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Transfer Fonksiyonlarının Gerçeklenmesi için İki Analitik Ayrıklaştırma Yönteminin Performans Değerlendirmesi

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Öz

Yakın tarihli araştırmalar kesirli aritmetiğin gerçek sistemlerin daha doğru modellemesini sağladığı rapor etmiştir. Bu nedenle, kesir dereceli sistem modelleri simülasyon ve nümerik analizlerde yaygın olarak faydalanılmaya başlandı. Ancak, ayrık zaman gerçeklemelerinin yüksek işlem karmaşıklığından dolayı mühendislik uygulamalarının çalışma aralıkları içinde kesir dereceli eleman ve transfer fonksiyonlarının yeterli doğrulukta sayısal olarak gerçeklenmesine ihtiyaç duyulmaktadır. Bu çalışma uygulama bakış açısı ile iki analitik ayrık yakınsama yaklaşımının frekans cevabı eşleşme özelliklerini incelemektedir: Bunlardan biri Tustin özyinelemeli yakınsaması yöntemi olarak bilinen doğrudan ayrıklaştırma yöntemidir ve diğeri kesir dereceli türev operatörünün sürekli kesir açılımından (CFE) faydalanan dolaylı bir ayrıklaştırma yaklaşımıdır. Bu iki yöntemin sonuçları karşılaştırılmakta ve yöntemlerin uygulanabilirlikleri, kontrol sistemleri ve filtre gerçekleme uygulamalarının çalışma frekans aralıkları gereksinimleri temelinde tartışılmaktadır.

Anahtar kelimeler: Kesir dereceli sistemler, kesir dereceli transfer fonksiyonu, analitik ayrıklaştırma yöntemleri, Tustin özyineleme

Performance Evaluation of Two Analytical Discretization Methods for Implementation of Fractional Order Transfer Functions

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Abstract

Recent researches reported that fractional calculus provides more accurate modeling of real systems. For this reason, fractional order system models begun to utilized widely in numerical analysis and system simulations. However, due to high computational complexity of discrete time domain realization, there is need for accurate digital implementation of fractional order elements and transfer functions in operating ranges of applications. This study investigates frequency response matching properties of two analytical discrete approximation approaches for the application point of view: One is a direct discretization method, which is known as Tustin recursive approximation, and the other is an indirect disretization method, which benefits from continued fraction expansion (CFE) of fractional order derivative operator. Results of these two methods are compared and applicability of the methods are discussed on the bases of operating range requirements of control and filter realization applications.

Keywords: Fractional order systems, fractional order transfer function, analytic discretization methods, Tustin recursive, continued fraction expansion.

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1. INTRODUCTION

Recently, there is a growing interest for utilization of fractional order system models in science and engineering problems [1-3]. Fractional order elements allow better representations of real word systems in frequency domain because the amplitude and phase response of fractional order derivative element s^r, $r \in R$ can provide a slope of a fraction of 20 dB ($20r$ dB) in amplitude response characteristics and a fraction of 90° ($90r$ decrees) in phase response characteristics [4,5].

Main concern in digital realization of fractional order models is the requirement of high computational power because fractional order derivative is not local operator in time domain because the current value of fractional order derivative of a function is dependent of the past values of this function, which was also referred as to long-memory effect [1-2]. In other words, computation of fractional order derivative is not localized to current values of a function and it spreads to previous values.

Therefore, realization of ideal fractional order elements in time domain solutions needs an increasing computation resource as time progresses so that the number of memory elements to hold past values increases. To deal with this problem, integer order approximate models are used to implement fractional order models.

Such approximate models provide an opportunity for digital realization with a low complexity and adequate accuracy within operating ranges of engineering applications. Since digital realization of ideal fractional order elements is not practical for real digital hardware, Finite Impulse Response (FIR) or Infinite Impulse response (IIR) filter approximations of fractional order models are commonly utilized for discrete implementation of fractional order elements or fractional order system models.

