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On Optimal Control of the Initial Status in a Hyperbolic System

Hiperbolik Bir Sistemde Başlangıç Konumunun Optimal Kontrolü Üzerine

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Abstract
In this study, optimal control problem governed by a hyperbolic problem with Dirichlet conditions is considered. It is demonstrated that the optimal solution for the considered optimal control problem is exist and unique and it is obtained adjoint problem. Derivative of the cost functional is calculated utilizing from adjoint problem. Finally, necessary optimality conditions for hyperbolic system are derived.

Keywords: Frechet Derivative, Hyperbolic Equations, Optimal Control

Öz
Bu makalede Dirichlet koşuluna sahip hiperbolik sistem ile yönetilen optimal kontrol problem göz önüne alınır. Optimal çözümün var ve tek olduğu kanıtlanır ve eşlenik problem elde edilir. Eşlenik problemden yararlanılarak amaç fonksiyonunun gradyeni hesaplanır. Hiperbolik sistem için gerekli optimallik şartları türetilir.

Anahtar kelimeler*: Frechet Türev, Hiperbolik Denklemler, Optimal Kontrol*

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1. Introduction

1.1. Formulation of the Problem

As it is known, hyperbolic partial differential equations can be used to describe many physical phenomena. Some of them are heat conduction, vibration of elastic material, diffusion-reaction processes, population systems, and many others. However, there has been much attention to studies related with optimal control problems involving hyperbolic equation in recent years. It has been found too many researches about these problems using various control functions in literature. When these researches are analyzed, it has been seen that the control function is at the right hand side of equation, in the coefficient or on boundary for hyperbolic equation. But, there are too little researches about the initial control for hyperbolic problem.

Some of these important studies can be summarized as follows.

The problem of controlling the coefficient function has been studied for linear hyperbolic equation using the cost functional in (Tagiyev, 2012). The coefficient function has controlled for nonlinear hyperbolic equation using the functional involved in (Kröner, 2011). In (Bahaa, 2012), various optimal boundary control problems for linear infinite order distributed hyperbolic systems involving constant time lags are considered. It has been studied optimal control problems for the hyperbolic equations with a damping term involving p-Laplacian in (Ju ve Jeong, 2013). In (Hwang ve Nakagiri 2006), it has been examined optimal control problems for the equation of motion of membrane with strong viscosity. The tensions of end points for the vibration problem have been controlled using the cost functional in (Subaşı vd., 2017). Similar problems with different controls and cost functionals have been examined in (Lions, 1971; Yeloğlu ve Subaşı, 2010; Bahaa, 2011).

In this study, we handle the following hyperbolic system

$$u_{tt}-u_{xx}+q(x)u=F(x,t),\quad (x,t)\in\Omega \tag{1}$$

$$u(x,0)=\varphi(x),\quad u_t(x,0)=\psi(x),\quad x\in(0,l) \tag{2}$$

$$u(0,t)=0,\quad u(l,t)=0,\quad t\in(0,T) \tag{3}$$

defined on the domain $\Omega:=(x,t)\in(0,l)\times(0,T)$. We control the initial status minimizing the cost functional

$$J_\alpha(\varphi)=\iint_\Omega\left[u(x,t)-y_1(x,t)\right]^2 dxdt \\ +\int_0^l\left[u_t(x,T)-y_2(x)\right]^2 dx+\alpha\|\varphi\|^2_{W_2^1(0,l)} \tag{4}$$

on the set ϕ which is closed, convex subset of $W_2^1(0,l)$.

Namely, aim of this study is to deal with the problem of

$$\inf_{\varphi\in\phi} J_\alpha(\varphi) \tag{5}$$

assuming that the conditions $\varphi(x)\in W_2^1(0,l)$, $\psi(x)\in L_2(0,l)$ are hold.

For cost functional, $y_1(x,t)\in L_2(\Omega)$ is the state function to which $u(x,t)$ must be close enough and $w(x)\in L_2(0,l)$ is the final speed to which $u_t(x,T)$ must be close enough. $\alpha>0$ is regularization parameter.

In brief, in this study, we show that the initial status of the system can be controlled for hyperbolic problem by minimizing the cost functional $J_\alpha(\varphi)$.

In literature, there is no much research on optimal control of the initial status for hyperbolic system. So, this study is important in view of making the contribution on the initial control for theoretical and numerical investigations.

We introduce the spaces used in this article.

