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On the Periodic Solutions of Some Systems of Difference Equations

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Abstract
In this paper, we study the solution of the systems of difference equations

$$x_{n+1}=\frac{1\pm(y_n+x_{n-1})}{y_{n-2}},\quad y_{n+1}=\frac{1\pm(x_n+y_{n-1})}{x_{n-2}},\ n=0,1,...,$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1},\ y_0$ are arbitrary non zero real numbers.

Keywords: Difference equation, Periodicity, System of difference equations
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1. Introduction

Difference equations enter as approximations of continuous problems and as models describing life situations in many directions. Recently, there has been great interest in studying difference equation systems. One of the reasons for this is a necessity for some techniques that can be used in investigating equations arising in mathematical models describing real-life situations in population biology, economic, probability theory, genetics and psychology see [1]-[25].

In [1] Alzahrani et al. found the form of solutions for the following systems of rational difference equations

$$x_{n+1}=\frac{y_ny_{n-2}}{\pm y_{n-2}\pm x_{n-3}},\quad y_{n+1}=\frac{x_nx_{n-2}}{\pm x_{n-2}\pm y_{n-3}}.$$

In [2] Asiri et al. studied the form of the solutions and the periodicity of the following third order systems of rational difference equations

$$x_{n+1}=\frac{y_{n-2}}{1-y_{n-2}x_{n-1}y_n},\quad y_{n+1}=\frac{x_{n-2}}{\pm 1\pm x_{n-2}y_{n-1}x_n}.$$

In [14] Elsayed et al. got the form of the solutions of the following difference equation systems of order four

$$x_{n+1}=\frac{x_{n-2}y_n}{y_{n-3}+y_n},\quad y_{n+1}=\frac{x_ny_{n-2}}{\pm x_{n-3}\pm x_n}.$$

In [4] Cinar studied the solutions of the systems of the difference equations.

$$x_{n+1}=\frac{1}{y_n},\quad y_{n+1}=\frac{y_n}{x_{n-1}y_{n-1}}.$$

In [23] Papaschinnopoulos and Schinas studied the oscillatory behavior, the boundedness of the solutions,and the global asymptotic stability of the positive equilibrium of the system of nonlinear difference equations

$$x_{n+1}=A+\frac{y_n}{x_{n-p}},\quad y_{n+1}=A+\frac{x_n}{y_{n-q}}.$$

In [13] Elsayed has obtained the solution of the following system of the difference equations

$$x_{n+1}=\frac{1}{y_{n-k}},\quad y_{n+1}=\frac{y_{n-k}}{x_ny_n}.$$

The behaviour of the positive solution of the following system

$$x_{n+1}=\frac{x_{n-1}}{1+x_{n-1}y_n},\quad y_{n+1}=\frac{y_{n-1}}{1+x_ny_{n-1}}$$

has been studied by Kurbanli et al. [19].

In [25] Yalcinkaya investigated the sufficient condition for the global asymptotic stability of the following system of difference equations

$$z_{n+1}=\frac{a+t_nz_{n-1}}{t_n+z_{n-1}},\quad t_{n+1}=\frac{a+z_nt_{n-1}}{z_n+t_{n-1}}.$$

The aim of this article is to obtain the expressions of the solutions of the following systems of difference equations

$$x_{n+1}=\frac{1\pm(y_n+x_{n-1})}{y_{n-2}},\quad y_{n+1}=\frac{1\pm(x_n+y_{n-1})}{x_{n-2}},\quad n=0,1,2,...,$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1},\ y_0$ are arbitrary non zero real numbers. Moreover, we obtain some numerical simulation to the equation are given to illustrate our results.

Definition (Periodicity)

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p}=x_n$ for all $n\geq -k$.

2. On the system $x_{n+1}=\frac{1+y_n+x_{n-1}}{y_{n-2}},\ y_{n+1}=\frac{1+x_n+y_{n-1}}{x_{n-2}}$

In this section, we study the solution of the following system of difference equations

$$x_{n+1}=\frac{1+y_n+x_{n-1}}{y_{n-2}},\quad y_{n+1}=\frac{1+x_n+y_{n-1}}{x_{n-2}},\tag{2.1}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1},\ y_0$ are arbitrary non zero real numbers.

2.1 Periodicity of the solutions of system (2.1)

The following theorem is devoted to the periodicity of the solutions of system (2.1).

Theorem 1. Suppose that $\{x_n,y_n\}_{n=1}^{\infty}$ be a solution of system (2.1). Then all solutions of system (2.1) are periodic with period eight.

