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A Two-parameter Deformation of Supergroup GL(1|2)

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ABSTRACT: A new super-Hopf algebra, denoted by , is obtained by using the standard method (the RTT-relation) with an *R*-matrix which is a solution of the quantum Yang-Baxter equation.

Keywords: Yang-Baxter equation, super-Hopf algebra, quantum supergroup.

GL(1|2) Süper Grubunun Bir İki-parametreli Deformasyonu

ÖZET: Kuantum Yang-Baxter denkleminin çözümü olan bir *R*-matrisi yardımıyla, standard RTT-bağıntısı kullanılarak ile gösterilen yeni bir süper-Hopf cebiri elde edilmiştir.

Keywords: Yang-baxter equation, super-hopf cebiri, kuantum super grup.

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INTRODUCTION

Quantum groups (Drinfeld, 1986) have a rich mathematical structure (Klimyk and Schmüdgen, 1997), (Majid, 1995). The standard method to construct a new algebra from a solution of the quantum Yang-Baxter equation (Yang, 1967) was initiated by Faddeev et al. in 1990. With this method, we will introduce a new superalgebra related to a $\mathbb{Z}_2$-graded *R*-matrix with two-parameter. The RTT-relation for the quantum supergroups has the same form as in the (Faddeev et al. in 1990), but matrix tensor product contains a factor (-1), as additional to the (Kulish and Sklyanin, 1982) related to $\mathbb{Z}_2$-grading (Berezin, 1987).

The tensor product of two even matrices *U* and *V* has the signs

$$(U\otimes V)_{ij,kl}=(-1)^{\tau(j)(\tau(\mathrm{i})+\tau(\mathrm{k}))}U_{ik}V_{jl}$$

where $\tau(U_{ij})=\tau(i)+\tau(j)$. Because of this description, a matrix in the form $I\otimes U$ has the same block-diagonal form as in the standard (no-grading) case while a matrix in the form $U\otimes I$ contains the factor (-1) for *odd* elements standing at odd rows of blocks. To give a little explanation, we consider the matrix $U=\begin{pmatrix} a & \alpha \\ \gamma & b \end{pmatrix}$ appearing as T_{33} on page 7, line 5. Then the tensor product of the matrices *U* and $I=(\delta_{ij})$ has the signs

$$(U\otimes I)_{ij,kl}=(-1)^{\tau(j)(\tau(i)+\tau(k))}U_{ik}\delta_{jl} \text{ and } (I\otimes U)_{ij,kl}=(-1)^{\tau(j)(\tau(i)+\tau(k))}\delta_{ik}U_{jl}=\delta_{ik}U_{jl}$$

where δ_{ij} denotes the kronecker delta. So, we have, for example

$$(U\otimes I)_{11,21}=+U_{12}\delta_{11}=\alpha,\ \ (\mathrm{U}\otimes I)_{12,22}=-U_{12}\delta_{22}=-\alpha,\ (\mathrm{U}\otimes I)_{22,12}=-U_{21}\delta_{22}=-\gamma, etc.$$

In this paper, we construct a two-parameter deformation of the supergroup $GL(1|2)$, denoted by $GL_{p,q}(1|2)$.

MATERIAL AND METHODS

Let $a,b,c,d,e,\alpha,\beta,\gamma,\delta$ be generators of an algebra A, where the generators a,b,c,d,e are of grade 0 and the generators $\alpha,\beta,\gamma,\delta$ are of grade 1. Let $O(M(1|2))$ be defined as the polynomial algebra $k[a,b,c,d,e,\alpha,\beta,\gamma,\delta]$. It will be sometimes more convenient and more illustrative to write a point $(a,b,c,d,e,\alpha,\beta,\gamma,\delta)$ of $O(M(1|2))$ in the matrix form, as a supermatrix,

$$T=\begin{pmatrix} a & \alpha & \beta \\ \gamma & b & c \\ \delta & d & e \end{pmatrix}=(t_{ij}). \tag{1}$$

We consider the R-matrix

$$R=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-1}p^{-1} & 0 & 0 & 0 & 1-p^{-1} & 0 & 0 \\ 0 & 1-p^{-1} & 0 & qp^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & qp^{-1} & 0 & p^{-1}-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^{-1} \end{pmatrix}$$

where $p,q\in\mathbb{C}-\{0\}$. This matrix satisfies the graded Yang-Baxter equation $R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}$ where $R_{12}=R\otimes I_3$, *etc* with the 3x3 identity matrix I_3.

