



Sufficient conditions of non global solution for fractional damped wave equations with non-linear memory

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Abstract

The focus of the current paper is to prove nonexistence results for the following Cauchy problem of a wave equation with fractional damping and non linear memory

$$u_{tt} - \Delta u + D_{0|t}^{\sigma} u_t = \int_0^t (t - \tau)^{-\gamma} |u(\tau, \cdot)|^p d\tau, \quad (1)$$

$$u(0, x) = u_0(x), u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^N, \quad (2)$$

where $p > 1$, $0 < \gamma < 1$ and Δ is the usual Laplace operator, $\sigma \in]0, 1[$ and $D_{0|t}^{\sigma}$ is the right hand side fractional operator of Riemann-Liouville. Our method of proof is based on suitable choices of the test functions in the weak formulation of the sought solutions.

Keywords: Damped wave equation, Fujita's exponent, fractional derivative, weak solution.

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1. Introduction

The classical heat equation with nonlinear memory (1.1) bellow was studied by Cazenave and al [4] in 2008 when they have generalized some results obtained by Fujita [3] in 1966

$$u_t(t, x) - \Delta u(t, x) = \int_0^t (t - \tau)^{-\gamma} |u|^{p-1} u(\tau, x) d\tau, \quad (1.1)$$

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where $0 \leq \gamma < 1$ and $u_0 \in C_0(\mathbb{R}^N)$. Their results are the following. Put

$$p_\gamma = 1 + \frac{2(2 - \gamma)}{(N - 2 + 2\gamma)_+} \text{ and } p^* = \max\left(p_\gamma, \frac{1}{\gamma}\right) \text{ with } (N - 2 + 2\gamma)_+ = \max(N - 2 + 2\gamma, 0),$$

hence

1. If $\gamma \neq 0, p \leq p^*$ and $u_0 > 0$, then the solution u of (1.1) blows up in finite time.
2. If $\gamma \neq 0, p > p^*$ and $u_0 \in L_{q^*}(\mathbb{R}^N)$ (where $q^* = \frac{(p-1)N}{4-2\gamma}$) with small data, that is $\|u_0\|_{L_{q^*}}$ small enough, then u exists globally.

In particular, they proved that the critical exponent in Fujita’s sense p^* is not the one predicted by scaling, and this is not a surprising result since it is well known that scaling is efficient only for parabolic equations and not for pseudo-parabolic ones. To show this, it is sufficient to note that equation (1.1) can be formally converted into

$$D_{0|t}^\alpha u_t - D_{0|t}^\alpha \Delta u = \Gamma(\alpha) |u|^{p-1} u,$$

where $\alpha = 1 - \gamma$ and $D_{0|t}^\alpha$ is the fractional derivative operator of order α ($\alpha \in]0, 1[$) of Riemann-Liouville defined by

$$D_{0|t}^\alpha u = \frac{d}{dt} I_{0|t}^{1-\alpha} u, \tag{1.2}$$

and $I_{0|t}^{1-\alpha}$ is the fractional integral of order $1 - \alpha$ defined by (2.3) bellow.

3. In the case of $\gamma = 0$, Souplet has showed in [18] that nonzero positive solution blows -up in finite time.

After that, the damped wave equation with nonlinear memory was treated by Fino [5] in 2010, when he investigated the global existence and blow-up of solutions for the following equation

$$u_{tt}(t, x) - \Delta u(t, x) + u_t(t, x) = \int_0^t (t - \tau)^{-\gamma} |u(\tau, x)|^p d\tau. \tag{1.3}$$

He used as a main tool in his work for the existence and uniqueness of solution to problem (1.3) the weighted energy method similar as the one introduced by G. Todorova and B. Yardanov [9] in 2001, while he employed the test function method to show the blow-up results. One can find his results in [5]. In particular he found the same p_γ and so the same critical exponent p^* founded by Cazenave and al in [4]. Our purpose of this work is to generalize some of the above results.

Remark 1.1. Throughout this work, the constants will be denoted C and are different from one place to another one.

2. Statement of the problem

In this section, we will prove blow-up results of the problem (1)-(2).

The method which we will use is the test function method considered by Mitidieri and Pohozaev ([13], [14]), Pohozaev and Tesei [12], Fino [5], Berbiche and Hakem [6] and by Zhang [10].

Before that, one can show, easily, that the problem (1)-(2) can be written as follows

$$u_{tt} - \Delta u + D_{0|t}^\sigma u_t = \Gamma(\alpha) I_{0|t}^\alpha (|u|^p), \tag{2.1}$$

with the initial data

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) \text{ for all } x \in \mathbb{R}^n, \tag{2.2}$$

where $\alpha = 1 - \gamma$, $\sigma \in]0, 1[$, $p > 1$ and $I_{0|t}^\alpha$ is the fractional integral of order α ($\alpha \in]0, 1[$) defined (See [15]), for all $v \in L^1_{loc}(\mathbb{R})$, by

$$I_{0|t}^\alpha v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(s)}{(t-s)^{1-\alpha}} ds, \tag{2.3}$$

and Δ is the usual Laplace operator defined, for all $v \in C^2(\mathbb{R}^n)$ by

$$\Delta v = \frac{\partial^2 v}{\partial x_1^2} + \dots + \frac{\partial^2 v}{\partial x_n^2}.$$

