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Homotopy Analysis Aboodh Transform Method for Nonlinear System of Partial Differential Equations

Djelloul Ziane[a] **and Mountassir Hamdi Cherif**[b*]

[a,b]*Laboratory of mathematics and its applications (LAMAP), University of Oran1 Ahmed Ben Bella, Oran, 31000, Algeria*
[b]*Department of Mathematics and Informatics, Electrical and Energy Engineering Graduate School (ESGEE), Oran, Algeria*
[*]*Corresponding author*

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Abstract

In this paper, a combined form of homotopy analysis method with Aboodh transform method is proposed to solve nonlinear system of partial differential equations. This method is called the homotopy analysis Aboodh transform method (HAATM). The homotopy analysis Aboodh transform method can easily be applied to many problems of nonlinear system, and is capable of reducing the size of computational work.

1. Introduction

The nonlinear evolution equations have attracted the attention of many researchers because of their wide applications in various fields such as physics, fluid mechanics, bio-mathematics, chemical physics and other areas of science and engineering. The investigation of exact solutions for the nonlinear evolution equations is a particularly hot topic [1]. So we find that a lot of researchers are working to develop new methods to solve this kind of equations. These efforts have strengthened this area of research through many methods, among them we find, homotopy analysis method (HAM). This method was developed in 1992 by Liao Shijun ([2], [3], [4], [5]), and was used by many researchers to solven nonlinear differential equations ([6], [7], [8]). Then, a new option emerged recently, includes the composition of Laplace transform, Sumudu transform, Natural transform or Aboodh transform with this method to solve nonlinear differential equations. Among which are the homotopy analysis method coupled with Laplace transform ([9], [10], [11]), homotopy analysis Sumudu transform method ([12], [13], [14]), homotopy Natural transform method ([15], [16]) and homotopy analysis Aboodh transform method [17].
The aim of this study is to combine homotopy analysis method and Aboodh transform method in order to obtain a more effective method, characterized by speed in solution and accuracy in the results obtained. The modified method is called homotopy analysis Aboodh transform method (HAATM). Three examples of nonlinear partial differential equations are given to re-confirm the strength and effectiveness of this modified method.
The present paper has been organized as follows: In Section 2 Some basic definitions and properties of the Aboodh transform method. In section 3 We give an analysis of the proposed method. In section 4 We present three examples explaining how to apply the proposed method. Finally, the conclusion follows.

2. Definitions and properties of the Aboodh transform

In this section, we give some basic definitions and properties of Aboodh transform which are used further in this paper.
A new transform called the Aboodh transform defined for function of exponential order, we consider functions in the set $\bar{A}$, defined by [18]:

$$\bar{A} = \left\{ f(t) : \exists\, M,\ k_1, k_2 > 0,\ |f(t)| < Me^{-vt} \right\}.$$

For given function in the set $\bar{A}$, the constant M must be finite number, k_1, k_2 my be finite or infinite.

Email addresses: **djeloulz@yahoo.com (D. Ziane), mountassir27@yahoo.fr (M. H. Cherif)**

The Aboodh transform denoted by the operator $A(\cdot)$ defined by the integral equation:

$$A[f(t)] = K(v) = \frac{1}{v}\int_0^\infty f(t)e^{-vt}dt, \quad t \geqslant 0, k_1 \leqslant v \leqslant k_2.$$

We will summarize here some results of simple functions related to Aboodh transform in the following table [18]:

$f(t)$	$A[f(t)]$	$f(t)$	$A[f(t)]$
1	$\frac{1}{v^2}$	$\sin at$	$\frac{a}{v(v^2+a^2)}$
t	$\frac{1}{v^3}$	$\cos at$	$\frac{1}{v^2+a^2}$
t^n	$\frac{n!}{v^{n+2}}$	$\sinh at$	$\frac{a}{v(v^2-a^2)}$
e^{at}	$\frac{1}{v^2-av}$	$\cosh at$	$\frac{1}{v^2-a^2}$

Theorem 2.1. *Let $K(v)$ is the Aboodh transform of $f(t)$, then one has:*

$$A\left[f'(t)\right] = vK(v) - \frac{f(0)}{v},$$

$$A\left[f''(t)\right] = v^2K(v) - \frac{f'(0)}{v} - f(0),$$

$$A\left[f^{(n)}(t)\right] = v^nK(v) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{v^{2-n+k}}.$$

Proof. (see [18]). □

Aboodh transform of partial derivative: To obtain Aboodh transform of partial derivative, we use integration by parts, and then we have:

$$A\left[\frac{\partial u(x,t)}{\partial t}\right] = vK(x,v) - \frac{u(x,0)}{v},$$

$$A\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right] = v^2K(x,v) - \frac{1}{v}\frac{\partial u(x,0)}{\partial t} - u(x,0),$$

For the proof of these formulas, you can see [19].