Proof. From Eq.(2.1), we see that

$$\begin{aligned}
x_{n+1}&=\frac{1+y_n+x_{n-1}}{y_{n-2}},\quad y_{n+1}=\frac{1+x_n+y_{n-1}}{x_{n-2}},\\
x_{n+2}&=\frac{1+y_{n+1}+x_n}{y_{n-1}}=\frac{1+x_{n-2}+x_n+y_{n-1}+x_nx_{n-2}}{y_{n-1}x_{n-2}},\\
y_{n+2}&=\frac{1+x_{n+1}+y_n}{x_{n-1}}=\frac{y_{n-2}+1+y_n+x_{n-1}+y_{n-2}y_n}{y_{n-2}x_{n-1}},\\
x_{n+3}&=\frac{1+y_{n+2}+x_{n+1}}{y_n}\\
&=\frac{y_{n-2}x_{n-1}+y_{n-2}+1+y_n+2x_{n-1}+y_{n-2}y_n+x_{n-1}y_n+(x_{n-1})^2}{y_{n-2}x_{n-1}y_n},\\
y_{n+3}&=\frac{1+x_{n+2}+y_{n+1}}{x_n}\\
&=\frac{y_{n-1}x_{n-2}+1+x_{n-2}+x_n+2y_{n-1}+x_nx_{n-2}+y_{n-1}x_n+(y_{n-1})^2}{y_{n-1}x_{n-2}x_n},
\end{aligned}$$

$$
\begin{aligned}
x_{n+4} &= \frac{1+y_{n+3}+x_{n+2}}{y_{n+1}} \\
&= \frac{\begin{array}{c} y_{n-1}x_{n-2}x_n+y_{n-1}x_{n-2}+1+x_{n-2}+x_n+2y_{n-1}+x_nx_{n-2} \\ +y_{n-1}x_n+(y_{n-1})^2+x_n(1+x_{n-2}+x_n+y_{n-1}+x_nx_{n-2}) \end{array}}{(1+x_n+y_{n-1})y_{n-1}x_n} \\
&= \frac{(1+x_{n-2}+x_n+y_{n-1}+x_nx_{n-2})(1+x_n+y_{n-1})}{(1+x_n+y_{n-1})y_{n-1}x_n} \\
&= \frac{1+x_{n-2}+x_n+y_{n-1}+x_nx_{n-2}}{y_{n-1}x_n},
\end{aligned}
$$

$$
\begin{aligned}
y_{n+4} &= \frac{1+x_{n+3}+y_{n+2}}{x_{n+1}} \\
&= \frac{\begin{array}{c} x_{n-1}y_ny_{n-2}+y_{n-2}x_{n-1}+y_{n-2}+1+y_n+2x_{n-1}+y_{n-2}y_n \\ +x_{n-1}y_n+(x_{n-1})^2+y_n(y_{n-2}+1+y_n+x_{n-1}+y_{n-2}y_n) \end{array}}{x_{n-1}y_n(1+y_n+x_{n-1})} \\
&= \frac{(1+y_n+x_{n-1})(y_{n-2}+1+y_n+x_{n-1}+y_{n-2}y_n)}{x_{n-1}y_n(1+y_n+x_{n-1})} \\
&= \frac{y_{n-2}+1+y_n+x_{n-1}+y_{n-2}y_n}{x_{n-1}y_n},
\end{aligned}
$$

$$
\begin{aligned}
x_{n+5} &= \frac{1+y_{n+4}+x_{n+3}}{y_{n+2}} \\
&= \frac{\begin{array}{c} y_{n-2}x_{n-1}y_n+y_{n-2}(y_{n-2}+1+y_n+x_{n-1}+y_{n-2}y_n)+ \\ y_{n-2}x_{n-1}+y_{n-2}+1+y_n+2x_{n-1}+y_{n-2}y_n+x_{n-1}y_n+(x_{n-1})^2 \end{array}}{y_n(y_{n-2}+1+y_n+x_{n-1}+y_{n-2}y_n)} \\
&= \frac{(x_{n-1}+y_{n-2}+1)(y_{n-2}+1+y_n+x_{n-1}+y_{n-2}y_n)}{y_n(y_{n-2}+1+y_n+x_{n-1}+y_{n-2}y_n)} \\
&= \frac{x_{n-1}+y_{n-2}+1}{y_n},
\end{aligned}
$$