2.1. Notations and definitions

Definition 2.1 (Weak solution). Let $T > 0$ and $\gamma \in]0, 1[$. A weak solution for the Cauchy problem (2.1)-(2.2) on $\mathbb{R}_+ \times \mathbb{R}^n$ with the initial data $u_0, u_1 \in L^1_{loc}(\mathbb{R}^N)$ is a locally integrable function $u \in L^p((0, T), L^p_{loc}(\mathbb{R}^N))$ such that

$$\begin{aligned} & \Gamma(\alpha) \int_0^T \int_{\mathbb{R}^n} I_{0|t}^\alpha (|u|^p) \varphi(t, x) dt dx + \int_{\mathbb{R}^n} u_1(x) \varphi(0, x) dx \\ & - \int_{\mathbb{R}^n} u_0(x) \varphi_t(0, x) dx + \int_{\mathbb{R}^n} u_0(x) D_{t|T}^\sigma \varphi(t, x)|_{t=0} dx \\ & = \int_0^T \int_{\mathbb{R}^n} u(t, x) \varphi_{tt}(t, x) dt dx + \int_0^T \int_{\mathbb{R}^n} u(t, x) D_{t|T}^{\sigma+1} \varphi(t, x) dt dx \\ & - \int_0^T \int_{\mathbb{R}^n} u(t, x) \Delta \varphi(t, x) dt dx, \end{aligned} \tag{2.4}$$

for all non-negative test function $\varphi \in C^2([0, T] \times \mathbb{R}^N)$ such that $\varphi(T, \cdot) = \varphi_t(T, \cdot) = D_{t|T}^\sigma \varphi(T, \cdot) = 0$ and $\alpha = 1 - \gamma$.

3. Main result

Our main result is the following. For all $\gamma, \sigma \in]0, 1[$ and $N \in \mathbb{N}$, we put

$$p_\gamma(\sigma) = 1 + \frac{2(2 - \gamma) + 2\sigma}{(N - 2 + 2\gamma + (N - 2)\sigma)_+}, \tag{3.1}$$

and

$$p^* = \max\{p_\gamma(\sigma), \gamma^{-1}\}. \tag{3.2}$$

Theorem 3.1. Let $0 < \gamma < 1$, $p \in (1, \infty)$ for $N = 1, 2$ and $1 < p < \frac{N}{N-2}$ for $N \geq 3$. Assume that $(u_0, u_1) \in H^s(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ and satisfy

$$\int_{\mathbb{R}^n} u_0(x) dx > 0, \quad \int_{\mathbb{R}^n} u_1(x) dx > 0. \tag{3.3}$$

Then, if $p \leq p^*$ then the solutions of the Cauchy problem (2.1)-(2.2) does not exist globally in time.

Proof. The theorem 3.1 will be demonstrated by absurd. So suppose that u is a global non trivial weak solution for problem (2.1)-(2.2). To prove Theorem 3.1 we need also to some results we will give them in the following section.

3.1. Preliminary results

As the principle of the method is the right choice of the test function, we chose it, for some $T > 0$, as follows:

$$\varphi(t, x) = D_{t|T}^\alpha \psi(t, x) = \varphi_1^r(x) D_{t|T}^\alpha \varphi_2(t), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N, \tag{3.4}$$

where $r > 1$ and $D_{t|T}^\alpha$ is the right fractional derivative operator of order α in the sense of Riemann-Liouville defined by

$$D_{t|T}^\alpha v(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_t^T \frac{v(s)}{(s-t)^\alpha} ds, \tag{3.5}$$

and the functions φ_1 and φ_2 are given by

$$\varphi_1(x) = \phi\left(\frac{x^2}{T^\theta}\right) \quad \text{and} \quad \varphi_2(t) = \left(1 - \frac{t}{T}\right)_+^\beta, \tag{3.6}$$

with $\beta > 1$, θ is a positive constant which will be chosen suitably later and ϕ is a cut-off non increasing function such that

$$\phi(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq 1 \\ 0 & \text{if } s \geq 2 \end{cases}, \quad 0 \leq \phi \leq 1 \text{ and } |\phi'(s)| \leq \frac{C}{s}. \tag{3.7}$$

We also denote by Ω_T for the support of φ_1 , that is

$$\Omega_T = \text{supp}\varphi_1 = \left\{x \in \mathbb{R}^N, \quad |x|^2 \leq 2T^\theta\right\}, \tag{3.8}$$

and by Δ_T for the set containing the support of $\Delta\varphi_1$ which is defined as follows

$$\Delta_T = \left\{x \in \mathbb{R}^N, \quad T^\theta \leq |x|^2 \leq 2T^\theta\right\}. \tag{3.9}$$

We will also use the fractional version of integration by parts (See [15])

$$\int_0^t f(t) D_{t|T}^\alpha g(t) dt = \int_0^t (D_{0|t}^\alpha f(t)) g(t) dt, \tag{3.10}$$

for all $f, g \in \mathcal{C}([0, T])$ such that $D_{0|t}^\alpha(f(t))$ and $D_{t|T}^\alpha g(t)$ exist and are continuous. The identities (See [15])

$$(D_{0|t}^\alpha \circ I_{0|t}^\alpha)(u) = u \text{ for all } u \in L^q([0, T]), \tag{3.11}$$

and

$$D_{t|T}^\sigma(D_{t|T}^\alpha) = D_{t|T}^{\sigma+\alpha}, \tag{3.12}$$

and also the following identity (see [15])

$$(-1)^n \partial_t^n D_{t|T}^\alpha u(t) = D_{t|T}^{\alpha+n} u(t), \quad n \in \mathbb{N}, \alpha \in]0, 1[, \tag{3.13}$$

which happens for all $u \in C^n[0, T]; T > 0$, where ∂_t^n is the n -times ordinary derivative with respect to t , will be strongly used in this work.