The space $L_2(\Omega)$ describes space of square integrable functions. The norm and inner product on this space are given by

$$\langle u,v\rangle_{L_2(\Omega)}=\iint_\Omega(u.v)\,dxdt\,,$$

$$\|u\|_{L_2(\Omega)}=\sqrt{\langle u,u\rangle_{L_2(\Omega)}}.$$

The space $W_2^1(\Omega)$ is a space consisting of all elements $L_2(\Omega)$ with generalized derivatives of first.

$$\langle f,g\rangle_{W_2^1(\Omega)}=\iint_\Omega\left(f.g+\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial x}+\frac{\partial f}{\partial t}\cdot\frac{\partial g}{\partial t}\right)dxdt\ ,$$
$$\|f\|_{W_2^1(\Omega)}=\sqrt{\langle f,f\rangle_{W_2^1(\Omega)}}.$$

We organize this paper as follows. Firstly, we introduce some definitions and preliminary results. Then, we show that the weak solution and optimal solution is exist and unique. Later, we obtain the adjoint problem for the considered problem and calculate Frechet derivative of the cost functional. Also, we establish necessary optimality conditions for optimal solution.

2. Material and Method

2.1. Existence and Uniqueness Theorem

Firstly, we show that the weak solution of hyperbolic problem is exist, unique and continuous dependence according to initial data under some conditions.

The generalized solution of the problem (1)-(3) is the function $u\in \overset{\circ}{W}{}_2^1(\Omega)$ satisfying the following integral equality;

$$\iint_\Omega[-u_t\eta_t+u_x\eta_x+q(x)u\eta]\,dxdt=\iint_\Omega F(x,t)\eta dxdt+\int_0^l\psi(x)\eta(x,0)dx \tag{6}$$

for $\forall\ \eta\in \overset{\circ}{W}{}_2^1(\Omega),\ \ \eta(x,T)=0.$

It can be seen in (Ladyzhenskaya, 1985) that weak solution is exist, unique and satisfies the inequality

$$\|u\|^2_{W_2^1(\Omega)}\le c_0\left(\|\varphi\|^2_{W_2^1(0,l)}+\|\psi\|^2_{L_2(0,l)}+\|F\|^2_{L_2(0,l)}\right) \tag{7}$$

Here c_0 is independent from φ and ψ.

Let give an increment $\delta\varphi(x)\in W_2^1(0,l)$ to the control function $\varphi(x)$ such as $\varphi+\delta\varphi\in\phi$. Then the difference function $\delta u=\delta u(x,t)=u(x,t;\varphi+\delta\varphi)-u(x,t;\varphi)$ is the solution of the following difference initial-boundary problem;

$$\delta u_{tt}-\delta u_{xx}+q(x)\delta u=0 \tag{8}$$

$$\delta u(x,0)=\delta\varphi,\quad \delta u_t(x,0)=0 \tag{9}$$

$$\delta u(0,t)=0,\ \ \delta u(l,t)=0 \tag{10}$$

It can easily be seen from (7) that the solution of above difference initial-boundary problem holds the inequality

$$\|\delta u(.,t)\|^2_{L_2(0,l)}\le c_1\|\delta\varphi\|^2_{W_2^1(0,l)},\ \ \forall t\in[0,T] \tag{11}$$

Here c_1 is independent from $\delta\varphi$.

We will benefit from Goebel's theorem in order to demonstrate the existence and the uniqueness of optimal solution for problem (1)-(5) (Goebel, 1970).

Theorem (Goebel Theorem) : Let H be a uniformly convex Banach space and the set ϕ be a closed, bounded and convex subset of H. If $\alpha>0$ and $\beta\ge1$ are given numbers and the functional $J(\varphi)$ is lower semi continuous and bounded from below on the set ϕ then there is a dense set G of H that the functional

$$J_\alpha(\varphi)=J(\varphi)+\alpha\|\varphi\|_H^\beta \tag{12}$$

takes its minimum on the set ϕ. If $\beta>1$ then minimum is unique.

Now, we can show that optimal solution is exist and unique using this theorem.

The set $W_2^1(0,l)$ is a uniformly convex Banach space (Yosida, 1980), the set ϕ is a closed, bounded and convex subset of $W_2^1(0,l)$.

On the other hand, for the increment of the functional $J(\varphi)$, the following inequality is valid;

$$|\delta J(\varphi)|\le c_2\left(\|\delta\varphi\|_{W_2^1(0,l)}+\|\delta\varphi\|^2_{W_2^1(0,l)}\right) \tag{13}$$

From this inequality, we can say that this functional is also lower semi continuous and bounded from below on the set ϕ. So, $J_\alpha(\varphi)$ has minimum on the set ϕ. Namely, the optimal solution is exist.

Finally, since $\beta=2$, the optimal solution is unique.