Theorem 2.2. *Let $K(x,v)$ is the Aboodh transform of $u(x,t)$, then one has:*

$$A\left[\frac{\partial^n u(x,t)}{\partial t^n}\right] = v^nK(x,v) - \sum_{k=0}^{n-1}\frac{1}{v^{2-n+k}}\frac{\partial^k u(x,0)}{\partial t^k}.$$

Proof. (see [17]). □

3. Homotopy analysis Aboodh transform method (HAATM)

To illustrate the basic idea of this method, we consider a general non-homogeneous, nonlinear partial diffrential equation

$$L_t[V(x,t)] + R[V(x,t)] + N[V(x,t)] = f(x,t), \tag{3.1}$$

where L_t denotes a first-order partial diffrential operator, R is the general linear operators, N is the nonlinear operator and $f(x,t)$ is the source terms.
Taking the Aboodh transform on both sides of (3.1), we get

$$A(L_t[V(x,t)]) + A(R[V(x,t)] + N[V(x,t)]) = A[f(x,t)]$$

Using the property of the Aboodh transform, we have

$$A[V(x,t)] - \frac{1}{v^2}V(x,0) + \frac{1}{v}(A[R(V(x,t)) + N(V(x,t)) - f(x,t)]) = 0$$

Define the nonlinear operators

$$N[\phi(x,t;p)] = A[\,\phi(x,t;p)] - \frac{1}{v^2}V(x,0;p) + \frac{1}{v}(A[R(\phi(x,t;p)) + N(\phi(x,t;p)) - f(x,t;p)])$$

By means of homotopy analysis method ([2], [3], [4], [5]), we construct the so-called the zero-order deformation equation

$$(1-q)A\left[\phi(x,t;p)-V_0(x,t)\right]=phH(x,t)N[\phi(x,t;p)], \tag{3.2}$$

where p is an embedding parameter and $p\in[0,\,1]$, $H(x,t)\neq 0$ is an auxiliary function, $h\neq 0$ is an auxiliary parameter, A is an auxiliary linear Aboodh operator. When $p=0$ and $p=1$, we have

$$\begin{cases}\phi(x,t;0)=V_0(x,t),\\ \phi(x,t;1)=V(x,t).\end{cases}$$

When P increases from 0 to 1, the $\phi(x,t,p)$ various from $V_0(x,t)$ to $V(x,t)$. Expanding $\phi(x,t;p)$ in Taylor series with respect to p, we have

$$\phi(x,t;p)=V_0(x,t)+\sum_{m=1}^{+\infty}V_m(x,t)p^m, \tag{3.3}$$

where

$$V_m(x,t)=\frac{1}{m!}\frac{\partial^m\phi(x,t;p)}{\partial p^m}\Big|_{p=0}$$

When $p=1$, the formula (3.3) becomes

$$V(x,t)=V_0(x,t)+\sum_{m=1}^{+\infty}V_m(x,t).$$

Define the vectors

$$\overrightarrow{V}=\{V_0(x,t),V_1(x,t),V_2(x,t),\ldots,V_m(x,t)\}.$$

Differentiating (3.2) $m-$times with respect to p, then setting $p=0$ and finally dividing them by $m!$, we obtain the so-called m^{th} order deformation equation

$$A[V_m(x,t)-\chi_m V_{m-1}(x,t)]=hH(x,t)\Re_m(\overrightarrow{V}_{m-1}(x,t)), \tag{3.4}$$

where

$$\Re_m(\overrightarrow{V}_{m-1}(x,t))=\frac{1}{(m-1)!}\frac{\partial^{m-1}N(x,t;p)}{\partial p^{m-1}}\Big|_{p=0},$$

and

$$\chi_m=\begin{cases}0, & m\leqslant 1,\\ 1, & m>1.\end{cases}$$

Applying the inverse Aboodh transform on both sides of (3.4), we can obtain

$$V_m(x,t)=\chi_m V_{m-1}(x,t)+hA^{-1}\left[H(x,t)\Re_m(\overrightarrow{V}_{m-1}(x,t))\right]. \tag{3.5}$$

The m^{th} deformation equation (3.5) is a linear which can be easily solved. So, the solution of (3.1) can be written into the following form

$$V(x,t)=\sum_{m=0}^{N}V_m(x,t),$$

when $N\to\infty$, we can obtain an accurate approximation solution of (3.1).
For the proof of the convergence of the homotopy analysis method see [3].

4. Application of this method

In this section, we apply the homotopy analysis method (HAM) coupled with Aboodh transform method for solving system of nonlinear partial differential equations.