$$
\begin{aligned}
y_{n+5} &= \frac{1+x_{n+4}+y_{n+3}}{x_{n+2}} \\
&= \frac{\begin{array}{c} y_{n-1}x_{n-2}x_n+x_{n-2}(1+x_{n-2}+x_n+y_{n-1}+x_nx_{n-2})+ \\ y_{n-1}x_{n-2}+1+x_{n-2}+x_n+2y_{n-1}+x_nx_{n-2}+y_{n-1}x_n+(y_{n-1})^2 \end{array}}{x_n(1+x_{n-2}+x_n+y_{n-1}+x_nx_{n-2})} \\
&= \frac{(1+y_{n-1}+x_{n-2})(1+x_{n-2}+x_n+y_{n-1}+x_nx_{n-2})}{x_n(1+x_{n-2}+x_n+y_{n-1}+x_nx_{n-2})} \\
&= \frac{1+y_{n-1}+x_{n-2}}{x_n},
\end{aligned}
$$

$$
\begin{aligned}
x_{n+6} &= \frac{1+y_{n+5}+x_{n+4}}{y_{n+3}} \\
&= \frac{y_{n-1}x_n+y_{n-1}(1+y_{n-1}+x_{n-2})+1+x_{n-2}+x_n+y_{n-1}+x_nx_{n-2}}{\frac{(y_{n-1}x_{n-2}+1+x_{n-2}+x_n+2y_{n-1}+x_nx_{n-2}+y_{n-1}x_n+(y_{n-1})^2)}{x_{n-2}}} = x_{n-2}, \\
y_{n+6} &= \frac{1+x_{n+5}+y_{n+4}}{x_{n+3}} \\
&= \frac{x_{n-1}y_n+x_{n-1}(x_{n-1}+y_{n-2}+1)+y_{n-2}+1+y_n+x_{n-1}+y_{n-2}y_n}{\frac{y_{n-2}x_{n-1}+y_{n-2}+1+y_n+2x_{n-1}+y_{n-2}y_n+x_{n-1}y_n+(x_{n-1})^2}{y_{n-2}}} = y_{n-2},
\end{aligned}
$$

$$x_{n+7} = \frac{1+y_{n+6}+x_{n+5}}{y_{n+4}} = \frac{y_n+y_ny_{n-2}+x_{n-1}+y_{n-2}+1}{\frac{(y_{n-2}+1+y_n+x_{n-1}+y_{n-2}y_n)}{x_{n-1}}} = x_{n-1},$$
$$y_{n+7} = \frac{1+x_{n+6}+y_{n+5}}{x_{n+4}} = \frac{x_n+x_nx_{n-2}+1+y_{n-1}+x_{n-2}}{\frac{1+x_{n-2}+x_n+y_{n-1}+x_nx_{n-2}}{y_{n-1}}} = y_{n-1},$$
$$x_{n+8} = \frac{1+y_{n+7}+x_{n+6}}{y_{n+5}} = \frac{1+y_{n-1}+x_{n-2}}{\frac{1+y_{n-1}+x_{n-2}}{x_n}} = x_n,$$
$$y_{n+8} = \frac{1+x_{n+7}+y_{n+6}}{x_{n+5}} = \frac{1+x_{n-1}+y_{n-2}}{\frac{x_{n-1}+y_{n-2}+1}{y_n}} = y_n.$$

Thus, the solutions are periodic with period eight.

2.2 The form of the solutions of system (2.1)

The following theorem describes the form of the solutions of system (2.1).

Theorem 2. Suppose that $\{x_n, y_n\}$ are solutions of the system (2.1). Then for $n = 0, 1, 2, \ldots$, we have the following formulas

$$x_{8n-2} = c, \quad x_{8n-1} = b, \quad x_{8n} = a, \quad x_{8n+1} = \frac{1+d+b}{f},$$
$$x_{8n+2} = \frac{ac+a+c+e+1}{ec}, \quad x_{8n+3} = \frac{b^2+bd+bf+df+2b+d+f+1}{fbd},$$
$$x_{8n+4} = \frac{ac+a+c+e+1}{ea}, \quad x_{8n+5} = \frac{1+f+b}{d},$$

$$y_{8n-2} = f, \quad y_{8n-1} = e, \quad y_{8n} = d, \quad y_{8n+1} = \frac{1+a+e}{c},$$
$$y_{8n+2} = \frac{df+b+d+f+1}{fb}, \quad y_{8n+3} = \frac{e^2+ac+ae+ce+2e+a+c+1}{cea},$$
$$y_{8n+4} = \frac{df+b+d+f+1}{bd}, \quad y_{8n+5} = \frac{1+c+e}{a},$$

where the initial conditions $x_{-2} = c,\ x_{-1} = b,\ x_0 = a,\ y_{-2} = f,\ y_{-1} = e,\ y_0 = d$.

