{Q}$ is not n-copure.

Proposition 2.5. Let M be an R-module and n be a positive integer. Then we have the following.

(a) If N is a submodule of M such that

$$(N :_M I_1I_2...I_n) = (N :_M I_1) + (N :_M I_2) + ... + (N :_M I_n)$$

for all proper ideals $I_1, I_2, ..., I_n$ of R, then N is a n-copure submodule of M.

(b) If R is a Noetherian ring and N is a n-copure submodule of M, then for each prime ideal P of R, N_P is a n-copure submodule of M_P as an R_P-module.

(c) If R is a Noetherian ring and N_P is a n-copure submodule of an R_P-module M_P for each maximal ideal P of R, then N is a n-copure submodule of M.

Proof. (a) Let $I_1, I_2, ..., I_n$ be proper ideals of R. Then

$$(N :_M I_1 I_2 ... I_n) = (N :_M I_1) + (N :_M I_2) + ... + (N :_M I_n)$$

by assumption. Thus

$$(0 :_M I_1 I_2 ... I_n) \subseteq (N :_M I_1 I_2 ... I_n) = (N :_M I_1) + (N :_M I_2) + ... + (N :_M I_n).$$

This implies that

$$(0 :_M I_1 I_2 ... I_n) + (N :_M I_1) + (N :_M I_2) + ... + (N :_M I_n) =$$

$$(N :_M I_1) + (N :_M I_2) + ... + (N :_M I_n).$$

Therefore,

$$(0 :_M I_1 I_2 ... I_n) + (N :_M I_1) + (N :_M I_2) + ... + (N :_M I_n) = (N :_M I_1 I_2 ... I_n),$$

as required.

(b) This follows from the fact that by [15, 9.13], if I is a finitely generated ideal of R, then $(N :_M I)_P = (N_P :_{M_P} I_P)$.

(c) Suppose that $I_1, I_2, ..., I_n$ are proper ideals of R. Since R is Noetherian, $I_1, I_2, ..., I_n$ are finitely generated. Hence by [15, 9.13], for each maximal ideal P of R, $(N :_M I_1 I_2 ... I_n)_P = (N_P :_{M_P} (I_1)_P (I_2)_P ... (I_n)_P)$. Thus by assumption,

$$\begin{aligned}(N :_M I_1 I_2 ... I_n)_P &= (N :_M I_1)_P + (N :_M I_2)_P + ... + (N :_M I_n)_P + (0 :_M I_1 I_2 ... I_n)_P \\ &= ((N :_M I_1) + (N :_M I_2) + ... + (N :_M I_n) + (0 :_M I_1 I_2 ... I_n))_P.\end{aligned}$$

Therefore

$$(N :_M I_1 I_2 ... I_n) = (N :_M I_1) + (N :_M I_2) + ... + (N :_M I_n) + (0 :_M I_1 I_2 ... I_n),$$

as desired. ■

Recall that an R-module M is said to be *fully copure* if every submodule of M is copure [7].

Definition 2.6. Let n be a positive integer. We say that an R-module M is *fully n-copure* if every submodule of M is n-copure.

An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$ [4]. It is easy to see that M is a comultiplication module if and only if $N = (0 :_M Ann_R(N))$ for each submodule N of M.

Let N and K be two submodules of M. The *coproduct* of N and K is defined by $(0 :_M Ann_R(N) Ann_R(K))$ and denoted by $C(NK)$ [5].

Theorem 2.7. Let M be a comultiplication R-module and n be a positive integer. Then the following statements are equivalent.

(a) For submodules $N_1, N_2, ..., N_n$ of M, we have

$$C(N_1N_2...N_n) = C(N_1N_2) + C(N_1N_3) + ... + C(N_1N_n) + C(N_2N_3...N_n).$$

(b) M is a fully n-copure R-module.