3. Main Results

3.1. Adjoint Problem and Derivative of the Functional

In this section, we write the Lagrange functional used for finding adjoint problem, before we show that the cost functional $J_\alpha(\varphi)$ is Frechet differentiability on the set ϕ. Augmented functional for the problem is

$$\tilde{J}_\alpha(u,\varphi,\eta)=\iint_\Omega\left[u(x,t)-y_1(x,t)\right]^2dx+\int_0^l\left[u_t(x,T)-y_2(x)\right]^2dxdt+\alpha\|\varphi\|^2_{W_2^1(0,l)}$$
$$+\int_0^T\int_0^l\left[u_{tt}-u_{xx}+q(x)u-F(x,t)\right]\eta_t dxdt \tag{14}$$

The first variation of this functional is obtained such as;

$$\delta\tilde{J}_\alpha(\varphi)=\int_0^l\int_0^T\left[2u(x,t)-2y_1(x,t)-\eta_{tt}+\eta_{xx}-q(x)\eta+\delta u(x,t)\right]\delta u(x,t)dxdt$$
$$+\int_0^l\left[2u_t(x,t)-2y_2(x)-\eta_t(x,T)+\delta u_t(x,T)\right]\delta u_t(x,T)dx$$
$$-\int_0^l\eta_x(x,0)\delta u_x(x,0)dx-\int_0^l\eta(x,0)\delta u(x,0)dx \tag{15}$$
$$+\alpha\int_0^l\left(2\varphi\delta\varphi+2\varphi_x\delta\varphi_x+\varphi_x^2\right)dx$$

By means of stationary condition $\delta\tilde{J}_\alpha=0$, the following adjoint boundary value problem is found;

$$\eta_{tt}-\eta_{xx}+q(x)\eta=2\left[u(x,t)-y_1(x,t)\right] \tag{16}$$
$$\eta_x(x,T)=0,\quad \eta_t(x,T)=2\left[u_t(x,T)-y_2(x)\right] \tag{17}$$
$$\eta(0,t)=0,\quad \eta(l,t)=0 \tag{18}$$

For $\forall\gamma\in\overset{o}{W}{}_2^1(\Omega)$, the function $\eta\in C^1\left([0,T],L_2(0,l)\right)\cap C^0\left([0,T],W_2^1(0,l)\right)$ which satisfies the following equality

$$\int_0^T\int_0^l\left[-\eta_t\gamma_t+\eta_x\gamma_x+q(x)\eta\gamma-2\left[u(x,t)-y_1(x,t)\right]\gamma\right]dxdt=\int_0^l\eta_t(x,0)\gamma(x,0)dx$$
$$-\int_0^l 2\left[u_t(x,T)-y_2(x)\right]\gamma(x,T)dx \tag{19}$$

is the solution of adjoint boundary value problem (16)-(18).

From the conditions of the difference problem, we rewrite the equality (1.15) as follows;

$$\delta J_\alpha(\varphi)=\int_0^l\left(-\eta(x,0)\delta\varphi-\eta_x(x,0)\delta\varphi_x+2\alpha\varphi\delta\varphi+2\alpha\varphi_x\delta\varphi_x\right)dx+o\left(\|\delta\varphi\|^2_{W_2^1(0,l)}\right)$$
$$=\left\langle-\eta(x,0)+2\alpha\varphi,\delta\varphi\right\rangle_{W_2^1(0,l)}+o\left(\|\delta\varphi\|^2_{W_2^1(0,l)}\right).$$

So, we obtain the derivative of the cost functional from definition of Frechet derivative such as;

$$J'_\alpha(\varphi)=-\eta(x,0)+2\alpha\varphi \tag{20}$$

3.2. Condition for Optimal Solution

After calculating the derivative of the functional, it can be said that the derivative $J'_\alpha(\varphi)$ is continuous on the set ϕ. The fact that the functional $J_\alpha(\varphi)$ is continuously differentiable on the set ϕ and the set ϕ is convex, in that case the following inequality is valid according to theorem in (Vasilyev, 1981);

$$\left\langle J'_\alpha(\varphi^*), \varphi-\varphi^*\right\rangle_{W_2^1(0,l)} \geq 0, \quad \forall\varphi\in\phi \qquad (21)$$

Therefore, necessary condition for optimal solution is given by the following inequality;

$$\left\langle -\eta(x,0)+2\alpha\varphi^*, \varphi-\varphi^*\right\rangle_{W_2^1(0,l)} \geq 0, \quad \forall\varphi\in\phi \quad (22)$$

4. Conclusion

This study presents the optimality conditions for an optimal control problem involving hyperbolic equation. The initial status has been chosen as the control function in this paper. We obtain an adjoint problem for problem (1)-(3) and calculate derivative of the cost functional via adjoint problem on the space $W_2^1(0,l)$. The derivative of the functional is highly important for obtaining a control which converges optimal solution. Therefore, this study leads to numerical investigations related to obtaining an optimal solution.

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