Or equivalently

$$\{x_n\}_{n=-2}^{+\infty} = \left\{ \begin{array}{c} c, b, a, \frac{1+d+b}{f}, \frac{ac+a+c+e+1}{ec}, \frac{b^2+bd+bf+df+2b+d+f+1}{fbd} \\ , \frac{ac+a+c+e+1}{ea}, \frac{1+f+b}{d}, c, b, a, \ldots \end{array} \right\},$$

$$\{y_n\}_{n=-2}^{+\infty} = \left\{ \begin{array}{c} f, e, d, \frac{1+a+e}{c}, \frac{df+b+d+f+1}{fb}, \frac{e^2+ac+ae+ce+2e+a+c+1}{cea}, \\ \frac{df+b+d+f+1}{bd}, \frac{1+c+e}{a}, f, e, d, \ldots \end{array} \right\}.$$

Proof. For $n = 0$ the result holds. Suppose that the result holds for $n-1$.

$$x_{8n-10} = c, \quad x_{8n-9} = b, \quad x_{8n-8} = a, \quad x_{8n-7} = \frac{1+d+b}{f},$$
$$x_{8n-6} = \frac{ac+a+c+e+1}{ec}, \quad x_{8n-5} = \frac{b^2+bd+bf+df+2b+d+f+1}{fbd},$$
$$x_{8n-4} = \frac{ac+a+c+e+1}{ea}, \quad x_{8n-3} = \frac{1+f+b}{d},$$

$$y_{8n-10} = f, \quad y_{8n-9} = e, \quad y_{8n-8} = d, \quad y_{8n-7} = \frac{1+a+e}{c},$$
$$y_{8n-6} = \frac{df+b+d+f+1}{fb}, \quad y_{8n-5} = \frac{e^2+ac+ae+ce+2e+a+c+1}{cea},$$
$$y_{8n-4} = \frac{df+b+d+f+1}{bd}, \quad y_{8n-3} = \frac{1+c+e}{a},$$

from system (2.1) we can prove as follow

$$\begin{aligned} x_{8n-2} &= \frac{1+y_{8n-3}+x_{8n-4}}{y_{8n-5}} = \frac{1+\frac{1+c+e}{a}+\frac{ac+a+c+e+1}{ea}}{\frac{e^2+ac+ae+ce+2e+a+c+1}{cea}} \\ &= \frac{c(ea+e(1+c+e)+ac+a+c+e+1)}{e^2+ac+ae+ce+2e+a+c+1} = c. \end{aligned}$$