Example 4.1. *We consider the following system of nonlinear coupled Burgers partial differential equations*

$$\begin{cases} U_t - U_{xx} - 2UU_x + (UV)_x = 0 \\ V_t - V_{xx} - 2VV_x + (UV)_x = 0 \end{cases}, \tag{4.1}$$

with the initial conditions

$$U(\varkappa,0) = \sin x,\ V(x,0) = \sin x.$$

The nonlinear operators are

$$\begin{cases} N[\phi(x,t,p)] = A[\phi(x,t;p)] - \frac{1}{v^2}\sin x \\ +\frac{1}{v}A\left[-\phi_{xx}(x,t;p) - 2\phi(x,t;p)\phi_x(x,t;p) + (\phi(x,t;p)\varphi(x,t;p))_x\right] \\ N[\varphi(x,t,p)] = A[\varphi(x,t;p)] - \frac{1}{v^2}\sin x \\ +\frac{1}{v}A\left[-\varphi_{xx}(x,t;p) - 2\varphi(x,t;p)\varphi_x(x,t;p) + (\phi(x,t;p)\varphi(x,t;p))_x\right] \end{cases}.$$

Thus, we obtain the m^{th} order deformation equations given by

$$\begin{cases} U_m(x,t) = \chi_m U_{m-1}(x,t) + hA^{-1}[\Re_m(\overrightarrow{U}_{m-1}(x,t))] \\ V_m(x,t) = \chi_m V_{m-1}(x,t) + hA^{-1}[\Re_m(\overrightarrow{V}_{m-1}(x,t))] \end{cases}. \tag{4.2}$$

with

$$\begin{cases} \Re_m(\overrightarrow{U}_{m-1}(x,t)) = A[U_{m-1}(x,t)] - \frac{1}{v^2}(1-\chi_m)\sin x \\ +\frac{1}{v}A\left[\sum_{i=0}^{m-1}(U_iV_{m-1-i})_\varkappa - 2\sum_{i=0}^{m-1}U_i(U_{m-1-i})_\varkappa - \sum_{i=0}^{m-1}(U_i)_{xx}\right], \\ \Re_m(\overrightarrow{V}_{m-1}(x,t)) = A[V_{m-1}(x,t)] - \frac{1}{v^2}(1-\chi_m)\sin x \\ +\frac{1}{v}A\left[\sum_{i=0}^{m-1}(U_iV_{m-1-i})_\varkappa - 2\sum_{i=0}^{m-1}V_i(V_{m-1-i})_\varkappa - \sum_{i=0}^{m-1}(V_i)_{xx}\right], \end{cases} \tag{4.3}$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases}$$

According to (4.2) and (4.3), the formulas of the first terms is given by

$$\begin{gathered} U_1(x,t) = hA^{-1}\left(\frac{1}{v}A\left[(U_0V_0)_x - 2U_0(U_0)_x - (U_0)_{xx}\right]\right), \\ U_2(x,t) = (1+h)U_1(x,t) \\ +hA^{-1}\left(\frac{1}{v}A\left[(U_0V_1 + U_1V_0)_x - 2(U_0U_{1x} + U_1U_{0x}) - (U_1)_{xx}\right]\right), \\ U_3(x,t) = (1+h)U_2(x,t) \\ +hA^{-1}\left(\frac{1}{v}A\left[(U_0V_2 + U_1V_1 + U_2V_0)_x - 2(2U_0U_{2x} + U_1U_{1x}) - (U_2)_{xx}\right]\right), \\ \vdots \end{gathered} \tag{4.4}$$

and

$$\begin{gathered} V_1(x,t) = hA^{-1}\left(\frac{1}{v}A\left[(U_0V_0)_x - 2V_0(V_0)_x - (V_0)_{xx}\right]\right), \\ V_2(x,t) = (1+h)V_1(x,t) \\ +hA^{-1}\left(\frac{1}{v}A\left[(U_0V_1 + U_1V_0)_x - 2(V_0V_{1x} + V_1V_{0x}) - (V_1)_{xx}\right]\right), \\ V_3(x,t) = (1+h)V_2(x,t) \\ +hA^{-1}\left(\frac{1}{v}A\left[(U_0V_2 + U_1V_1 + U_2V_0)_x - 2(2V_0V_{2x} + V_1V_{1x}) - (V_2)_{xx}\right]\right), \\ \vdots \end{gathered} \tag{4.5}$$