A simple and immediate calculation, using (3.5) and the identity (3.13) serves to the following proposition:

Proposition 3.2. *Given $\beta > 1$. Let φ_2 be the function defined by*

$$\varphi_2(t) = \left(1 - \frac{t}{T}\right)_+^\beta,$$

then for all $\alpha \in]0, 1[$, we have

$$D_{t|T}^\alpha \varphi_2(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} T^{-\beta} (T-t)_+^{\beta-\alpha} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} T^{-\alpha} \left(1 - \frac{t}{T}\right)_+^{\beta-\alpha},$$

and

$$D_{t|T}^{\alpha+1} \varphi_2(t) = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha)} T^{-\beta} (T - t)_+^{\beta - \alpha - 1} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha)} T^{-\alpha - 1} \left(1 - \frac{t}{T}\right)_+^{\beta - \alpha - 1},$$

also

$$D_{t|T}^{\alpha+2} \varphi_2(t) = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha - 1)} T^{-\beta} (T - t)_+^{\beta - \alpha - 2} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha - 1)} T^{-\alpha - 2} \left(1 - \frac{t}{T}\right)_+^{\beta - \alpha - 2}.$$

Therefore, for all $\alpha, \sigma \in]0, 1[$, we have

$$D_{t|T}^{\sigma+\alpha} \varphi_2(t) = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 - \sigma - \alpha)} T^{-\beta} (T - t)_+^{\beta - \alpha - \sigma} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 - \sigma - \alpha)} T^{-\sigma - \alpha} \left(1 - \frac{t}{T}\right)_+^{\beta - \alpha - \sigma}.$$

Proof. The proof of the proposition 3.2 is a simple and immediate verification. We have by definition (3.5)

$$D_{t|T}^\alpha \varphi_2(t) = -\frac{1}{\Gamma(1 - \alpha)} \frac{\partial}{\partial t} \int_t^T \frac{\varphi_2(s)}{(s - t)^\alpha} ds,$$

using the Euler’s change of variable

$$s \mapsto y = \frac{s - t}{T - t}.$$

1. We deduce, using formula (3.14) bellow,

$$\begin{aligned} D_{t|T}^\alpha \varphi_2(t) &= \frac{1}{\Gamma(1 - \alpha)} \frac{\partial}{\partial t} \int_t^T \frac{\left(1 - \frac{s}{T}\right)^\beta}{(s - t)^\alpha} ds \\ &= \frac{T^{-\beta}}{\Gamma(1 - \alpha)} \frac{\partial}{\partial t} \left((T - t)^{\beta - \alpha + 1} \int_0^1 y^{-\alpha} (1 - y)^\beta dy \right) \\ &= \frac{(\beta - \alpha + 1) \mathcal{B}(1 - \alpha, \beta + 1)}{\Gamma(1 - \alpha)} T^{-\beta} (T - t)^{\beta - \alpha} \\ &= \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} T^{-\beta} (T - t)^{\beta - \alpha}, \end{aligned}$$

where \mathcal{B} is the famous *Beta’s function* defined by

$$\mathcal{B}(u, v) = \int_0^1 t^{u-1} (1 - t)^{v-1} dt,$$

and satisfies in particular

$$\mathcal{B}(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u + v)}. \tag{3.14}$$

2. We apply directly formula (3.13) to show that

$$\forall t \in [0, T] : D_{t|T}^{\alpha+1} \varphi_2(t) = -\partial_t D_{t|T}^\alpha \varphi_2(t) \text{ et } D_{t|T}^{\alpha+2} \varphi_2(t) = \partial_t^2 D_{t|T}^\alpha \varphi_2(t),$$

hence the result is conclude.

3. We apply the identity (3.12) and formula (3.14) we get directly the desired result.

□

3.2. Treatment of the weak formulation (2.4)

3.2.1. Treatment of the left-hand side

Introducing the test function defined by (3.4), we get using the formula of integration by parts (3.10) and the identity (3.11)

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} I_{0|t}^\alpha(|u|^p)\varphi(t, x) dt dx &= \int_0^T \int_{\mathbb{R}^n} I_{0|t}^\alpha(|u|^p)D_{t|T}^\alpha\psi(t, x) dt dx \\ &= \int_0^T \int_{\mathbb{R}^n} D_{0|T}^\alpha I_{0|T}^\alpha(|u|^p)\psi(t, x) dt dx = \int_0^T \int_{\mathbb{R}^n} |u|^p\psi(t, x) dt dx. \end{aligned} \tag{3.15}$$

For the 2nd term of the left-hand side of equality (2.4), we use the Proposition 3.2 to obtain

$$\begin{aligned} \int_{\mathbb{R}^n} u_1(x)\varphi(0, x) dx &= \int_{\mathbb{R}^n} u_1(x)\varphi_1^r(x)D_{t|T}^\alpha\varphi_2(t)|_{t=0} dx \\ &= C_1T^{-\alpha} \int_{\mathbb{R}^n} u_1(x)\varphi_1^r(x) dx, \end{aligned} \tag{3.16}$$

since

$$D_{t|T}^\alpha\varphi_2(t)|_{t=0} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)}T^{-\alpha} = C_1T^{-\alpha}.$$

For the 3rd term, noting that

$$\varphi_t(t, x) = \frac{\partial\varphi}{\partial t}(t, x) = -\varphi_1^r(x)D_{t|T}^{\alpha+1}\varphi_2(t),$$

using always the Proposition 3.2 we get the following estimate

$$\int_{\mathbb{R}^n} u_0(x)\varphi_t(0, x) dx = C_2T^{-\alpha-1} \int_{\mathbb{R}^n} u_0(x)\varphi_1^r(x) dx, \tag{3.17}$$

since

$$D_{t|T}^{\alpha+1}\varphi_2(t)|_{t=0} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha)}T^{-\alpha-1} = C_2T^{-\alpha-1}.$$

