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An Algorithm for Solutions to the Elliptic Quaternion Matrix Equation $AX = B$

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Abstract: In this paper, the existence of solution to the elliptic quaternion matrix equations $AX = B$ is characterized and solutions of this matrix equation are derived by means of real representations. Also, our results are illustrated with an example.

Keywords: Elliptic quaternion, Elliptic quaternion matrix, Real representation of elliptic quaternion matrices.

1 Introduction and Preliminaries

The well-known concept of the real quaternion was first introduced by Hamilton in 1843 [1]. It has four components, i.e.

$$q = q_r + q_i i + q_j j + q_k k$$

where $q_r, q_i, q_j, q_k \in R$ and i, j and k satisfy

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ ik = -ki = -j,\ jk = -kj = i.$$

The real quaternion algebra plays an important role in matrix analysis, quantum physics, kinematics, differential geometry, game development, image and signal processing etc. Thus, there are number of studies associated with real quaternions [2, 3]. Since the multiplication of real quaternions is non-commutative, all results about the complex numbers cannot be generalized in real quaternions. This problem restricts the applications of real quaternions. In addition, this can increase the complexity of many processes.

The concept of commutative quaternions was first introduced by Schütte and Wenzel [4]. The major difference between commutative quaternions and real quaternions is commutativeness of the multiplication, which are commutative for commutative quaternions. There are many studies on commutative quaternions in literature. Catoni et al. studied the functions of commutative quaternions variable and obtained generalized Cauchy-Riemann conditions [5]. Pei et. al introduced digital signal and image processing using commutative quaternions. For color image processing, they defined a simplified polar form of commutative quaternions to represent the color image and showed that this representation is useful to process color images in the brightness-hue-saturation color space [4]. In [6], Isokawa et al. investigated two types of multistate Hopfield neural networks based on commutative quaternions. Moreover, Kosal and Tosun investigated some algebraic properties of commutative quaternion matrices by means of complex representation of commutative quaternion matrices [7]. In [8], Kosal et al. constructed some explicit expression of the solution of the Kalman-Yakubovich-conjugate commutative quaternion matrix equations, by means of real representation of a commutative quaternion matrices. In [9], Kosal and Tosun studied some equivalence relations and related to results over the commutative quaternions and their matrices. In this sense, they defined consimilarity, semisimilarity and consemisimilarity over the commutative quaternions algebra and their matrix algebra and determined the equalities of these equivalence relations. In [10], Kosal and Tosun established universal similarity factorization equalities over the commutative quaternions and their matrices. Based on these equalities, real matrix representations of commutative quaternions and their matrices have been derived, and their algebraic properties and fundamental equations have been determined.

In this study, the existence of solution to the elliptic quaternion matrix equations $AX = B$ is characterized and solutions of this matrix equation are derived by means of real representations. Elliptic quaternions are generalized form of commutative quaternions and so complex numbers [5]. Thus, the obtained results extend, generalize and complement some known commutative quaternions matrices and complex matrices results from the literature.

A set of elliptic quaternions is denoted by [5]

$$H_p = \{a = a_0 + a_1 i + a_2 j + a_3 k :\ a_0, a_1, a_2, a_3 \in R,\ i, j, k \notin R\}$$

where

$$i^2 = k^2 = \alpha,\ j^2 = 1,\ ij = ji = k,\ jk = kj = i,\ ki = ik = \alpha j,\ \alpha < 0.$$

There are three types of conjugate of $a = a_0 + a_1 i + a_2 j + a_3 k \in H_p$. They are

$$ {}^{1}\bar{a} = a_0 - a_1 i + a_2 j - a_3 k, $$
$$ {}^{2}\bar{a} = a_0 + a_1 i - a_2 j - a_3 k, \ . $$
$$ {}^{3}\bar{a} = a_0 - a_1 i - a_2 j + a_3 k $$

and norm of a is defined

$$ \begin{aligned} \|a\| &= \sqrt[4]{\left| a \left({}^{1}\bar{a}\right) \left({}^{2}\bar{a}\right) \left({}^{3}\bar{a}\right) \right|} \\ &= \sqrt[4]{\left[(a_0 + a_2)^2 - \alpha (a_1 + a_3)^2 \right] \left[(a_0 - a_2)^2 - \alpha (a_1 - a_3)^2 \right]}. \end{aligned} $$