From the equations (4.4) and (4.5), the first solution terms of homotopy analysis Aboodh transform method of the system (4.1), is given by

$$
\begin{aligned}
U_0(x,t) &= \sin x,\\
V_0(x,t) &= \sin x,\\
U_1(x,t) &= (h)\sin(x)t,\\
V_1(x,t) &= (h)\sin(x)t,\\
U_2(x,t) &= (h)(1+h)\sin(x)t+(h^2)\sin(x)\tfrac{t^2}{2!},\\
V_2(x,t) &= (h)(1+h)\sin(x)t+(h^2)\sin(x)\tfrac{t^2}{2!},\\
U_3(x,t) &= (h)(1+h)^2\sin(x)t+2(1+h)(h^2)\sin(x)\tfrac{t^2}{2!}+(h^3)\sin(x)\tfrac{t^3}{3!},\\
V_3(x,t) &= (h)(1+h)^2\sin(x)t+2(1+h)(h^2)\sin(x)\tfrac{t^2}{2!}+(h^3)\sin(x)\tfrac{t^3}{3!},\\
&\vdots
\end{aligned}
$$

and so on.
The other components of the (HAATM) can be determined in a similar way. Finally, the approximate solution (U,V) of the system (4.1)in a series form, is given by

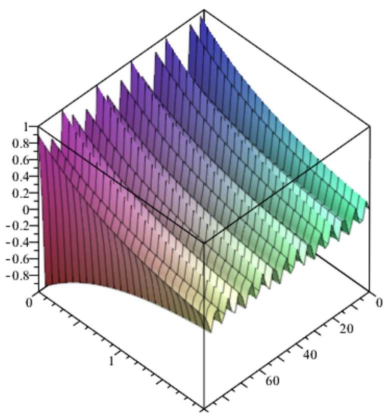
$$
\begin{cases}
U(x,t)=\sin x\left(1+h(3+3h+h^2)t+(3+2h)h^2\frac{t^2}{2!}+h^3\frac{t^3}{3!}+\cdots\right)\\
V(x,t)=\sin x\left(1+h(3+3h+h^2)t+(3+2h)h^2\frac{t^2}{2!}+h^3\frac{t^3}{3!}+\cdots\right)
\end{cases}
$$

*Substiting $h=-1$ in (**??**), the approximate solution of the system (4.1) is given as follows*

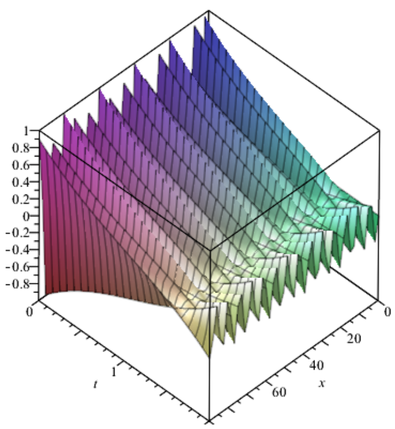
$$
\begin{cases}
U(x,t)=\sin x\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\cdots\right)\\
V(x,t)=\sin x\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\cdots\right)
\end{cases}
$$

And in the closed form, the solution (U,V) is given by

$$
\begin{cases}
U(x,t)=\sin(x)e^{-t}\\
V(x,t)=\sin(x)e^{-t}
\end{cases}.
$$



(a)



(b)

Figure 4.1: (a) Exact solution for $U(x,t)$ and $V(x,t)$, (b) Approximate solution $U(x,t)$ and $V(x,t)$ when $h\longrightarrow -0.99$.

Example 4.2. *Consider the nonlinear system of inhomogeneous partial differential equations [20]*

$$
\begin{cases}
U_t+U_xV+U=1\\
V_t-UV_x-V=1
\end{cases}. \tag{4.6}
$$

with the initial conditions

$$U(\varkappa,0)=e^x,\ V(x,0)=e^{-x}.$$

The nonlinear operators are

$$
\begin{cases}
N[\phi(x,t,p)]=A\left[\phi(x,t;p)\right]-\frac{1}{v^2}e^x+\frac{1}{v}A\left[\phi_x(x,t;p)\varphi(x,t;p)+\phi(x,t;p)-1\right]\\
N[\varphi(x,t,p)]=A\left[\varphi(x,t;p)\right]-\frac{1}{v^2}e^{-x}+\frac{1}{v}A\left[-\phi(x,t;p)\varphi_x(x,t;p)-\varphi(x,t;p)-1\right]
\end{cases}
$$