Proof. $(a) \Rightarrow (b)$. Let N be a submodule of M and $I_1, I_2, ..., I_n$ be proper ideals of R. Then as M is a comultiplication R-module, for each i $(1 \leq i \leq n)$

$$\begin{aligned} C(N(0 :_M I_i)) &= (0 :_M Ann_R(N)Ann_R((0 :_M I_i))) \\ &= ((0 :_M Ann_R((0 :_M I_i))) :_M Ann_R(N)) \\ &= ((0 :_M I_i) : Ann_R(N)) = (N :_M I_i). \end{aligned}$$

Now by part (a) and the fact that M is a comultiplication R-module,

$$\begin{aligned} &(N :_M I_1) + (N :_M I_2) + ... + (N :_M I_n) + (0 :_M I_1I_2...I_n) \\ &= C(N(0 :_M I_1)) + C(N(0 :_M I_2)) + ... + C(N(0 :_M I_n)) + C((0 :_M I_1)(0 :_M I_2)...(0 :_M I_n)) \\ &= C(N(0 :_M I_1)(0 :_M I_2)...(0 :_M I_n)) \\ &= (N :_R I_1I_2...I_n). \end{aligned}$$

$(b) \Rightarrow (a)$. As M is a comultiplication R-module, we have $C(N_1N_i) = (N_1 :_M Ann_R(N_i))$ for all $2 \leq i \leq n$. Now since by part (b), N_1 is a n-copure submodule of M,

$$\begin{aligned} &C(N_1N_2) + C(N_1N_3) + ... + C(N_1N_n) + C(N_2N_3...N_n) = \\ &(N_1 :_M Ann_R(N_2)) + ... + (N_1 :_M Ann_R(N_n)) + \\ &(0 :_M Ann_R(N_2)Ann_R(N_3)...Ann_R(N_n)) \\ &(N_1 :_M Ann_R(N_2)Ann_R(N_3)...Ann_R(N_n)) = C(N_1N_2...N_n) \end{aligned}$$

■

Let R be a be a principal ideal domain and M be an R-module. By [6, 2.12], every submodule of M is pure if and only if it is copure. But the following examples shows that it is not true for n-pure and n-copure submodules.

Example 2.8. Let $n > 1$ be an integer. Consider the submodule $G_1 := \langle 1/p + \mathbb{Z} \rangle$ of the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$. Then the submodule G_1 of the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is a n-pure submodule but it is not n-copure.

Example 2.9. Let $n > 1$ be an integer. The submodule $2\mathbb{Z}$ of the $\mathbb{Z}$-module $\mathbb{Z}$ is a n-copure submodule but it is not n-pure.

A proper submodule N of an R-module M is said to be *completely irreducible* if $N = \bigcap_{i\in I} N_i$, where $\{N_i\}_{i\in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [12].

Remark 2.10. Let N and K be two submodules of an R-module M. To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$.

An R-module M satisfies *the double annihilator conditions* (*DAC* for short) if for each ideal I of R, we have $I = Ann_R((0 :_M I))$. M is said to be a *strong comultiplication module* if M is a comultiplication R-module which satisfies the double annihilator conditions [6].

A family $\{N_i\}_{i\in I}$ of submodules of an R-module M is said to be an *inverse family of submodules of* M if the intersection of two of its submodules again contains a module in $\{N_i\}_{i\in I}$. Also M *satisfies the property* $AB5^*$ if for every submodule K of M and every inverse family $\{N_i\}_{i\in I}$ of submodules of M, $K + \cap_{i\in I} N_i = \cap_{i\in I}(K + N_i)$ [16]. For example, every strong comultiplication R-module satisfies the property $AB5^*$ by using Lemma [11, 2.2] and [2, 2.9].

Theorem 2.11. Let M be an R-module which satisfies the property $AB5^*$ and let n be a positive integer. Then we have the following.

(a) If $\{N_\lambda\}_{\lambda\in\Lambda}$ is a chain of n-copure submodules of M, then $\cap_{\lambda\in\Lambda} N_\lambda$ is a n-copure submodule of M.

(b) If $\{N_\lambda\}_{\lambda\in\Lambda}$ is a chain of submodules of M and K is a n-copure submodule of N_λ for each $\lambda \in \Lambda$, then K is a n-copure submodule of $\cap_{\lambda\in\Lambda} N_\lambda$.