During the last decade, several analytical discretization methods have been proposed for digital realization of fractional order systems [6-9]. These methods are mainly classified in two groups: One is direct methods that can allow directly discretization of original fractional order functions in the form of a discrete filter. The other is indirect methods that are carried out in more steps. A continuous integer order approximation of original fractional order transfer functions is first obtained, and then those integer order approximations are discretized by using one of well known direct discretization methods.

In this study, direct Tustin recursive discretization and indirect CFE approximation methods are compared, and performances of these methods are evaluated for applicability in control and signal processing applications.

2. PROBLEM FORMULATON AND THEORETICAL BACKGROUND

2.1. Fundamentals of Fractional Calculus and Fractional Order System Modeling

Fractional calculus is an extension of integer order integration and differentiation operations to non-integer order operators, which was denoted by ${}_aD_t^{\alpha}$ [6].

$$
{}_aD_t^{\alpha} = \begin{cases} \dfrac{d^{\alpha}}{dt^{\alpha}} & \alpha > 0, \\ 1 & \alpha = 0, \\ \int\limits_a^{\alpha} (dt)^{-\alpha} & \alpha < 0, \end{cases} \tag{1}
$$

For fractional order derivative and integration operation, a and t are the lower and upper bounds of the operation, and $\alpha \in R$ is the fractional order (non-integer order). This general description of ${}_aD_t^{\alpha}$ implements classical integer order differentiation and integration operations for integer values of α. The Caputo definition gains significance in system analysis and design problems. The Caputo definition of fractional order differentiation was given based on Euler's gamma function $\Gamma(.)$, as follows [2,3],

$$
{}_aD_t^{\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{n}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau , \tag{2}
$$

where $n-1 \leq \alpha < n$. Laplace transform of ${}_0D_t^{\alpha} f(t)$ was written as $L(D^{\alpha} f(t)) = s^{\alpha} F(s)$ for zero initial conditions [2,3,6]. By using this property, time domain fractional order system models was written by fractional order differential equations in a general form as [1-3,6],

$$
\begin{aligned}
& a_n D^{\alpha_n} y(t) + a_{n-1} D^{\alpha_{n-1}} y(t) + \\
& ... + a_1 D^{\alpha_1} y(t) + a_0 y(t) \\
& = b_m D^{\varphi_m} u(t) + b_{m-1} D^{\varphi_{m-1}} u(t) + . , \\
& .. + b_1 D^{\varphi_1} u(t) + b_0 u(t)
\end{aligned} \tag{3}
$$

By applying Laplace transform, it is expressed in s-domain as [1-3,6]

$$
T(s) = \frac{Y(s)}{U(s)} = \frac{\sum_{i=0}^{m} b_i s^{\varphi_i}}{\sum_{i=0}^{n} a_i s^{\alpha_i}} . \tag{4}
$$

This is a general form of transfer function modeling of fractional order systems, and it facilitates design and analysis of fractional order control systems in frequency domain by using $s^{\alpha} = (j\omega)^{\alpha} = \omega^{\alpha}(\cos(\frac{\pi}{2}\alpha) + j\sin(\frac{\pi}{2}\alpha))$ [1-3,6]. The denominator polynomial coefficients, a_i and coefficients,

b_i are positive real numbers, and fractional orders are $\alpha_i \in R$, and $\varphi_i \in R$. For the constant terms of denominator polynomial and numerator polynomial orders, one can set $\alpha_0 = 0$, and $\varphi_0 = 0$.