Addition, multiplication and scalar multiplication of the elliptic quaternions $a = a_0 + a_1 i + a_2 j + a_3 k, b = b_0 + b_1 i + b_2 j + b_3 k \in H_p$ and $\lambda \in R$ are defined by

$$ a + b = (a_0 + b_0) + (a_1 + b_1)\, i + (a_2 + b_2)\, j + (a_3 + b_3)\, k, $$

$$ \begin{aligned} pq = &(a_0 b_0 + \alpha a_1 b_1 + a_2 b_2 + \alpha a_3 b_3) + (a_1 b_0 + a_0 b_1 + a_3 b_2 + a_2 b_3)\, i \\ &+ (a_0 b_2 + a_2 b_0 + \alpha a_1 b_3 + \alpha a_3 b_1)\, j + (a_3 b_0 + a_0 b_3 + a_1 b_2 + a_2 b_1)\, k, \end{aligned} $$

and

$$ \lambda a = \lambda \left(a_0 + a_1 i + a_2 j + a_3 k \right) = \lambda a_0 + \lambda a_1 i + \lambda a_2 j + \lambda a_3 k $$

respectively.
If $a = a_0 + a_1 i + a_2 j + a_3 k \in H_p$ and $\|a\| \neq 0$ then a has multiplicative inverses. Multiplicative inverse of a is given by

$$ a^{-1} = \frac{\left({}^{1}\bar{a}\right)\left({}^{2}\bar{a}\right)\left({}^{3}\bar{a}\right)}{\|a\|^4}. $$

2 Elliptic Quaternion Matrices

The set of $H_p^{m \times n}$ denotes all $m \times n$ type matrices with elliptic quaternion entries. For $A = \left(a_{ij}\right),\ B = \left(b_{ij}\right) \in H_p^{m \times n},\ C = \left(c_{jk}\right) \in H_p^{n \times l}$ and $\lambda \in R$, the ordinary matrix addition, scalar multiplication and multiplication are defined by

$$ A + B = \left(a_{ij}\right) + \left(b_{ij}\right) = \left(a_{ij} + b_{ij}\right) \in H_p^{m \times n}, $$

$$ \lambda A = \lambda \left(a_{ij}\right) = \left(\lambda a_{ij}\right) \in H_p^{m \times n} $$

and

$$ AC = \left(\sum_{j=1}^{n} a_{ij} c_{jk} \right) \in H_p^{m \times l} $$

respectively.
There are three types' of conjugate of $A = \left(a_{ij}\right) \in H_p^{m \times n}$. They are

$$ {}^{1}\bar{A} = \left({}^{1}\overline{a_{ij}}\right) \in H_p^{m \times n},\ {}^{2}\bar{A} = \left({}^{2}\overline{a_{ij}}\right) \in H_p^{m \times n}\, and\, {}^{3}\bar{A} = \left({}^{3}\overline{a_{ij}}\right) \in H_p^{m \times n}. $$

A matrix $A^T \in H_p^{n \times m}$ is transpose of $A \in H_p^{m \times n}$. Also $A^{*_s} = \left({}^{s}\overline{A}\right)^T \in H_p^{m \times n},\ s = 1, 2, 3,$ is called conjugate transpose according to the s^{th} conjugate of $A \in H_p^{m \times n}$.

Theorem 1. *Let A and B be elliptic quaternion matrices of appropriate sizes. Then followings are satisfied:*

1. $\left({}^{s}\bar{A}\right)^T = {}^{s}\overline{\left(A^T\right)}$,
2. $(AB)^{*_s} = B^{*_s} A^{*_s}$,
3. $(AB)^T = B^T A^T$,
4. ${}^{s}\overline{(AB)} = \left({}^{s}\bar{A}\right)\left({}^{s}\bar{B}\right)$,
5. *If A^{-1} and B^{-1} exist then* $(AB)^{-1} = B^{-1} A^{-1}$,
6. *If A^{-1} exists* $\left(A^{*_s}\right)^{-1} = \left(A^{-1}\right)^{*_s}$,
7. $\left({}^{s}\bar{A}\right)^{-1} = {}^{s}\overline{\left(A^{-1}\right)}$.

This theorem can also be easily proved by direct calculation.

3 Real Representation of Elliptic Quaternion Matrices

Let $A = A_0 + A_1 i + A_2 j + A_3 k \in H_p^{m\times n}$ where $A_0, A_1, A_2, A_3 \in R^{m\times n}$. For $X \in H_p^{n\times n}$, we will define the linear transformations

$$\eta_A(X) = AX.$$

Then, we get real representations of elliptic quaternion matrix $A = A_0 + A_1 i + A_2 j + A_3 k \in H_p^{m\times n}$

$$\eta_p(A) = \begin{pmatrix} A_0 & \alpha A_1 & A_2 & \alpha A_3 \\ A_1 & A_0 & A_3 & A_2 \\ A_2 & \alpha A_3 & A_0 & \alpha A_1 \\ A_3 & A_2 & A_1 & A_0 \end{pmatrix} \in R^{4m\times 4n}.$$