Thus, we obtain the m^{th} order deformation equations given by

$$\left\{\begin{array}{l} U_m(x,t)=\chi_m U_{m-1}(x,t)+hA^{-1}[\Re_m(\overrightarrow{U}_{m-1}(x,t))] \\ V_m(x,t)=\chi_m V_{m-1}(x,t)+hA^{-1}[\Re_m(\overrightarrow{V}_{m-1}(x,t))] \end{array}\right. . \tag{4.7}$$

with

$$\left\{\begin{array}{l} \Re_m(\overrightarrow{U}_{m-1}(x,t))=A\left[U_{m-1}(x,t)\right]-\frac{1}{v^2}\left(1-\chi_m\right)e^{x} \\ \quad +\frac{1}{v}A\left[\sum_{i=0}^{m-1}(U_i)_x V_{m-1-i}+\sum_{i=0}^{m-1}U_i-1\right], \\ \Re_m(\overrightarrow{V}_{m-1}(x,t))=A\left[V_{m-1}(x,t)\right]-\frac{1}{v^2}\left(1-\chi_m\right)e^{-x} \\ \quad +\frac{1}{v}A\left[-\sum_{i=0}^{m-1}U_i(V_{m-1-i})_{x}-\sum_{i=0}^{m-1}V_i-1\right], \end{array}\right. \tag{4.8}$$

and

$$\chi_m=\left\{\begin{array}{l} 0,\ \ m\leqslant 1, \\ 1,\ \ m>1. \end{array}\right.$$

According to (4.7) and (4.8), the formulas of the first terms is given by

$$\begin{array}{c} U_1(x,t)=hA^{-1}\left(\frac{1}{v}A\left[(U_0)_x V_0+U_0-1\right]\right), \\ U_2(x,t)=(1+h)U_1(x,t)+hA^{-1}\left(\frac{1}{v}A\left[(U_0)_x V_1+(U_1)_x V_0+U_1\right]\right), \\ U_3(x,t)=(1+h)U_2(x,t) \\ +hA^{-1}\left(\frac{1}{v}A\left[(U_0)_x V_2+(U_1)_x V_1+(U_2)_x V_0+U_2\right]\right), \\ \vdots \end{array} \tag{4.9}$$

and

$$\begin{array}{c} V_1(x,t)=hA^{-1}\left(\frac{1}{v}A\left[-U_0(V_0)_x-V_0-1\right]\right)\ , \\ V_2(x,t)=(1+h)V_1(x,t)+hA^{-1}\left(\frac{1}{v}A\left[-U_0(V_1)_x-U_1(V_0)_x-V_1\right]\right), \\ V_3(x,t)=(1+h)V_2(x,t) \\ +hA^{-1}\left(\frac{1}{v}A\left[-U_0(V_2)_x-U_1(V_1)_x-U_2(V_0)_x-V_2\right]\right), \\ \vdots \end{array} \tag{4.10}$$

From the equations (4.9) and (4.10), the first solution terms of homotopy analysis Aboodh transform method of the system (4.6), is given by

$$\begin{array}{c} U_0(x,t)=e^{x}, \\ V_0(x,t)=e^{-x}, \\ U_1(x,t)=(h)e^{x}t, \\ V_1(x,t)=(-h)e^{-x}t, \\ U_2(x,t)=(h)(1+h)e^{x}t+(h^2)e^{x}\frac{t^2}{2!}, \\ V_2(x,t)=(-h)(1+h)e^{-x}t+(h^2)e^{-x}\frac{t^2}{2!}, \\ U_3(x,t)=(h)(1+h)^2e^{x}t+2(1+h)(h^2)e^{x}\frac{t^2}{2!}+(h^3)e^{x}\frac{t^3}{3!}, \\ V_3(x,t)=(-h)(1+h)^2e^{-x}t+2h^2(1+h)e^{-x}\frac{t^2}{2!}+(-h^3)e^{-x}\frac{t^3}{3!}, \\ \vdots \end{array}$$

and so on.
The other components of the (HAATM) can be determined in a similar way. Finally, the approximate solution (U,V) of the system (4.6)in a series form, is given by

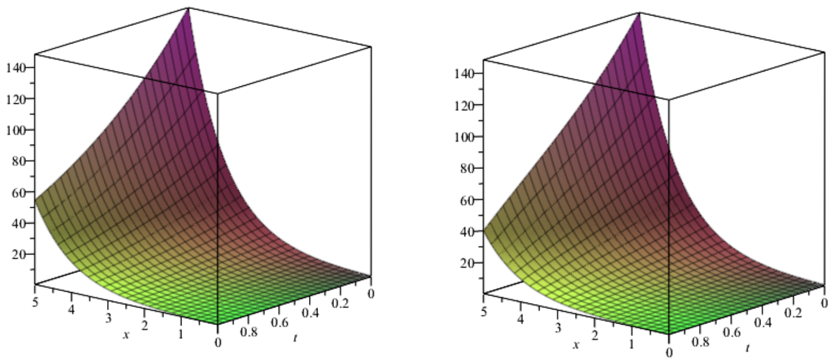
$$\left\{\begin{array}{l} U(x,t)=e^{x}\left(1+h(3+3h+h^2)t+(3+2h)h^2\frac{t^2}{2!}+h^3\frac{t^3}{3!}+\cdots\right) \\ V(x,t)=e^{-x}\left(1+(-h)(3+3h+h^2)t+(3+2h)h^2\frac{t^2}{2!}+(-h^3)\frac{t^3}{3!}+\cdots\right) \end{array}\right. ,$$