Proof. (a) Let $I_1, I_2, ..., I_n$ be proper ideals of R. Clearly,

$$(\cap_{\lambda\in\Lambda} N_\lambda :_M I_1) + (\cap_{\lambda\in\Lambda} N_\lambda :_M I_2) + ... + (\cap_{\lambda\in\Lambda} N_\lambda :_M I_n) + (0 :_M I_1 I_2 ... I_n) \subseteq$$
$$(\cap_{\lambda\in\Lambda} N_\lambda :_M I_1 I_2 ... I_n).$$

Let L be a completely irreducible submodule of M such that

$$(\cap_{\lambda\in\Lambda} N_\lambda :_M I_1) + (\cap_{\lambda\in\Lambda} N_\lambda :_M I_2) + ... + (\cap_{\lambda\in\Lambda} N_\lambda :_M I_n) + (0 :_M I_1 I_2 ... I_n) \subseteq L.$$

Then we have

$$(\cap_{\lambda\in\Lambda} N_\lambda :_M I_1) + (\cap_{\lambda\in\Lambda} N_\lambda :_M I_2) + ... + (\cap_{\lambda\in\Lambda} N_\lambda :_M I_n) + (0 :_M I_1 I_2 ... I_n) + L = L.$$

Since M satisfies the property $AB5^*$, we have

$$\cap_{\lambda\in\Lambda}((N_\lambda :_M I_1) + (\cap_{\lambda\in\Lambda} N_\lambda :_M I_2) + ... + (\cap_{\lambda\in\Lambda} N_\lambda :_M I_n) + (0 :_M I_1 I_2 ... I_n) + L) = L.$$

Now as L is a completely irreducible submodule of M, there exists $\alpha_1 \in \Lambda$ such that

$$(N_\alpha :_M I_1) + (\cap_{\lambda\in\Lambda} N_\lambda :_M I_2) + ... + (\cap_{\lambda\in\Lambda} N_\lambda :_M I_n) + (0 :_M I_1 I_2 ... I_n) + L = L.$$

Since M satisfies the property $AB5^*$,

$$\cap_{\lambda\in\Lambda}((N_\alpha :_M I_1)+(N_\lambda :_M I_2)+...+(\cap_{\lambda\in\Lambda}N_\lambda :_M I_n)+(0:_M I_1I_2...I_n)+L)=L.$$

Now again as L is a completely irreducible submodule of M, there exists $\alpha_2\in\Lambda$ such that

$$(N_{\alpha_1} :_M I_1)+(N_{\alpha_2} :_M I_2)+...+(\cap_{\lambda\in\Lambda}N_\lambda :_M I_n)+(0:_M I_1I_2...I_n)+L=L.$$

By continuing in this way, we have there exist $\alpha_3,...\alpha_n\in\Lambda$ such that

$$(N_{\alpha_1} :_M I_1)+(N_{\alpha_2} :_M I_2)+...+(N_{\alpha n} :_M I_n)+(0:_M I_1I_2...I_n)+L=L.$$

We can assume that $N_{\alpha 1}\subseteq N_{\alpha 2}\subseteq ...\subseteq N_{\alpha n}$. Therefore,

$$(N_{\alpha_1} :_M I_1)+(N_{\alpha_1} :_M I_2)+...+(N_{\alpha 1} :_M I_n)+(0:_M I_1I_2...I_n)+L\subseteq L.$$

It follows that $(N_{\alpha 1} :_M I_1I_2...I_n)\subseteq L$ since $N_{\alpha 1}$ is a n-copure submodule of M. Hence, $(\cap_{\lambda\in\Lambda}N_\lambda :_M I_1I_2...I_n)\subseteq L$. This implies that

$$(\cap_{\lambda\in\Lambda}N_\lambda :_M I_1I_2...I_n)\subseteq$$
$$(\cap_{\lambda\in\Lambda}N_\lambda :_M I_1)+(\cap_{\lambda\in\Lambda}N_\lambda :_M I_2)+...+(\cap_{\lambda\in\Lambda}N_\lambda :_M I_n)+(0:_M I_1I_2...I_n).$$

by Remark 2.10.