Theorem 2. *Let $A, B \in H_p^{m\times n}$, $C \in H_p^{n\times p}$ and $\lambda \in R$. Then following identities are satisfied:*

1. $A = B \Leftrightarrow \eta_p(A) = \eta_p(B)$, $\eta_p(A+B) = \eta_p(A) + \eta_p(B)$,
2. $\eta_p(AC) = \eta_p(A)\eta_p(C)$, $\eta_p(\lambda A) = \eta_p(A\lambda) = \lambda\eta_p(A)$,
3. $A = \frac{1}{2-2\alpha} E_{4m}\eta_p(A) E_{4n}^{*_1}$ *where* $E_{4t} = \begin{pmatrix} I_t & iI_t & jI_t & kI_t \end{pmatrix} \in H^{t\times 4t}$,
4. *If A is a nonsingular matrix of size m then*

$$\eta_p\left(A^{-1}\right) = \eta_p^{-1}(A),\ A^{-1} = \frac{1}{2-2\alpha} E_{4m}\eta_p^{-1}(A) E_{4m}^{*_1},$$

5. $\eta_p(A) = R_{4m}^{-1}\eta_p(A) R_{4n},\ \eta_p(A) = S_{4m}^{-1}\eta_p(A) S_{4n}$ *and* $\eta_p(A) = T_{4m}^{-1}\eta_p(A) T_{4n}$ *where*

$$Q_{4t} = \begin{pmatrix} 0 & \alpha I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha I_t \\ 0 & 0 & I_t & 0 \end{pmatrix},\quad S_{4t} = \begin{pmatrix} 0 & 0 & I_t & 0 \\ 0 & 0 & 0 & I_t \\ I_t & 0 & 0 & 0 \\ 0 & I_t & 0 & 0 \end{pmatrix},\quad T_{4t} = \begin{pmatrix} 0 & 0 & 0 & \alpha I_t \\ 0 & 0 & I_t & 0 \\ 0 & \alpha I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \end{pmatrix}.$$

4 On Solutions to the Elliptic Quaternion Matrix Equation $AX = B$

Now, we consider the solution of the

$$AX = B \tag{1}$$

by means of the real representation, where $A \in H_p^{m\times n}$, $B \in H_p^{m\times p}$. We define the real representation of the matrix equation (1) by

$$\eta_p(A) Y = \eta_p(B). \tag{2}$$

Proposition 1. *The equation (1) has a solution X if and only if the equation (2) has a solution $Y = \eta_p(X)$.*

Theorem 3. *The equation (2) has a solution $Y \in R^{4n\times 4p}$ if and only if the equation (1) has a solution $X \in H_p^{n\times p}$. In that case, if $Y \in R^{4n\times 4p}$ is a solution of (2), then the matrix*

$$X = \frac{1}{8-8\alpha}\left(I_m\, iI_m\, jI_m\, kI_m\right)\left(Y + Q_{4m}^{-1} Y Q_{4p} + S_{4m}^{-1} Y S_{4p} + T_{4m}^{-1} Y T_{4p}\right)\begin{pmatrix} I_p \\ -iI_p \\ jI_p \\ -kI_p \end{pmatrix} \tag{3}$$

is a solution of (1).

Proof:
We show that if the real matrix

$$Y = \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{pmatrix},\ Y_{uv} \in R^{n\times p},\ u, v = 1, 2, 3, 4 \tag{4}$$

is a solution to (2), then the matrix given in (3) is a solution to (1). Since $Q_m^{-1} Y Q_n = Y, \quad R_m^{-1} Y R_n = Y, \quad S_m^{-1} Y S_n = Y$, we have

$$\eta_p(A) Q_{4m}^{-1} Y Q_{4p} = \eta_p(B),\ \ \eta_p(A) R_{4m}^{-1} Y R_{4p} = \eta_p(B),\ \ \eta_p(A) S_{4m}^{-1} Y S_{4p} = \eta_p(B).$$