Always by Proposition 3.2, the following estimate will be obtained for the 4th term of the left hand-side of the weak formulation(2.4)

$$\int_{\mathbb{R}^n} u_0(x)D_{t|T}^\sigma\varphi(t, x)|_{t=0} dx = CT^{-\sigma-\alpha} \int_{\mathbb{R}^n} u_0(x)\varphi_1^r(x) dx, \tag{3.18}$$

since

$$D_{t|T}^\sigma\varphi(t, x)|_{t=0} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \sigma - \alpha + 1)}T^{-\sigma-\alpha}\varphi_1^r(x) = CT^{-\sigma-\alpha}\varphi_1^r(x). \tag{3.19}$$

3.2.2. Treatment of the right-hand side

Taking into account the formula (3.13) we easily get

$$\varphi_{tt}(t, x) = \frac{\partial^2\varphi}{\partial t^2}(t, x) = \varphi_1^r(x)\partial_t^2D_{t|T}^\alpha\varphi_2(t) = \varphi_1^r(x)D_{t|T}^{\alpha+2}\varphi_2(t),$$

and then

$$\int_0^T \int_{\mathbb{R}^n} u(t, x)\varphi_{tt}(t, x) dt dx = \int_0^T \int_{\mathbb{R}^n} u(t, x)\varphi_1^r(x)D_{t|T}^{\alpha+2}\varphi_2(t) dt dx. \tag{3.20}$$

Using formula (3.12) and (3.13) we show firstly that

$$D_{t|T}^{\sigma+1}\varphi(t, x) = \varphi_1^r(x)D_{t|T}^{\sigma+\alpha+1}\varphi_2(t),$$

and then

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^n} u(t, x) D_{t|T}^{\sigma+1} \varphi(t, x) dt dx \\ &= \int_0^T \int_{\mathbb{R}^n} u(t, x) \varphi_1^r(x) D_{t|T}^{\sigma+\alpha+1} \varphi_2(t) dt dx. \end{aligned} \tag{3.21}$$

Finally for the third term of the right-hand side of the formulation (2.4), using the following identity

$$\Delta(\varphi_1^r) = r\varphi_1^{r-1} \Delta\varphi_1 + r(r-1)\varphi_1^{r-2} |\nabla\varphi_1|^2,$$

we get

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^n} u(t, x) \Delta\varphi(t, x) dt dx = \\ & \int_0^T \int_{\mathbb{R}^n} u(t, x) (r\varphi_1^{r-1} \Delta\varphi + r(r-1)\varphi_1^{r-2} |\nabla\varphi_1|^2) D_{t|T}^\alpha \varphi_2(t) dt dx. \end{aligned} \tag{3.22}$$

Inserting formulas (3.15), (3.16), (3.17), (3.18), (3.20), (3.21) and (3.22) in the formulation (2.4) we obtain

$$\begin{aligned} & \Gamma(\alpha) \int_0^T \int_{\mathbb{R}^n} |u|^p \psi(t, x) dt dx + C_1 T^{-\alpha} \int_{\mathbb{R}^n} u_1(x) \varphi_1^r(x) dx \\ &+ C_2 T^{-\alpha-1} \int_{\mathbb{R}^n} u_0(x) \varphi_1^r(x) dx + C T^{-\sigma-\alpha} \int_{\mathbb{R}^n} u_0(x) \varphi_1^r(x) dx \\ &= \int_0^T \int_{\mathbb{R}^n} u(t, x) \varphi_1^r(x) D_{t|T}^{\alpha+2} \varphi_2(t) dt dx \\ &- \int_0^T \int_{\mathbb{R}^n} u(t, x) \left[r\varphi_1^{r-1} \Delta\varphi_1 \right. \\ &+ \left. r(r-1)\varphi_1^{r-2} |\nabla\varphi_1|^2 \right] (x) D_{t|T}^\alpha \varphi_2(t) dt dx \\ &+ \int_0^T \int_{\mathbb{R}^n} u(t, x) \varphi_1^r(x) D_{t|T}^{\sigma+\alpha+1} \varphi_2(t) dt dx. \end{aligned} \tag{3.23}$$

The facts that $\varphi_1 \leq 1$ and

$$\left| r\varphi_1^{r-1} \Delta\varphi + r(r-1)\varphi_1^{r-2} |\nabla\varphi_1|^2 \right| \leq \varphi_1^{r-2} (|\Delta\varphi_1| + |\nabla\varphi_1|^2),$$

allows us to deduce from the formula (3.23) the following inequality

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^n} |u|^p \psi(t, x) dt dx + C T^{-\alpha} \int_{\mathbb{R}^n} u_1(x) \varphi_1^r(x) dx \\ &+ C T^{-\alpha-1} \int_{\mathbb{R}^n} u_0(x) \varphi_1^r(x) dx + C T^{-\sigma-\alpha} \int_{\mathbb{R}^n} u_0(x) \varphi_1^r(x) dx \\ &\leq C \int_0^T \int_{\mathbb{R}^n} |u(t, x)| \varphi_1^r(x) |D_{t|T}^{\alpha+2} \varphi_2(t)| dt dx \\ &+ C \int_0^T \int_{\mathbb{R}^n} |u(t, x)| \varphi_1^{r-2} (|\Delta\varphi_1| + |\nabla\varphi_1|^2) |D_{t|T}^\alpha \varphi_2(t)| dt dx \\ &+ \int_0^T \int_{\mathbb{R}^n} |u(t, x)| \varphi_1^r(x) |D_{t|T}^{\sigma+\alpha+1} \varphi_2(t)| dt dx, \end{aligned} \tag{3.24}$$

for some constant $C > 0$. Next, applying the following ε - Young inequality

$$AB \leq \varepsilon A^p + C(\varepsilon) B^q, \quad pq = p + q,$$

to the terms of the right-hand side of inequality (3.24) we get

$$\begin{aligned}
 & \int_0^T \int_{\mathbb{R}^n} |u(t, x)| \varphi_1^r(x) |D_{t|T}^{\alpha+2} \varphi_2(t)| dt dx \\
 &= \int_0^T \int_{\mathbb{R}^n} |u(t, x)| \psi^{\frac{1}{p}} \psi^{-\frac{1}{p}} \varphi_1^r(x) |D_{t|T}^{\alpha+2} \varphi_2(t)| dt dx \\
 &\leq \varepsilon \int_0^T \int_{\mathbb{R}^n} |u|^p \psi dt dx + C(\varepsilon) \int_0^T \int_{\mathbb{R}^n} \varphi_1^r \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{\alpha+2} \varphi_2 \right|^{\frac{p}{p-1}} dt dx.
 \end{aligned}
 \tag{3.25}$$