2.2. Direct Discretization by Recursive Tustin Transformation

Direct discretization by recursive Tustin transformation differentiator was explained in Ref [6,7]. This method is based on Muir-recursion scheme, which was originally used in geophysical data processing with applications to petroleum prospecting [7,10]. This scheme was used in recursive discretization of fractional-order differentiator according to Tustin generating function. The Tustin generating function for discretization of integer order derivative element was given for a data sampling period of T as,

$$s \approx \left(\frac{2}{T}\right)\frac{1-z^{-1}}{1+z^{+1}} \tag{5}$$

One of the promising property of Tustin transformation, which is also called Bilinear transformation, is the stability preservation. It maps left half plane of continuous complex domain to unit circle of discrete complex domain and it preserves stability of model in this transformation. Tustin generating function takes a fractional order and discretization of fractional order derivative was expressed as [7],

$$s^r \approx \left(\frac{2}{T}\right)^r \frac{(1-z^{-1})^r}{(1+z^{-1})^r} = \left(\frac{2}{T}\right)^r \lim_{n\to\infty} \frac{A_n\left(z^{-1},r\right)}{A_n\left(z^{-1},-r\right)} \tag{6}$$

Due to finite memory recourses of digital hardware, for a given fractional order r, discrete approximation of s^r was written for n^{th} order of discrete IIR filter implementation as follows [7].

$$s^r \approx \left(\frac{2}{T}\right)^r \frac{A_n\left(z^{-1},r\right)}{A_n\left(z^{-1},-r\right)}, \tag{7}$$

where, $A_n\left(z^{-1},r\right)$ terms are calculated recursively [7].

$$A_0\left(z^{-1},r\right)=1,\; A_n\left(z^{-1},r\right)= A_{n-1}\left(z^{-1},r\right)-c_n z^n A_{n-1}\left(z^{-1},r\right)$$

$$c_n = \begin{Bmatrix} r/n, & n \text{ is odd} \\ 0 & n \text{ is even} \end{Bmatrix} \tag{8}$$

Table 1 lists the expressions of $A_n\left(z^{-1},r\right)$ up to 9^{th} order IIR filter approximation. A fractional order transfer function is discretized by substituting each fractional order elements s^r terms in transfer function with discrete IIR filter approximations.

Table 1. List of formula for different orders [6,7]

n	$A_n\left(z^{-1},r\right)$
0	1
1	$-rz^{-1}+1$
3	$-\frac{1}{3}rz^{-3}+\frac{1}{3}r^2z^{-2}-rz^{-1}+1$
5	$-\frac{1}{5}rz^{-5}+\frac{1}{5}r^2z^{-4}-\left(\frac{1}{3}r+\frac{1}{15}r^3\right)z^{-3}$ $+\frac{2}{5}r^2z^{-2}-rz^{-1}+1$
7	$-\frac{1}{7}rz^{-7}+\frac{1}{7}r^2z^{-6}-\left(\frac{1}{5}r+\frac{2}{35}r^3\right)z^{-5}$ $+\left(\frac{26}{105}r^2+\frac{1}{105}r^4\right)z^{-4}$ $-\left(\frac{1}{3}r+\frac{2}{21}r^3\right)z^{-3}+\frac{3}{7}r^2z^{-2}-rz^{-1}+1$
9	$-\frac{1}{9}rz^{-9}+\frac{1}{9}r^2z^{-8}-\left(\frac{1}{7}r+\frac{1}{21}r^3\right)z^{-7}$ $+\left(\frac{34}{189}r^2+\frac{2}{189}r^4\right)z^{-6}$ $-\left(\frac{1}{5}r+\frac{16}{189}r^3+\frac{1}{945}r^5\right)z^{-5}$ $+\left(\frac{17}{63}r^2+\frac{1}{63}r^4\right)z^{-4}$ $-\left(\frac{1}{3}r+\frac{1}{9}r^3\right)z^{-3}+\frac{4}{9}r^2z^{-2}-rz^{-1}+1$

2.3. Indirect Discrete Approximations by Using CFE Method

Series expansion methods are frequently used to obtain approximate expressions of functions. The CFE is a series expansion and finite truncation of a continuous function by the CFE method is a rational function [5,11,12,13]. The CFE method provides approximation to a fractional order element in limited frequency ranges.