This equation shows that if Y is a solution to (2), then $Q_{4m}^{-1} Y Q_{4p}, \quad R_{4m}^{-1} Y R_{4p}$ and $S_{4m}^{-1} Y S_{4p}$ are also solutions to (2). Thus the following real matrix:

$$Y' = \frac{1}{4}\left(Y + Q_{4m}^{-1} Y Q_{4p} + R_{4m}^{-1} Y R_{4p} + S_{4m}^{-1} Y S_{4p}\right) \tag{5}$$

is a solution to (2). Now substituting (4) in (5) and the simplifying the expression, we easily get

$$Y' = \begin{pmatrix} Z_0 & \alpha Z_1 & Z_2 & \alpha Z_3 \\ Z_1 & Z_0 & Z_3 & Z_2 \\ Z_2 & \alpha Z_3 & Z_0 & \alpha Z_1 \\ Z_3 & Z_2 & Z_1 & Z_0 \end{pmatrix},$$

where

$$Z_0 = \tfrac{1}{4}\left(Y_{11} + Y_{22} + Y_{33} + Y_{44}\right), \quad Z_1 = \tfrac{1}{4}\left(\tfrac{Y_{12}}{\alpha} + Y_{21} + \tfrac{Y_{34}}{\alpha} + Y_{43}\right),$$
$$Z_2 = \tfrac{1}{4}\left(Y_{13} + Y_{24} + Y_{31} + Y_{42}\right), \quad Z_3 = \tfrac{1}{4}\left(\tfrac{Y_{14}}{\alpha} + Y_{23} + \tfrac{Y_{32}}{\alpha} + Y_{41}\right).$$

Thus we obtain

$$X = Z_1 + Z_2 i + Z_3 j + Z_4 k = \frac{1}{8-8\alpha}\left(I_m \, iI_m \, jI_m \, kI_m\right)\left(Y + Q_{4m}^{-1} Y Q_{4p} + S_{4m}^{-1} Y S_{4p} + T_{4m}^{-1} Y T_{4p}\right)\begin{pmatrix} I_p \\ -iI_p \\ jI_p \\ -kI_p \end{pmatrix}.$$

□

5 Numerical Algorithms

Based on the discussions in the previous section, in this section we provide numerical algorithms for solving elliptic quaternion matrix equation $AX = B$.

1. Input A_0, A_1, A_2, A_3 and B_0, B_1, B_2, B_3.
2. Form $\eta_p(A)$ and $\eta_p(B)$.
3. Compute Y and $Y' = \frac{1}{4}\left(Y + Q_{4m}^{-1} Y Q_{4p} + R_{4m}^{-1} Y R_{4p} + S_{4m}^{-1} Y S_{4p}\right)$.
4. Calculate

$$X = \frac{1}{8-8\alpha}\left(I_m \, iI_m \, jI_m \, kI_m\right)\left(Y + Q_{4m}^{-1} Y Q_{4p} + S_{4m}^{-1} Y S_{4p} + T_{4m}^{-1} Y T_{4p}\right)\begin{pmatrix} I_p \\ -iI_p \\ jI_p \\ -kI_p \end{pmatrix}.$$

6 Numerical Examples

Let us for solve the elliptic quaternion matrix equation

$$\begin{pmatrix} 1 & 1+i \\ j & k \end{pmatrix} X = \begin{pmatrix} 1+2i+j & 2+j+k \\ -1+j+2k & -1+i+2j \end{pmatrix}.$$

Under consideration of the Theorem 3, real representation of given matrix equation is

$$\begin{pmatrix} 1 & 1 & 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \alpha \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & \alpha \\ 1 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} Y = \begin{pmatrix} 1 & 2 & 2\alpha & 0 & 1 & 1 & 0 & \alpha \\ -1 & -1 & 0 & \alpha & 1 & 2 & 2\alpha & 0 \\ 2 & 0 & 1 & 2 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & -1 & 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & \alpha & 1 & 2 & 2\alpha & 0 \\ 1 & 2 & 2\alpha & 0 & -1 & -1 & 0 & \alpha \\ 0 & 1 & 1 & 1 & 2 & 0 & 1 & 2 \\ 2 & 0 & 1 & 2 & 0 & 1 & -1 & -1 \end{pmatrix}$$

If we solve this equation, we have

$$Y = \begin{pmatrix} 1 & 2 & 2\alpha & 0 & -1 & -1 & -2\alpha & -\alpha \\ 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\ 2 & 0 & 1 & 2 & -2 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 \\ -1 & -1 & -2\alpha & -\alpha & 1 & 2 & 2\alpha & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & -1 & -1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Finally, we obtain

$$\begin{aligned} X &= \tfrac{1}{8-8\alpha}\left(I_m \, iI_m \, jI_m \, kI_m\right)\left(Y + Q_{4m}^{-1} Y Q_{4p} + S_{4m}^{-1} Y S_{4p} + T_{4m}^{-1} Y T_{4p}\right)\begin{pmatrix} I_p \\ -iI_p \\ jI_p \\ -kI_p \end{pmatrix} \\ &= \begin{pmatrix} 1+2i-j-2k & 2-j-k \\ 2j & 2j \end{pmatrix}. \end{aligned}$$

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7 References

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