and in the case $h=-1$, the approximate solution is given as follows

$$\left\{\begin{array}{l} U(x,t)=e^{x}\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\cdots\right) \\ V(x,t)=e^{-x}\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right) \end{array}\right.$$

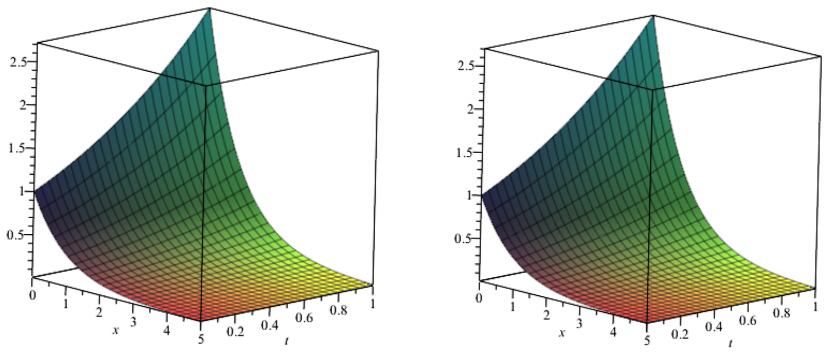
And in the closed form, the solution (U,V) is given by

$$\left\{\begin{array}{l} U(x,t)=e^{x-t} \\ V(x,t)=e^{-x+t} \end{array}\right.$$



(a) (b)

Figure 4.2: (a) Exact solution $U(x,t)$. (b) Approximate solution $U(x,t)$ when $h \longrightarrow -1.09$.



(c) (d)

Figure 4.3: (c) Exact solution $V(x,t)$. (d) Approximate solution $V(x,t)$ when $h \longrightarrow -1.09$.

Example 4.3. *Consider the system of nonlinear coupled partial differential equations [21]*

$$\left\{ \begin{array}{l} U_t(x,y,t) - V_x(x,y,t)W_y(x,,y,t) = 1 \\ V_t(x,y,t) - W_x(x,y,t)U_y(x,,y,t) = 5 \\ W_t(x,y,t) - U_x(x,y,t)V_y(x,,y,t) = 5 \end{array} \right. , \tag{4.11}$$

with the initial conditions

$$U(\varkappa,y,0) = x+2y,\ V(x,y,0) = x-2y,\ W(x,y,0) = -x+2y.$$

The nonlinear operators are

$$\left\{ \begin{array}{c} N[\phi(x,t,p)] = A\left[\phi(x,t;p)\right] - \frac{1}{v^2}\left(x+2y\right) - \frac{1}{v}A\left[\varphi_x(x,t;p)\psi_y(x,t;p)+1\right] \\ N[\varphi(x,t,p)] = A\left[\varphi(x,t;p)\right] - \frac{1}{v^2}\left(x-2y\right) - \frac{1}{v}A\left[\psi_x(x,t;p)\phi_y(x,t;p)+5\right] \\ N[\psi(x,t,p)] = A\left[\psi(x,t;p)\right] - \frac{1}{v^2}\left(-x+2y\right) - \frac{1}{v}A\left[\phi_x(x,t;p)\varphi_y(x,t;p)+5\right] \end{array} \right.$$

Thus, we obtain the m^{th} order deformation equations given by

$$\left\{ \begin{array}{c} U_m(x,t) = \chi_m U_{m-1}(x,t) + hA^{-1}[\Re_m(\overrightarrow{U}_{m-1}(x,t))] \\ V_m(x,t) = \chi_m V_{m-1}(x,t) + hA^{-1}[\Re_m(\overrightarrow{V}_{m-1}(x,t))] \\ W_m(x,t) = \chi_m W_{m-1}(x,t) + hA^{-1}[\Re_m(\overrightarrow{W}_{m-1}(x,t))] \end{array} \right. \tag{4.12}$$