Similarly, we have

$$\begin{aligned}
 & \int_0^T \int_{\mathbb{R}^n} |u| \varphi_1^{r-2} (|\Delta \varphi_1| + |\nabla \varphi_1|^2) |D_{t|T}^{\alpha} \varphi_2| dt dx \leq \varepsilon \int_0^T \int_{\mathbb{R}^n} |u|^p \psi dt dx \\
 &+ C(\varepsilon) \int_0^T \int_{\mathbb{R}^n} \lambda(\varphi_1) \varphi_1^{r-2\frac{p}{p-1}} \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha} \varphi_2|^{\frac{p}{p-1}} dt dx,
 \end{aligned}
 \tag{3.26}$$

with

$$\lambda(\varphi_1) = |\Delta \varphi_1|^q + |\nabla \varphi_1|^{2q}.$$

For the third term of the right-hand side we obtain

$$\begin{aligned}
 & \int_0^T \int_{\mathbb{R}^n} |u(t, x)| \varphi_1^r(x) |D_{t|T}^{\sigma+\alpha+1} \varphi_2(t)| dt dx \\
 &\leq \int_0^T \int_{\mathbb{R}^n} |u(t, x)|^p \psi(t, x) dt dx \\
 &+ \int_0^T \int_{\mathbb{R}^n} \varphi_1^r(x) \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\sigma+\alpha+1} \varphi_2(t)|^{\frac{p}{p-1}} dt dx.
 \end{aligned}
 \tag{3.27}$$

Using the fact that (3.3) implies that

$$\int_{\mathbb{R}^n} u_i(x) \varphi^r(x) dx > 0, \quad i = 1, 2,$$

we conclude from (3.24), (3.25), (3.26) and (3.27), for ε small enough

$$\begin{aligned}
 & \int_0^T \int_{\mathbb{R}^n} |u|^p \psi(t, x) dt dx \leq C \left(\int_0^T \int_{\mathbb{R}^n} \varphi_1^r \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+2} \varphi_2|^{\frac{p}{p-1}} dt dx \right. \\
 &+ \int_0^T \int_{\mathbb{R}^n} \lambda(\varphi_1) \varphi_1^{r-2\frac{p}{p-1}} \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha} \varphi_2|^{\frac{p}{p-1}} dt dx \\
 &\left. + \int_0^T \int_{\mathbb{R}^n} \varphi_1^r(x) \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\sigma+\alpha+1} \varphi_2(t)|^{\frac{p}{p-1}} dt dx \right) \leq C(I_1 + I_2 + I_3),
 \end{aligned}
 \tag{3.28}$$

for some positive constant C . Now, to estimate integrals I_1, I_2 and I_3 we consider the scaled variables

$$x = T^{\frac{\theta}{2}} y \quad \text{and} \quad t = T\tau,
 \tag{3.29}$$

and noting that they are null outside Ω_T (defined by (3.8)), then, using Fubini’s theorem, we get, for I_1

$$\begin{aligned}
 & \int_0^T \int_{\Omega_T} \varphi_1^r \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+2} \varphi_2|^{\frac{p}{p-1}} dt dx = J_{11} J_{12} \\
 &= \left(\int_{\Omega_T} \varphi_1^r dx \right) \left(\int_0^T \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+2} \varphi_2|^{\frac{p}{p-1}} dt \right).
 \end{aligned}
 \tag{3.30}$$

We have

$$J_{11} = \int_{\Omega_T} \varphi_1^r dx = T^{\frac{N\theta}{2}} \int_0^2 \phi^r(y^2) dy = CT^{\frac{N\theta}{2}}, \tag{3.31}$$

and using *Proposition 3.2*, we get

$$J_{12} = \int_0^T \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+2} \varphi_2|^{\frac{p}{p-1}} dt = CT^{1-(\alpha+2)\frac{p}{p-1}}. \tag{3.32}$$

Combining (3.31) and (3.32) into (3.30) we obtain then

$$\int_0^T \int_{\Omega_T} \varphi_1^r \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+2} \varphi_2|^{\frac{p}{p-1}} dt dx = CT^{-(\alpha+2)\frac{p}{p-1} + \frac{N\theta}{2} + 1}. \tag{3.33}$$

By the same way we have for I_2

$$\begin{aligned} & \int_0^T \int_{\Omega_T} \lambda(\varphi_1) \varphi_1^{r-2q} \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^\alpha \varphi_2|^q dt dx = J_{21} J_{22} \\ & = \left(\int_{\Omega_T} \lambda(\varphi_1) \varphi_1^{r-2q} dx \right) \left(\int_0^T \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^\alpha \varphi_2|^q dt \right). \end{aligned} \tag{3.34}$$