(b) Let $I_1,I_2,...,I_n$ be proper ideals of R. Clearly,

$$(K:_{\cap_{\lambda\in\Lambda}N_\lambda} I_1)+(K:_{\cap_{\lambda\in\Lambda}N_\lambda} I_2)+...+(K:_{\cap_{\lambda\in\Lambda}N_\lambda} I_n)+(0:_{\cap_{\lambda\in\Lambda}N_\lambda} I_1I_2...I_n)\subseteq .$$
$$(K:_{\cap_{\lambda\in\Lambda}N_\lambda} I_1I_2...I_n).$$

To see the reverse inclusion, let L be a completely irreducible submodule of M such that

$$(K:_{\cap_{\lambda\in\Lambda}N_\lambda} I_1)+(K:_{\cap_{\lambda\in\Lambda}N_\lambda} I_2)+...+(K:_{\cap_{\lambda\in\Lambda}N_\lambda} I_n)+(0:_{\cap_{\lambda\in\Lambda}N_\lambda} I_1I_2...I_n)\subseteq L.$$

Then

$$\cap_{\lambda\in\Lambda}(K:_{N_\lambda} I_1)+(K:_{\cap_{\lambda\in\Lambda}N_\lambda} I_2)+...+(K:_{\cap_{\lambda\in\Lambda}N_\lambda} I_n)+(0:_{\cap_{\lambda\in\Lambda}N_\lambda} I_1I_2...I_n)+L=L.$$

Since M satisfies the property $AB5^*$, we have

$$\cap_{\lambda\in\Lambda}((K:_{N_\lambda} I_1)+(K:_{\cap_{\lambda\in\Lambda}N_\lambda} I_2)+...+(K:_{\cap_{\lambda\in\Lambda}N_\lambda} I_n)+(0:_{\cap_{\lambda\in\Lambda}N_\lambda} I_1I_2...I_n))+L)=L.$$

Now as L is a completely irreducible submodule of M, there exists $\alpha_1\in\Lambda$ such that

$$(K:_{N_{\alpha 1}} I_1)+(K:_{\cap_{\lambda\in\Lambda}N_\lambda} I_2)+...+(K:_{\cap_{\lambda\in\Lambda}N_\lambda} I_n)+(0:_{\cap_{\lambda\in\Lambda}N_\lambda} I_1I_2...I_n)+L=L.$$

By similar argument, since M satisfies the property $AB5^*$ and L is a completely irreducible submodule of M, there exist $\alpha_2, \alpha_3, ...\alpha_n \in \Lambda$ such that,

$$(K :_{N_{\alpha 1}} I_1) + (K :_{N_{\alpha 2}} I_2) + ... + (K :_{N_{\alpha n}} I_n) + (0 :_{\cap_{\lambda\in\Lambda} N_\lambda} I_1 I_2 ... I_n) + L = L.$$

Since $\{N_\lambda\}_{\lambda\in\Lambda}$ is a chain, we can assume that $N_{\alpha 1} \subseteq N_{\alpha 2} \subseteq ... \subseteq N_{\alpha n}$. Therefore,

$$(K :_{N_{\alpha 1}} I_1) + (K :_{N_{\alpha 1}} I_2) + ... + (K :_{N_{\alpha 1}} I_n) + (0 :_{N_{\alpha 1}} I_1 I_2 ... I_n) + L = L.$$

It follows that $(K :_{N_{\alpha 1}} I_1 I_2 ... I_n) \subseteq L$ since K is a n-copure submodule of N_α. Therefore, $(K :_{\cap_{\lambda\in\Lambda} N_\lambda} I_1 I_2 ... I_n) \subseteq L$. This implies that

$$(K :_{\cap_{\lambda\in\Lambda} N_\lambda} I_1 I_2 ... I_n) \subseteq$$
$$(K :_{\cap_{\lambda\in\Lambda} N_\lambda} I_1) + (K :_{\cap_{\lambda\in\Lambda} N_\lambda} I_2) + ... + (K :_{\cap_{\lambda\in\Lambda} N_\lambda} I_n) + (0 :_{\cap_{\lambda\in\Lambda} N_\lambda} I_1 I_2 ... I_n).$$

by Remark 2.10. ■

Theorem 2.12. Let M be an R-module which satisfies the property $AB5^*$, N a submodule of M, and let n be a positive integer. Then there is a submodule K of M minimal with respect to $N \subseteq K$ and K is a n-copure submodule of M.