CFE of fractional order elements is based on continuous fraction expansion of the term $(1+x)^\alpha$ [5,8,14,15].

$$(1+x)^{\alpha} = b_0 + \cfrac{a_1 x}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_2 x}{b_3 + \cfrac{a_3 x}{.....}}}} \qquad (9)$$

The terms of this expansion are written $\{b_0; \frac{a_1 x}{b_1}; \frac{a_2 x}{b_2}; \frac{a_3 x}{b_3};....; \frac{a_1 x}{b_1}....\}$ [5,15].

It has been reported in several works that integer order approximate model of fractional order derivatives by using CFE provides satisfactory accuracy for control application [14]. In this study, we used 3[th] order continuous CFE approximation of fractional order derivative, which was written for $0<\alpha<1$ as [14]:

$$s^{\alpha} \cong \frac{(\alpha^3+6\alpha^2+11\alpha+6)s^3+(-3\alpha^3-6\alpha^2+27\alpha+54)s^2+}{(-\alpha^3+6\alpha^2-11\alpha+6)s^3+(3\alpha^3-6\alpha^2-27\alpha+54)s^2+}\cdots \quad (10)$$
$$\frac{(3\alpha^3-6\alpha^2-27\alpha+54)s+(-\alpha^3+6\alpha^2-11\alpha+6)}{(-3\alpha^3-6\alpha^2+27\alpha+54)s+(\alpha^3+6\alpha^2+11\alpha+6)}$$

Steps of indirect discrete approximations of fractional order derivative are as follows [16]:
Step 1: Obtain integer order approximate model of fractional order derivative term s^{α} by using equation (10).

Step 2: Then, these integer order CFE approximations are discretized according to Tustin transformation by using c2d(.) function in Matlab.

3.ILLUSTRATIVE DISCRETIZATION EXAMPLE

In this section, we give an illustrative example to demonstrate application of direct Tustin recursive discretization and indirect CFE approximation methods for fractional order transfer function discretization. The amplitude and phase response approximation performances of these methods and stability of resulting IIR filter approximations are compared to discuss their feasibility in applications.
Let us obtain discrete IIR filter implementation of a fractional order transfer function, give by

$$F_c(s) = \frac{1}{s^{0.8}-1} \quad . \qquad (11)$$

For this purpose, one first obtains IIR filter approximation of $s^{0.8}$ and then substitutes it in the function $F_c(s)$.

(i) For application of the direct Tustin recursive method, $A_3(z^{-1},0.8)$ and $A_3(z^{-1},-0.8)$ can be written by using Table 1 as follows,

$$A_3(z^{-1},0.8) = -0.2667\,z^{-3} + 0.2133\,z^{-2} - 0.8\,z^{-1} + 1 \quad (12)$$
$$A_3(z^{-1},-0.8) = 0.2667\,z^{-3} + 0.2133\,z^{-2} + 0.8\,z^{-1} + 1 \quad (13)$$

Then, the term of $s^{0.8}$ is discretized for a sampling period (T) of 0.01 by using equation (7) as,

$$s^{0.8} \approx \left(\frac{2}{0.01}\right)^{0.8} \frac{-0.2667\,z^{-3} + 0.2133\,z^{-2} - 0.8\,z^{-1} + 1}{0.2667\,z^{-3} + 0.2133\,z^{-2} + 0.8\,z^{-1} + 1} \,. \quad (14)$$

For the third order IIR filter implementation of $F_c(s)$, the equation (14) is used in equation (11) and one obtains,

$$F_{tustin}(z) = \frac{1+0.8z^{-1}+0.2133z^{-2}+0.2667z^{-3}}{68.31-56.25z^{-1}+14.57z^{-2}-18.75z^{-3}} \,. \quad (15)$$

(ii) For application of the indirect CFE approximation method, the third order CFE approximation of $s^{0.8}$ can be obtained in continuous frequency domain by using equation (10) as,

$$s^{0.8} \cong \frac{19.15s^3+70.22s^2+30.1s+0.528}{0.528s^3+30.1s^2+70.22s+19.15} \quad . \qquad (16)$$