So, if we replace q by its value $\frac{p}{p-1}$ we get

$$J_{21} = \int_{\Omega_T} \left(|\Delta \varphi_1|^{\frac{p}{p-1}} + |\nabla \varphi_1|^{2\frac{p}{p-1}} \right) \varphi_1^{r-2\frac{p}{p-1}} dx = CT^{-\theta\frac{p}{p-1} + \frac{N\theta}{2}}, \tag{3.35}$$

and

$$J_{22} = \int_0^T \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^\alpha \varphi_2|^{\frac{p}{p-1}} dt = CT^{-\alpha\frac{p}{p-1} + 1}. \tag{3.36}$$

We replace (3.35) and (3.36) into (3.34) we find

$$\int_0^T \int_{\Omega_T} \lambda(\varphi_1) \varphi_1^{r-2q} \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^\alpha \varphi_2|^q dt dx = CT^{-(\alpha+\theta)\frac{p}{p-1} + \frac{N\theta}{2} + 1}. \tag{3.37}$$

For I_3 we have

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^n} \varphi_1^r(x) \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\sigma+\alpha+1} \varphi_2(t)|^{\frac{p}{p-1}} dt dx = J_{31} J_{32} \\ & = \left(\int_{\Omega_T} \varphi_1^r dx \right) \left(\int_0^T |\varphi_2|^{-\frac{1}{p-1}} |D_{t|T}^{\sigma+\alpha+1} \varphi_2(t)|^{\frac{p}{p-1}} dt \right). \end{aligned} \tag{3.38}$$

Then we find for J_{31}

$$J_{31} = J_{11} = CT^{\frac{N\theta}{2}}, \tag{3.39}$$

and as usual

$$J_{32} = \int_0^T |\varphi_2|^{-\frac{1}{p-1}} |D_{t|T}^{\sigma+\alpha+1} \varphi_2(t)|^{\frac{p}{p-1}} dt = CT^{1-(\sigma+\alpha+1)\frac{p}{p-1}}. \tag{3.40}$$

Hence by inserting (3.39) and (3.40) in 3.38 we get

$$\int_0^T \int_{\mathbb{R}^n} \varphi_1^r(x) \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\sigma+\alpha+1} \varphi_2(t)|^{\frac{p}{p-1}} dt dx = CT^{-(\sigma+\alpha+1)\frac{p}{p-1} + 1 + \frac{N\theta}{2}}. \tag{3.41}$$

Finally, we replace (3.33),(3.37) and (3.41) into (3.28) we obtain

$$\int_0^T \int_{\Omega_T} |u|^p \psi(t, x) dt dx \leq C \left(T^{-(\alpha+2)\frac{p}{p-1} + \frac{N\theta}{2} + 1} + T^{-(\alpha+\theta)\frac{p}{p-1} + \frac{N\theta}{2} + 1} + T^{-(\sigma+\alpha+1)\frac{p}{p-1} + 1 + \frac{N\theta}{2}} \right). \quad (3.42)$$

Now, since θ is arbitrary and it must only be positive, we choose it as follows

$$\theta = \sigma + 1 > 0 \quad \text{since } \sigma \in]0, 1[.$$

This choice of θ allows us to have

$$-(\alpha + \theta)\frac{p}{p-1} + \frac{N\theta}{2} + 1 = -(\sigma + \alpha + 1)\frac{p}{p-1} + 1 + \frac{N\theta}{2}, \quad (3.43)$$

with this choice of θ , and by (3.43) we get from (3.42)

$$\int_0^T \int_{\Omega_T} |u|^p \psi(t, x) dt dx \leq CT^\delta, \quad (3.44)$$

where

$$\begin{aligned} \delta &= \max \left(-(\alpha + 2)\frac{p}{p-1} + \frac{N\theta}{2} + 1, -(\sigma + \alpha + 1)\frac{p}{p-1} + \frac{N\theta}{2} + 1 \right) \\ &= -(\sigma + \alpha + 1)\frac{p}{p-1} + \frac{N\theta}{2} + 1 = -(\sigma + \alpha + 1)\frac{p}{p-1} + (\sigma + 1)\frac{N}{2} + 1. \end{aligned}$$

At this stage, to prove the first result in *Theorem.3.1*, we distinguish two cases.

Case of $p \leq p_\gamma(\sigma)$

This case itself is divided into two subcases as follows

1. i. Subcase of $p < p_\gamma(\sigma)$.

In this case, one can remark that the condition $p < p_\gamma(\sigma)$ is equivalent to $\delta < 0$, then we pass to the limit as $T \rightarrow \infty$ in (3.44), we get

$$\lim_{T \rightarrow +\infty} \int_0^T \int_{\Omega_T} |u|^p \psi(t, x) dt dx = 0. \quad (3.45)$$

Using the dominated convergence theorem of Lebesgue (*Theorem 1.1.4* in [1]), the continuity of u with respect to t and x and the fact that

$$\lim_{T \rightarrow +\infty} \psi(t, x) = 1, \quad (3.46)$$

we obtain

$$\int_0^{+\infty} \int_{\mathbb{R}^N} |u|^p dt dx = 0,$$

and this implies that $u = 0$, which is a contradiction because we have supposed that the solution u is not trivial.

1. ii. Subcase of $p = p_\gamma(\sigma)$.