with,

$$\left\{\begin{array}{l} \Re_m(\overrightarrow{U}_{m-1}(x,t)) = A\left[U_{m-1}(x,t)\right] - \frac{1}{v^2}\left(1-\chi_m\right)(x+2y) \\ \qquad -\frac{1}{v}A\left[\sum_{i=0}^{m-1}(V_i)_x(W_{m-1-i})_y+1\right], \\ \Re_m(\overrightarrow{V}_{m-1}(x,t)) = A\left[V_{m-1}(x,t)\right] - \frac{1}{v^2}\left(1-\chi_m\right)(x-2y) \\ \qquad -\frac{1}{v}A\left[\sum_{i=0}^{m-1}(W_i)_x(U_{m-1-i})_y+5\right], \\ \Re_m(\overrightarrow{W}_{m-1}(x,t)) = A\left[W_{m-1}(x,t)\right] - \frac{1}{v^2}\left(1-\chi_m\right)(-x+2y) \\ \qquad -\frac{1}{v}A\left[\sum_{i=0}^{m-1}(U_i)_x(V_{m-1-i})_y+5\right], \end{array}\right. \tag{4.13}$$

$$\chi_m = \left\{\begin{array}{l} 0, \quad m \leqslant 1, \\ 1, \quad m > 1. \end{array}\right.$$

According to (4.12) and (4.13), the formulas of the first terms is given by

$$\begin{array}{c} U_1(x,t) = -hA^{-1}\left(\frac{1}{v}A\left[(V_0)_x(W_0)_y+1\right]\right), \\ U_2(x,t) = (1+h)U_1(x,t) - hA^{-1}\left(\frac{1}{v}A\left[(V_0)_x(W_1)_y+(V_1)_x(W_0)_y\right]\right), \\ U_3(x,t) = (1+h)U_2(x,t) \\ -hA^{-1}\left(\frac{1}{v}A\left[(V_0)_x(W_2)_y+(V_1)_x(W_1)_y+(V_2)_x(W_0)_y\right]\right), \\ \vdots \end{array}$$

$$\begin{array}{c} V_1(x,t) = -hA^{-1}\left(\frac{1}{v}A\left[(W_0)_x(U_0)_y+5\right]\right), \\ V_2(x,t) = (1+h)V_1(x,t) - hA^{-1}\left(\frac{1}{v}A\left[(W_0)_x(U_1)_y+(W_1)_x(U_0)_y\right]\right), \\ V_3(x,t) = (1+h)V_2(x,t) \\ -hA^{-1}\left(\frac{1}{v}A\left[(W_0)_x(U_2)_y+(W_1)_x(U_1)_y+(W_2)_x(U_0)_y\right]\right), \\ \vdots \end{array}$$

and

$$\begin{array}{c} W_1(x,t) = -hA^{-1}\left(\frac{1}{v}A\left[(U_0)_x(V_0)_y+5\right]\right), \\ W_2(x,t) = (1+h)W_1(x,t) - hA^{-1}\left(\frac{1}{v}A\left[(U_0)_x(V_1)_y+(U_1)_x(V_0)_y\right]\right), \\ W_3(x,t) = (1+h)W_2(x,t) \\ -hA^{-1}\left(\frac{1}{v}A\left[(U_0)_x(V_2)_y+(U_1)_x(V_1)_y+(U_2)_x(V_0)_y\right]\right), \\ \vdots \end{array}$$

From the equations (4.4) and (4.5), the first solution terms of homotopy analysis Aboodh transform method of the system (4.1), is given by

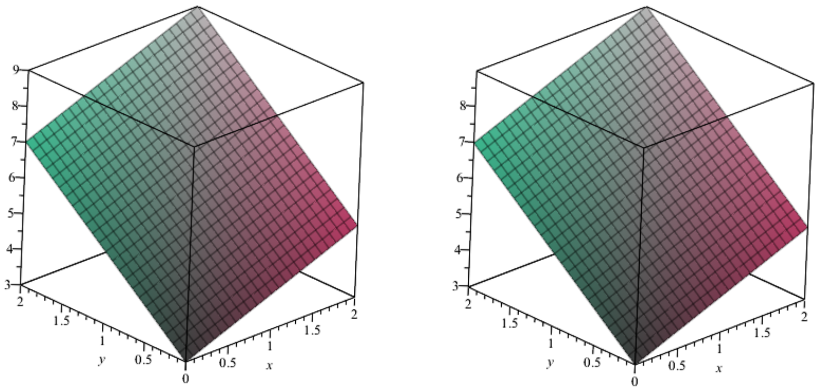
$$\begin{array}{c} U_0(\varkappa,y,t) = x+2y, \qquad V_0(x,y,t) = x-2y, \\ W_0(x,y,t) = -x+2y, \\ U_1(x,y,t) = -3(h)t, \qquad V_1(x,y,t) = -3(h)t, \\ W_1(x,y,t) = -3(h)t, \\ U_2(x,y,t) = -3h(1+h)t, \qquad V_2(x,y,t) = -3h(1+h)t, \\ W_2(x,y,t) = -3h(1+h)t, \\ U_3(x,t) = -3h(1+h)^2t, \qquad V_3(x,t) = -3h(1+h)^2t, \\ W_3(x,t) = -3h(1+h)^2t, \\ \vdots \end{array}$$