Proof. Let
$$\Sigma = \{N \leq H | H \ is \ a \ n-copure \ submodule \ of \ M\}.$$

Then $M \in \Sigma$ and so $\Sigma \neq \emptyset$. Let $\{N_\lambda\}_{\lambda\in\Lambda}$ be a totally ordered subset of Σ. Then $N \leq \cap_{\lambda\in\Lambda} N_\lambda$ and by Theorem 2.11 (a), $\cap_{\lambda\in\Lambda} N_\lambda$ is a n-copure submodule of M. Therefore by using Zorn's Lemma, one can see that Σ has a minimal element, K say as disired. ■

Theorem 2.13. Let M be a strong comultiplication R-module, N a submodule of M, and let n be a positive integer. Then N is a n-copure submodule of M if and only if $Ann_R(N)$ is a n-pure ideal of R.

Proof. Since M is a comultiplication R-module, $N = (0 :_M Ann_R(N))$. Let N be a n-copure submodule of M and let $I_1, I_2, ..., I_n$ be proper ideals of R. Then

$$(N :_M I_1 I_2 ... I_n) = (N :_M I_1) + (N :_M I_2) + ... + (N :_M I_n) + (0 :_M I_1 I_2 ... I_n)$$

implies that

$$((0 :_M Ann_R(N)) :_M I_1 I_2 ... I_n) =$$
$$((0 :_M Ann_R(N)) :_M I_1) + ((0 :_M Ann_R(N)) :_M I_2) + ...+$$
$$((0 :_M Ann_R(N)) :_M I_n) + (0 :_M I_1 I_2 ... I_n)$$

It follows that

$$(0:_M Ann_R(N)I_1I_2...I_n) = (0:_M Ann_R(N)I_1) + (0:_M Ann_R(N)I_2) + ... +$$
$$(0:_M Ann_R(N)I_n) + (0:_M I_1I_2...I_n)$$

Thus by [11, 2.2],

$$(0:_M Ann_R(N)I_1I_2...I_n) =$$
$$(0:_M Ann_R(N)I_1 \cap Ann_R(N)I_2 \cap ... \cap Ann_R(N)I_n \cap (I_1I_2...I_n)).$$

This implies that

$$Ann_R(N)I_1I_2...I_n = Ann_R(N)I_1 \cap Ann_R(N)I_2 \cap ... \cap Ann_R(N)I_n \cap (I_1I_2...I_n).$$

since M is a strong comultiplication R-module. Hence $Ann_R(N)$ is a n-pure ideal of R. Conversely, let $Ann_R(N)$ be a n-pure ideal of R and let $I_1, I_2, ..., I_n$ be proper ideals of R. Then

$$Ann_R(N)I_1I_2...I_n = Ann_R(N)I_1 \cap Ann_R(N)I_2 \cap ... \cap Ann_R(N)I_n \cap I_1I_2...I_n.$$

Hence by using [11, 2.2],

$$(0:_M Ann_R(N)I_1I_2...I_n) = (0:_M Ann_R(N)I_1) + (0:_M Ann_R(N)I_2) + ... +$$
$$(0:_M Ann_R(N)I_n) + (0:_M I_1I_2...I_n).$$

Therefore, as M is a comultiplication R-module,

$$(N:_M I_1I_2...I_n) = (N:_M I_1) + (N:_M I_2) + ... + (N:_M I_n) + (0:_M I_1I_2...I_n),$$

as desired. ■

Acknowledgments. The author would like to thank Prof. Habibollah Ansari-Toroghy for his helpful suggestions and useful comments.

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