Firstly, we remark that the condition $p = p_\gamma(\sigma)$ is equivalent to $\delta = 0$. Next, taking the limit as $T \rightarrow \infty$ in (3.44) with the consideration $\delta = 0$ we get

$$\int_0^{+\infty} \int_{\mathbb{R}^N} |u|^p dt dx < +\infty,$$

from which we can deduce that

$$\lim_{T \rightarrow \infty} \int_0^{+\infty} \int_{\Delta_T} |u|^p \psi dt dx = 0, \tag{3.47}$$

where Δ_T is defined by (3.9). Fixing arbitrarily R in $]0, T[$ for some $T > 0$ such that when $T \rightarrow \infty$ we don't have $R \rightarrow \infty$ at the same time and choosing φ_1 as

$$\varphi_1(x) = \phi\left(\frac{|x|^2}{T^{\frac{\theta}{2}} R^{-\frac{\theta}{2}}}\right), \tag{3.48}$$

with θ is an arbitrary positive constant and ϕ is the cut-off function defined by (3.7). Using the following Hölder's inequality

$$\int_X uv d\mu \leq \left(\int_X u^p d\mu\right)^{\frac{1}{p}} \left(\int_X v^q d\mu\right)^{\frac{1}{q}}; \quad u \in L^p(X), \quad v \in L^q(X), \quad p, q > 0, \quad pq = p + q,$$

instead of the ε -Young's one to estimate integral I_2 in (3.28) on the set

$$\Omega_{TR^{-1}} = \left\{x \in \mathbb{R}^N : |x|^2 \leq 2T^\theta R^{-\theta}\right\} = \text{supp}\varphi_1,$$

and noting that $\text{supp}\Delta\varphi_1 \subset \Delta_{TR^{-1}} \subset \Omega_{TR^{-1}}$ where

$$\Delta_{TR^{-1}} = \left\{x \in \mathbb{R}^N : T^\theta R^{-\theta} \leq |x|^2 \leq 2T^\theta R^{-\theta}\right\}, \tag{3.49}$$

we get

$$\begin{aligned} & \int_0^T \int_{\Omega_{TR^{-1}}} |u| \varphi_1^{r-2} \left[|\Delta\varphi_1|^2 + |\nabla\varphi_1|^2\right] \left|D_{t|T}^\alpha \varphi_2\right| dt dx \\ & \leq \left(\int_0^T \int_{\Delta_{TR^{-1}}} |u|^p \psi dt dx\right)^{\frac{1}{p}} \left(\int_0^T \int_{\Delta_{TR^{-1}}} \psi^{\frac{q}{p}} \varphi_1^{r-2q} \left(|\Delta\varphi_1|^q + |\nabla\varphi_1|^{2q}\right) \left|D_{t|T}^\alpha \varphi_2\right|^q dt dx\right)^{\frac{1}{q}}. \end{aligned}$$

Recalling Integrals I_1, I_3 in page 231 and \tilde{I}_2 such that

$$\tilde{I}_2 = \left(\int_0^T \int_{\Delta_{TR^{-1}}} \psi^{\frac{q}{p}} \varphi_1^{r-2q} \left(|\Delta\varphi_1|^q + |\nabla\varphi_1|^{2q}\right) \left|D_{t|T}^\alpha \varphi_2\right|^q dt dx\right)^{\frac{1}{q}}.$$

To estimate them, we use at this stage the change of variables $x = T^{\frac{\theta}{2}} R^{-\frac{\theta}{2}} y$, and $t = T\tau$ on the set $\Omega_{TR^{-1}}$. We have firstly

$$I_1 + I_3 \leq C \left(T^{-(\alpha+2)\frac{p}{p-1} + N\frac{\theta}{2} + 1} + T^{-(\sigma+\alpha+1)\frac{p}{p-1} + \frac{N\theta}{2} + 1}\right) R^{-N\theta}, \tag{3.50}$$

and using the hypothesis $\delta = 0$ we conclude from (3.50)

$$I_1 + I_3 \leq CR^{-N\theta/2}. \tag{3.51}$$

Calculating the integral \tilde{I}_2 using the same change of variables and the same form of function φ_1 and using (3.51) we obtain from (3.28)

$$\begin{aligned} \int_0^T \int_{\Omega_{TR^{-1}}} |u|^p \psi dt dx & \leq CR^{-N\theta/2} \\ & + CR^{\theta - \frac{N\theta}{2q}} \left(\int_0^T \int_{\Delta_{TR^{-1}}} |u|^p \psi dt dx\right)^{\frac{1}{p}}. \end{aligned} \tag{3.52}$$

Now taking the limit as $T \rightarrow +\infty$ in (3.52) and using (3.47) and the fact that $\lim_{T \rightarrow +\infty} \psi(t, x) = 1$, we arrive at

$$\int_0^\infty \int_{\mathbb{R}^N} |u|^p dt dx \leq CR^{-N\theta/2},$$

which means that necessarily $R \rightarrow +\infty$ and this is a contradiction.

Case of $p \leq \frac{1}{\gamma}$.

Even this case is divided into two subcases as follows

2. i. Subcase of $p < \frac{1}{\gamma}$.

In this case we recall (3.28), we take $\varphi_1(x) = \phi\left(\frac{|x|^2}{R^\theta}\right)$ where ϕ is the function defined by (3.7) and R is a fixed positive number. Trying to calculate generalized integrals I_1, I_2 and I_3 (page 231) with respect to x on the set

$$\Sigma_R = \left\{x \in \mathbb{R}^N : |x| \leq 2R^{\theta/2}\right\} = \text{supp}\varphi_1.$$

Employing the scaled variables

$$x = R^{\frac{\theta}{2}}y \quad \text{and} \quad t = T\tau,$$

for the first integral we have

$$\begin{aligned} & \int_0^T \int_{\Sigma_R} \varphi_1^r \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+2} \varphi_2|^{\frac{p}{p-1}} dt dx \\ &= \left(\int_{\Sigma_R} \varphi_1^r dx \right) \left(\int_0^T \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+2} \varphi_2|^{\frac{p}{p-1}} dt \right) \\ &= \left(R^{N\theta/2} \int_0^1 \phi^r(y^2) dy \right) \\ &\times \left(T^{1-(\alpha+2)\frac{p}{p-1}} \int_0^1 (1-\tau)^{-\frac{\beta}{p-1} + (\beta-\alpha-2)\frac{p}{p-1}} d\tau \right) \\ &= CR^{\frac{N\theta}{2}} T^{1-(\alpha+2)\frac{p}{p-1}} \end{aligned} \tag{3.53}$$