Also, we get

$$\begin{aligned} x_{8n-1} &= \frac{1+y_{8n-2}+x_{8n-3}}{y_{8n-4}} = \frac{1+f+\frac{1+f+b}{d}}{\frac{df+b+d+f+1}{bd}} \\ &= \frac{b(d+fd+1+f+b)}{fd+b+d+f+1} = b, \\ x_{8n} &= \frac{1+y_{8n-1}+x_{8n-2}}{y_{8n-3}} = \frac{1+e+c}{\frac{1+c+e}{a}} = a, \\ x_{8n+1} &= \frac{1+y_{8n}+x_{8n-1}}{y_{8n-2}} = \frac{1+d+b}{f}, \\ x_{8n+2} &= \frac{1+y_{8n+1}+x_{8n}}{y_{8n-1}} = \frac{1+\frac{1+a+e}{c}+a}{e} = \frac{1+a+e+ac+c}{ce}, \\ x_{8n+3} &= \frac{1+y_{8n+2}+x_{8n+1}}{y_{8n}} = \frac{1+\frac{df+b+d+f+1}{fb}+\frac{1+d+b}{f}}{d} \\ &= \frac{fb+df+b+d+f+1+b(1+d+b)}{fbd} \\ &= \frac{b^2+bd+bf+df+2b+d+f+1}{fbd}, \\ x_{8n+4} &= \frac{1+y_{8n+3}+x_{8n+2}}{y_{8n+1}} = \frac{1+\frac{e^2+ac+ae+ce+2e+a+c+1}{cea}+\frac{1+a+e+ac+c}{ce}}{\frac{1+a+e}{c}} \\ &= \frac{cea+e^2+ae+ce+e+(ac+e+a+c+1)+a(1+a+e+ac+c)}{ea(1+a+e)} \\ &= \frac{ac+a+c+e+1}{ea}, \\ x_{8n+5} &= \frac{1+y_{8n+4}+x_{8n+3}}{y_{8n+2}} = \frac{1+\frac{df+b+d+f+1}{bd}+\frac{b^2+bd+bf+df+2b+d+f+1}{fbd}}{\frac{df+b+d+f+1}{fb}} \\ &= \frac{b^2+bd+bf+df+2b+d+f+1+f(df+b+d+f+1)+fbd}{d(df+b+d+f+1)} \\ &= \frac{1+f+b}{d}, \\ y_{8n-2} &= \frac{1+x_{8n-3}+y_{8n-4}}{x_{8n-5}} = \frac{1+\frac{b+f+1}{d}+\frac{f+1+d+b+df}{bd}}{\frac{b^2+bd+bf+df+2b+d+f+1}{fbd}} = \frac{\frac{b^2+bf+2b+bd+f+d+1+df}{bd}}{\frac{b^2+bd+bf+df+2b+d+f+1}{fbd}} = f, \\ y_{8n-1} &= \frac{1+x_{8n-2}+y_{8n-3}}{x_{8n-4}} = \frac{1+c+\frac{c+e+1}{a}}{\frac{ac+a+c+e+1}{ea}} = \frac{e(a+ca+c+e+1)}{ac+a+c+e+1} = e, \\ y_{8n} &= \frac{1+x_{8n-1}+y_{8n-2}}{x_{8n-3}} = \frac{1+b+f}{\frac{1+b+f}{d}} = d, \\ y_{8n+1} &= \frac{1+x_{8n}+y_{8n-1}}{x_{8n-2}} = \frac{1+e+a}{c}, \\ y_{8n+2} &= \frac{1+x_{8n+1}+y_{8n}}{x_{8n-1}} = \frac{1+\frac{1+d+b}{f}+d}{b} = \frac{f+1+d+b+df}{fb}, \end{aligned}$$

$$y_{8n+3} = \frac{1+x_{8n+2}+y_{8n+1}}{x_{8n}} = \frac{1+\frac{1+a+e+ac+c}{ce}+\frac{1+e+a}{c}}{a}$$
$$= \frac{e^2+ac+ae+ce+2e+a+c+1}{cea},$$

$$y_{8n+4} = \frac{1+x_{8n+3}+y_{8n+2}}{x_{8n+1}} = \frac{1+\frac{b^2+bd+bf+df+2b+d+f+1}{fbd}+\frac{f+1+d+b+df}{fb}}{\frac{1+d+b}{f}}$$
$$= \frac{b^2+fbd+b+bd+bf+(df+b+d+f+1)+d(f+1+d+b+df)}{bd(1+d+b)}$$
$$= \frac{f+1+d+b+df}{bd},$$
$$y_{8n+5} = \frac{1+x_{8n+4}+y_{8n+3}}{x_{8n+2}} = \frac{1+\frac{ac+a+c+e+1}{ea}+\frac{e^2+ac+ae+ce+2e+a+c+1}{cea}}{\frac{1+a+e+ac+c}{ce}}$$
$$= \frac{c(ac+a+c+e+1)+e^2+cea+ce+ae+e+ac+e+a+c+1}{a(1+a+e+ac+c)}$$
$$= \frac{c+e+1}{a}.$$

This completes the proof.

2.3 Numerical examples

For confirming the results of this section, we consider the following numerical example which represent solutions to system (2.1).

Example 1. We consider interesting numerical example for the difference equations system (2.1) with the initial conditions $x_{-2}=2,\ x_{-1}=-0.7,\ x_0=0.9,\ y_{-2}=3,\ y_{-1}=-0.5$ *and* $y_0=11$. (See Fig. 2.1).