Then, we used CFE approximation of $s^{0.8}$ (equation (16)) in equation (11) and obtained integer order approximation of $F_c(s)$ as,

$$F_{\text{int}} \cong \frac{0.528s^3+30.1s^2+70.22s+19.15}{18.62s^3+40.12s^2-40.12s-18.62} \quad . \qquad (17)$$

After using c2d() function of Matlab, which use basic Tustin generating function (Equation (5)), a third order IIR filter implementation of $F_c(s)$ by indirect CFE method can be obtained as,

$$F_{cfe}(z) = \frac{z^3-2.555z^2+2.121z-0.5655}{35.27z^3-105z^2+104.3z-34.5} \,. \quad (18)$$

Figure 1 illustrates the amplitude responses of original continuous FOTF ($F_c(s)$), indirect CFE method and direct Tustin recursive method. Figure 2 illustrates the phase response results of those methods.

We observed that the indirect CFE approximation method can better converges to original fractional order transfer function in the low frequency region, however the direct Tustin recursive discretization method can provide an approximation to the original fractional order transfer function at high frequency region.

Consequently, indirect CFE approximation method can be effective for applications with low frequency operating ranges such as control system applications. It is useful to check stability of discrete filter approximations of $F_c(s)$ function. For signal processing applications, filters should be designed stable. Otherwise, the filter is useless for practical application.

However, stability of controlled system models is not crucial for control applications because controllers are designed to stabilize the control system. Here, the model accuracy is more substantial for control application so that design of effective controllers depends on plant model accuracy.

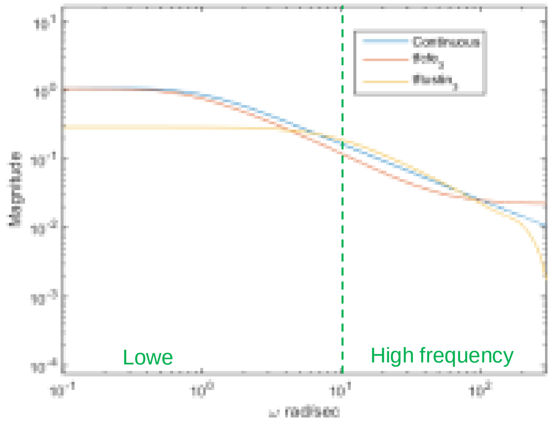

Figure 1. Comparison of amplitude responses of original continuous FOTF, the IIR filter implementation by indirect CFE method ($tfcfe_3$) and the IIR filter implementation by direct Tustin recursive method ($tftustin_3$).

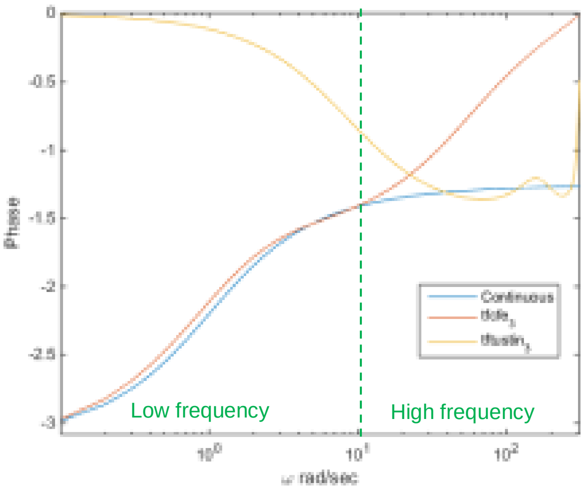

Figure 2. Comparison of phase responses of original continuous FOTF, the IIR filter implementation by indirect CFE method ($tfcfe_3$) and the IIR filter implementation by direct Tustin recursive method ($tftustin_3$).

Figure 3 and 4 show pole placement in complex Z plane to check stability of resulting IIR filters. Since all poles of $F_{tustin}(z)$ is in unit circle, $F_{tustin}(z)$ is stable and provides stable IIR filter approximation to $F_c(s)$ function.