By the same way, we obtain

$$\begin{aligned} & \int_0^T \int_{\Sigma_R} \lambda(\varphi_1) \varphi_1^{r-2q} \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^\alpha \varphi_2|^q dt dx \\ &= \left(\int_{\Sigma_T} \lambda(\varphi_1) \varphi_1^{r-2q} dx \right) \left(\int_0^T \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^\alpha \varphi_2|^q dt \right) \\ &= CR^{\frac{N\theta}{2} - \theta\frac{p}{p-1}} T^{1-\alpha\frac{p}{p-1}}. \end{aligned} \tag{3.54}$$

Finally, for the third integral, we have

$$\int_0^T \int_{\mathbb{R}^n} \varphi_1^r(x) \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\sigma+\alpha+1} \varphi_2(t)|^{\frac{p}{p-1}} dt dx = CR^{\frac{N\theta}{2}} T^{-(\sigma+\alpha+1)\frac{p}{p-1}+1}. \tag{3.55}$$

Using formula (3.53), (3.54) and (3.55) we get

$$\begin{aligned} \int_0^T \int_{\Sigma_R} |u|^p \psi(t, x) dt dx &= CR^{\frac{N\theta}{2}} \left(T^{1-(\alpha+2)\frac{p}{p-1}} + T^{-(\sigma+\alpha+1)\frac{p}{p-1}+1} \right) \\ &+ CR^{\left(\frac{N}{2} - \frac{p}{p-1}\right)\theta} T^{1-\alpha\frac{p}{p-1}}. \end{aligned} \tag{3.56}$$

Firstly, we note that $p < \frac{1}{\gamma}$ implies that $1 - \alpha \frac{p}{p-1} < 0$. So, the facts that $(\alpha + 2) \frac{p}{p-1} > \alpha \frac{p}{p-1}$ and $(\sigma + \alpha + 1) \frac{p}{p-1} > \alpha \frac{p}{p-1}$ allow us the fact that

$$\lim_{T \rightarrow +\infty} \psi(t, x) = \varphi_1^r(x), \tag{3.57}$$

hence, by taking the limit as $T \rightarrow +\infty$ in (3.56), we arrive at

$$\int_0^{+\infty} \int_{\Sigma_R} |u|^p \varphi_1^r(x) dt dx = 0. \tag{3.58}$$

Next, taking the limit in (3.58) as $R \rightarrow +\infty$ and taking into account the fact that $\lim_{R \rightarrow +\infty} \varphi_1^r(x) = 1$, we get

$$\int_0^{+\infty} \int_{\mathbb{R}^N} |u|^p dt dx = 0.$$

This implies that $u = 0$ which is a contradiction.

2. ii. Subcase of $p = \frac{1}{\gamma}$.

In this case, we assume furthermore that

$$p < \frac{N}{N-1}. \tag{3.59}$$

First, we observe that (3.59) implies that

$$\frac{N}{2} - \frac{p}{p-1} < 0. \tag{3.60}$$

Under these assumptions, we have

$$\begin{aligned} 1 - (\alpha + 2) \frac{p}{p-1} &= -\frac{2}{\alpha} < 0, & 1 - \alpha \frac{p}{p-1} &= 0, \\ 1 - (\sigma + \alpha + 1) \frac{p}{p-1} &= -(\sigma + 1) \frac{p}{p-1} < 0. \end{aligned} \tag{3.61}$$

Hence, taking the limit as $T \rightarrow \infty$ in (3.56) with the considerations (3.61) and (3.57) we obtain

$$\int_0^\infty \int_{\Sigma_R} |u|^p \varphi_1^r(x) dt dx = CR^{(\frac{N}{2} - \frac{p}{p-1})\theta}. \tag{3.62}$$

Finally, one can remark easily that if $N = 1, 2$ then $\frac{N}{2} - \frac{p}{p-1} < 0$ for all $p > 1$, then by taking the limit as $R \rightarrow \infty$ in (3.62) and using the facts that $\theta > 0$ and $\lim_{R \rightarrow +\infty} \varphi_1^r(x) = 1$, we get

$$\int_0^\infty \int_{\mathbb{R}^N} |u|^p dt dx = 0, \tag{3.63}$$

which implies that $u = 0$ and this is a contradiction.

If $N \geq 3$ then $\frac{N}{2} - \frac{p}{p-1}$ will be negative and we can get (3.63) by letting $R \rightarrow \infty$ in (3.62), if we assume furthermore that (3.59) or equivalently (3.60) is satisfied. This achieved the proof of *Theorem 3.1*. \square

Remark 3.3. We remark that the condition (3.59) is needed only in the case of $p = \frac{1}{\gamma}$ and $N \geq 3$ and not otherwise. We, also, point out that the condition (3.59) is equivalent to

$$\frac{N-2}{N} < \gamma < 1.$$

4. Conclusion

First, one can show that if $\sigma \rightarrow 0$ then $p_\gamma(\sigma) \rightarrow p_\gamma$, and we find the same critical exponent obtained by Fino ([5]), and this is reasonable because if $\sigma = 0$ then $D_{0|t}^\sigma u_t = u_t$. In other word, our result is a generalization of the result of ([5]). Also, thanks to the presence of the term u_t in the model of ([5]), one can show using Fourier transform, for example, or by scaling argument, that this model is parabolic like and then it tends to the model of Cazenave and al ([4]) as $t \rightarrow +\infty$. For this reason, the two problems have the same critical exponent p^* .

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