The matrix $\hat{R}$ satisfies the $\mathbb{Z}_2$-graded braid relation

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12}=\hat{R}_{23}\hat{R}_{12}\hat{R}_{23}$$

and the $\mathbb{Z}_2$-graded Hecke condition

$$(\hat{R}-I_9)(\hat{R}+p^{-1}I_9)=0.$$

The eigenvalues of $\hat{R}$ are 1 and $-p^{-1}$ and it can be written, as a sum of projectors, in the form

$$\hat{R}=-p^{-1}P_-+P_+$$

where

$$P_-=\frac{-\hat{R}+I_9}{1+p^{-1}}, \quad P_+=\frac{\hat{R}+p^{-1}I_9}{1+p^{-1}} \tag{2}$$

provided that $1+p^{-1}\neq 0$. The projectors obey $P_iP_j=\delta_{ij}P_j$ (no summation) and $P_-+P_+=I_9$.

RESULTS AND DISCUSSION

In this section, we get the $(p,q)-$commutation relations of the elements of the supermatrix T given in (1) and show that the algebra $O(GL_{p,q}(1|2))$ is a super-Hopf algebra.

Theorem 3.1. A 3x3-supermatrix T is a $\mathbb{Z}_2$-graded quantum matrix if and only if

$$\hat{R}T_1T_2=T_1T_2\hat{R} \tag{3}$$

where $T_2=I_3\otimes T$, $T_1=PT_2P$ and $\hat{R}=PR$ with the super permutation matrix P. As a result of (3), the elements of the supermatrix T satisfy the relations

$$ab = ba + q(1- p^{-1})\,\gamma\alpha, \quad ac = p^{-1} ca, \quad ad = p\, da,$$

$$ae = ea + q(1-p)\,\delta\beta, \quad bc = q\, cb, \quad bd = p q^{-1} db,$$

$$be = eb + q^{-1}(p-1)\, dc,\; cd = p q^{-2} dc,\; ce = p q^{-1} ec,\; de = q\, ed,$$

$$a\alpha = p q^{-1} \alpha a, \quad a\beta = q^{-1} p^{-1} \beta a, \quad a\gamma = q\,\gamma a, \quad a\delta = q\,\delta a,$$

$$b\alpha = p q^{-1} \alpha b, \quad b\beta = \beta b + q^{-1}(p-1)\,\alpha c, \quad b\gamma = q\,\gamma b, \tag{4}$$

$$b\delta = \delta b + q(1-p)\,\gamma d, \quad c\alpha = pq\,\alpha c, \quad c\beta = pq\,\beta c, \quad c\gamma = q\,\gamma c,$$

$$c\delta = p\,\delta c, \quad d\alpha = q^{-1} p^{-1} \alpha d, \quad d\beta = p^{-1}\beta d, \quad d\gamma = q^{2} p^{-1} \gamma d, \quad d\delta = q\delta d,$$

$$e\alpha = q^{-2}\alpha e + q^{-1}(p^{-1}-1)\,\beta d, \quad e\beta = q^{-1} p^{-1} \beta e, \quad e\gamma = q^{2}\gamma e + q(1-p)\,\delta c,$$

$$e\delta = q\delta e, \quad \alpha\beta = -q\, p^{-1}\beta\alpha, \quad \alpha\gamma = -q^{2} p^{-1}\gamma\alpha, \quad \alpha\delta = -q^{2}\delta\alpha + q(p-1)\, da,$$

$$\beta\gamma = -q^{2}\gamma\beta + q(1-p)\, ac, \quad \beta\delta = -p\, q^{2}\delta\beta, \; \gamma\delta = -q^{-1}\delta\gamma,$$

$$\alpha^2 = \beta^2 = \gamma^2 = \delta^2 = 0.$$

Proof. Results can be obtained by making direct calculations. □

One can see that when $p = q$, these relations coincide with those of $GL_{p,q}(1|2)$ given in (Celik, 2016).

Definition 3.1. The superalgebra $O(\mathrm{M}_{p,q}(1|2))$ is the quotient of the free algebra $k < a,b,c,d,e,\alpha,\beta,\gamma,\delta >$ by the two-sided ideal $J_{p,q}$ constituted by the relations in (4) of Theorem 3.1.

Let A and B be two superalgebras. Then their tensor product $A \otimes B$ is a superalgebra with respect to tensor product of A and B. The product rule for tensor product of superalgebras is given in the following definition. We denote by $\tau(a)$ the *grade* of an element $a \in A$.

Definition 3.2. If A is a superalgebra, then the product rule in the superalgebra $A \otimes A$ is described by

$$(a_1 \otimes a_2)(a_3 \otimes a_4) = (-1)^{\tau(a_2)\tau(a_3)} a_1 a_3 \otimes a_2 a_4$$

where a_i 's are homogeneous elements in the superalgebra A.

The quantum superdeterminant for the supermatrix T in the block form is given by (cf. Kobayashi and Uematsu, 1992)

$$s\det(T) = \det(A - BD^{-1}C)(\det(D))^{-1}$$

and it is not a central element. If the inverse of the quantum superdeterminant $s\det(T)$ exists, then the algebra $O(GL_{p,q}(1|2))$ has a super-Hopf algebra structure. The super-Hopf algebra structure of $O(GL_{p,q}(1|2))$ is given in below.