In this example, some poles of $F_{cfe}(z)$ place out of the unit circle as shown in Figure 4, and this indicates that $F_{cfe}(z)$ is not a stable filter solution. So, unstable $F_{cfe}(z)$ function is not useful for signal processing applications. Figure 5 shows step responses of approximate filters and confirms stability status of $F_{tustin}(z)$ and $F_{cfe}(z)$ filter functions. Since $F_{cfe}(z)$ provides satisfactory frequency response approximation at the low frequency region, it can be useful for control applications.

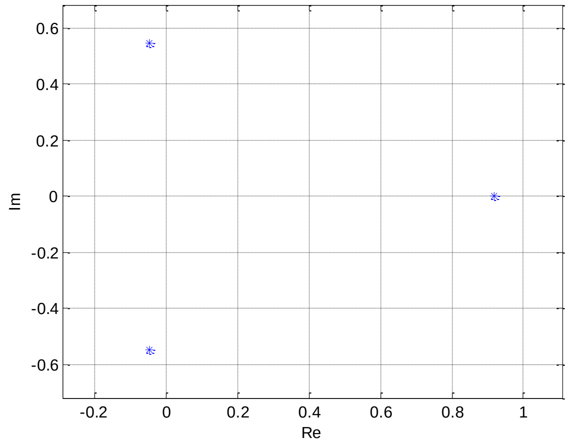

Figure 3. Poles of $F_{tustin}(z)$ function

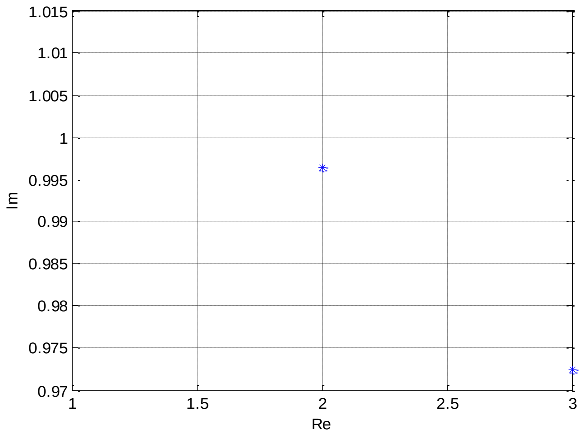

Figure 4. Poles of $F_{cfe}(z)$ function

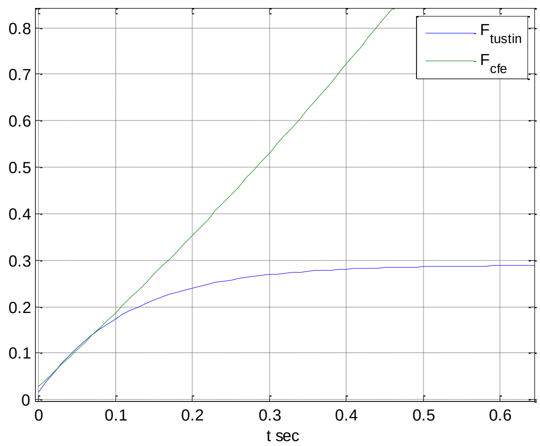

Figure 5. Step responses of $F_{tustin}(z)$ and $F_{cfe}(z)$ functions

4. CONCLUSIONS

Comparisons of these two approaches indicate that indirect CFE approximation method can be used for low-frequency applications such as control system applications. However, direct Tustin recursive discretization can more effective for high frequency applications such as filter realization applications. These results show that, depending on operating frequency ranges of applications, more convenient analytical discretization method should be preferred based on consideration of frequency response matching and stability status. Otherwise, practical performance of fractional order system deteriorates because its approximate implementation purely represents the behavior of original fractional order function in operating ranges. Therefore, investigation of approximation performances of discretization methods should be carried out, and effective methods regarding to application requirements should be chosen. Otherwise, one may not adequately benefit from advantages of fractional order systems in engineering applications.

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