and so on.
The other components of the (HAATM) can be determined in a similar way. Finally, the approximate solution (U,V,W) of the system (4.11)in a series form, is given by

$$\left\{\begin{array}{l} U(x,y,t) = x+2y-3(h)t-3h(1+h)^2t-3h(1+h)^3t+\cdots \\ V(x,y,t) = x-2y-3(h)t-3h(1+h)^2t-3h(1+h)^3t+\cdots \\ W(x,y,t) = -x+2y-3(h)t-3h(1+h)^2t-3h(1+h)^3t+\cdots \end{array}\right.$$

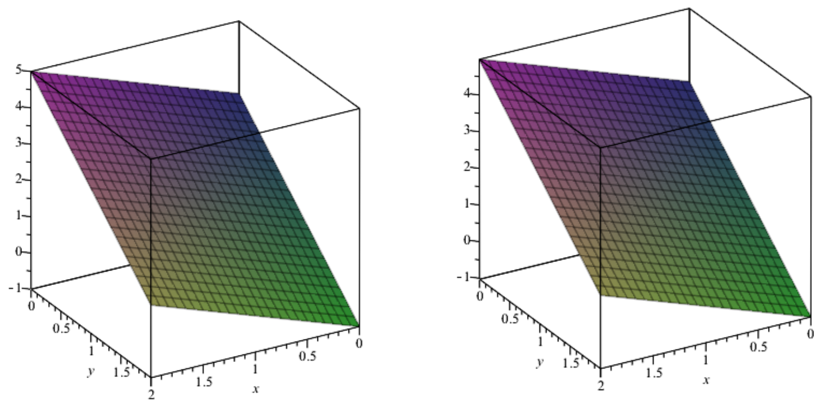
*Substiting $h=-1$ in (**??**), the exact solution of the system (4.11) is given by*

$$\left\{\begin{array}{l} U(x,y,t) = x+2y+3t \\ V(x,y,t) = x-2y+3t \\ W(x,y,t) = -x+2y+3t \end{array}\right. \tag{4.14}$$



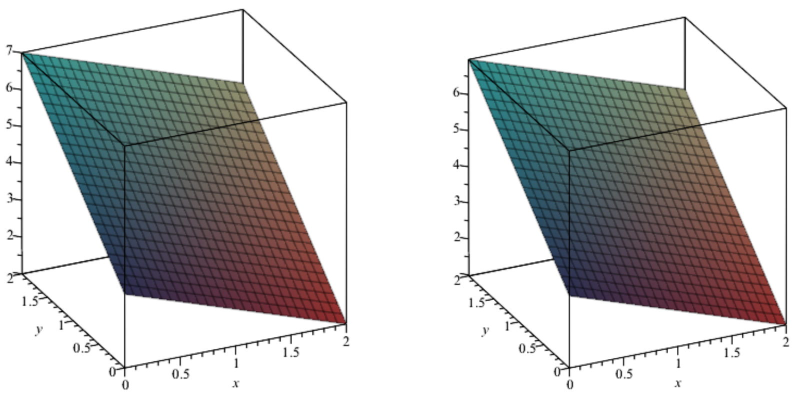
(a) (b)

Figure 4.4: (a) Exact solution $U(x,y,t)$ at the moment $t=1$. (b) Approximate solution $U(x,y,t)$ at the moment $t=1$ when $h \longrightarrow -0.99$.



(c) (d)

Figure 4.5: (c) Exact solution $V(x,y,t)$ at the moment $t=1$. (d) Approximate solution $V(x,y,t)$ at the moment $t=1$ when $h \longrightarrow -0.99$.



(e) (f)

Figure 4.6: (e) Exact solution $W(x,y,t)$ at the moment $t=1$. (f) Approximate solution $W(x,y,t)$ at the moment $t=1$ when $h \longrightarrow -0.99$.

5. Conclusion

In this paper, we have seen that the coupling of homotopy analysis method (HAM) and the Aboodh transform method, proved very effective to solve nonlinear system of partial differential equations. The proposed algorithm (HAATM) is suitable for such problems and is very user friendly. The advantage of this method is its ability to combine two powerful methods to obtain exact solutions of nonlinear system of partial differential equations. The results obtained in the examples presented shows that this modified method is very powerful and efficient technique in finding exact solutions for wide classes of problems.

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