Theorem 3.2. The algebra $O(GL_{p,q}(1|2))$ has a unique super-Hopf algebra structure with co-maps Δ, ε and S such that

$$\Delta(t_{ij}) = \sum_{k=1}^{3} t_{ik} \otimes t_{kj}, \; \varepsilon(t_{ij}) = \delta_{ij} \text{ and } \mathrm{S}(T) = T^{-1}.$$

Proof. The following properties of the co-structures can easily verified:

The comultiplication Δ is coassociative in the sense that

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$

where $\mathrm{id}: A \to A$ denotes the identity map and $\Delta(uv) = \Delta(u)\Delta(v)$, $\Delta(1) = 1 \otimes 1$.

The counit ε has the property

$$m \circ (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = m \circ (\mathrm{id} \otimes \varepsilon) \circ \Delta$$

where $m : A \otimes A \to A$ and $\varepsilon(uv) = \varepsilon(u)\varepsilon(v)$, $\varepsilon(1) = 1$.

The coinverse S satisfies

$$m\circ(S\otimes \mathrm{id})\circ\Delta=\varepsilon=m\circ(\mathrm{id}\otimes S)\circ\Delta$$

and $S(uv)=(-1)^{\tau(u)\tau(v)}S(v)S(u),\ \ S(1)=1$. □

Definition 3.3. The super-Hopf algebra $O(GL_{p,q}(1|2))$ is called the coordinate algebra of the quantum supergroup $GL_{p,q}(1|2)$.

A discussion of some submatrices

Here are a few comments about some submatrices of T.

1. Let us first consider the even 2x2-submatrix $T_{33}=\begin{pmatrix} a & \alpha \\ \gamma & b\end{pmatrix}$ which forms subgroup $GL_{p,q}(1|1)$ with the commutation rules

$$a\alpha=p\,q^{-1}\alpha a,\quad a\gamma=q\,\gamma a,\quad b\alpha=p\,q^{-1}\alpha b,\quad b\gamma=q\,\gamma b,$$

$$ab=ba+q(1-\ p^{-1})\,\gamma\alpha,\quad \alpha\gamma=-\ q^{2}p^{-1}\gamma\alpha,\quad \alpha^{2}=\gamma^{2}=0.$$

These relations coincide with relations in (Dabrowski and Wang, 1991) when p is replaced by pq. If we assume that the formal inverse b^{-1} of b exists, then the quantum superdeterminant is given by the expression

$$s\det(T_{33})=ab^{-1}-\alpha b^{-1}\gamma b^{-1}$$

and it is a central element of the quantum superalgebra $O(GL_{p,q}(1|1))$.

It can be seen in a similar way that the even 2x2-submatrix $T_{22}=\begin{pmatrix} a & \beta \\ \delta & e\end{pmatrix}$ forms subgroup $GL_{p,q}(1|1)$ with the defining commutation relations.

2. We now consider an algebra A generated by the elements a,α,δ,d and defining commutation rules

$$a\beta = q^{-1}p^{-1}\beta a,\quad a\delta = q\,\delta a,\quad e\beta = q^{-1}p^{-1}\beta e,\qquad e\delta = q\delta e,$$

$$ae = ea + q(1-p)\,\delta\beta,\quad \beta\delta = -p\,q^{2}\,\delta\beta,\quad \beta^{2} = \delta^{2} = 0\,.$$

Obviously these relations represent a two-parameter deformation of the algebra A. Here the generators a and d are almost even (bosonic) and the generators α and δ are almost odd (fermionic). Indeed, as $p,q \to 1$ the algebra A with these relations becomes a superalgebra. However, submatrices of the form $T_{23} = \begin{pmatrix} a & \alpha \\ \delta & d \end{pmatrix}$ with the defined relations (except for p=q=1) do not form a subgroup $GL_{p,q}(1|1)$. It seems that such matrices are related to the super braided matrices (Majid, 1991). If so, this will be addressed in another study.

3. The 2x2-submatrix $T_{23} = \begin{pmatrix} b & c \\ d & e \end{pmatrix}$ forms subgroup $GL_{p,q}(2)$ subject to the relations

$$bc = qcb,\qquad bd = p\,q^{-1}db,\qquad ce = p\,q^{-1}ec,\qquad de = qed,$$

$$be = eb + q^{-1}(p-1)\,dc,\qquad cd = p\,q^{-2}\,dc.$$

These relations coincide with relations given in (Schirrmacher et al., 1991) when q is replaced by p and pq^{-1} is replaced by q. The quantum determinant is given by

$$\det(T_{11}) = be - qcd = eb - q^{-1}dc$$

and it is not in the centre of the algebra $O(GL_{p,q}(2))$, but it becomes central if $p = q^{2}$.

CONCLUSION

An *R*-matrix satisfying quantum Yang-Baxter equation was found, and using this matrix, deformation of the supergroup with a two-parameter was obtained and it shown that has a super-Hopf algebra structure, as usual

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