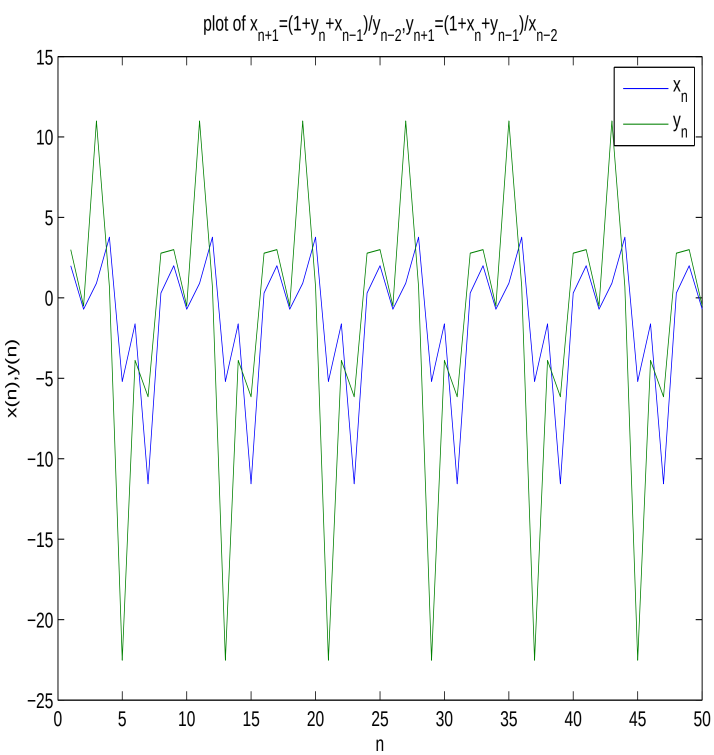

Figure 2.1

The following cases can be proved similarly.

3. On The System $x_{n+1} = \frac{1-(y_n+x_{n-1})}{y_{n-2}}, \quad y_{n+1} = \frac{1+(x_n+y_{n-1})}{x_{n-2}}$

In this section we study the solution of the following system of difference equations

$$x_{n+1} = \frac{1-(y_n+x_{n-1})}{y_{n-2}}, \quad y_{n+1} = \frac{1+(x_n+y_{n-1})}{x_{n-2}}, \tag{3.1}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1},\ y_0$, are arbitrary non zero real numbers.

Theorem 3. Let $\{x_n, y_n\}_{n=-2}^{+\infty}$ be solutions of system (3.1). Then

1- $\{x_n\}_{n=-2}^{+\infty}$ and $\{y_n\}_{n=-2}^{+\infty}$ and are periodic with period eight i.e.,

$$x_{n+8} = x_n, y_{n+8} = y_n$$

for $n \geq -2$.

2- We have the following form of the solutions

$$\begin{aligned}
x_{8n-2} &= c, \quad x_{8n-1} = b, \quad x_{8n} = a, \quad x_{8n+1} = -\frac{-1+d+b}{f}, \\
x_{8n+2} &= -\frac{ac+a-c+e+1}{ec}, \quad x_{8n+3} = \frac{b^2+bd+bf-df+d-f-1}{fbd}, \\
x_{8n+4} &= \frac{ac+a-c-e+1}{ea}, \quad x_{8n+5} = \frac{1+f+b}{d},
\end{aligned}$$

$$\begin{aligned}
y_{8n-2} &= f, \quad y_{8n-1} = e, \quad y_{8n} = d, \quad y_{8n+1} = \frac{1+a+e}{c}, \\
y_{8n+2} &= -\frac{-df+b+d-f-1}{fb}, \quad y_{8n+3} = -\frac{-e^2+ac-ae-ce+a-c+1}{cea}, \\
y_{8n+4} &= -\frac{df+b-d+f+1}{bd}, \quad y_{8n+5} = -\frac{c+e-1}{a},
\end{aligned}$$

or equivalently

$$\{x_n\}_{n=-2}^{+\infty} = \left\{ \begin{array}{c} c, b, a, -\frac{-1+d+b}{f}, -\frac{ac+a-c+e+1}{ec}, \frac{b^2+bd+bf-df+d-f-1}{fbd}, \\ \frac{ac+a-c-e+1}{ea}, \frac{1+f+b}{d}, c, b, a, \ldots \end{array} \right\},$$

$$\{y_n\}_{n=-2}^{+\infty} = \left\{ \begin{array}{c} f, e, d, \frac{1+a+e}{c}, -\frac{-df+b+d-f-1}{fb}, -\frac{-e^2+ac-ae-ce+a-c+1}{cea}, \\ -\frac{df+b-d+f+1}{bd}, -\frac{c+e-1}{a}, f, e, d, \ldots \end{array} \right\}.$$

where the initial conditions $x_{-2} = c,\ x_{-1} = b,\ x_0 = a, y_{-2} = f,\ y_{-1} = e,\ y_0 = d$.

Example 2. We consider example for the difference system (3.1) where the initial conditions $x_{-2} = -5,\ x_{-1} = 7,\ x_0 = 2,\ y_{-2} = -3,\ y_{-1} = 1.3$ *and* $y_0 = 3$. (See Fig. 3.1).

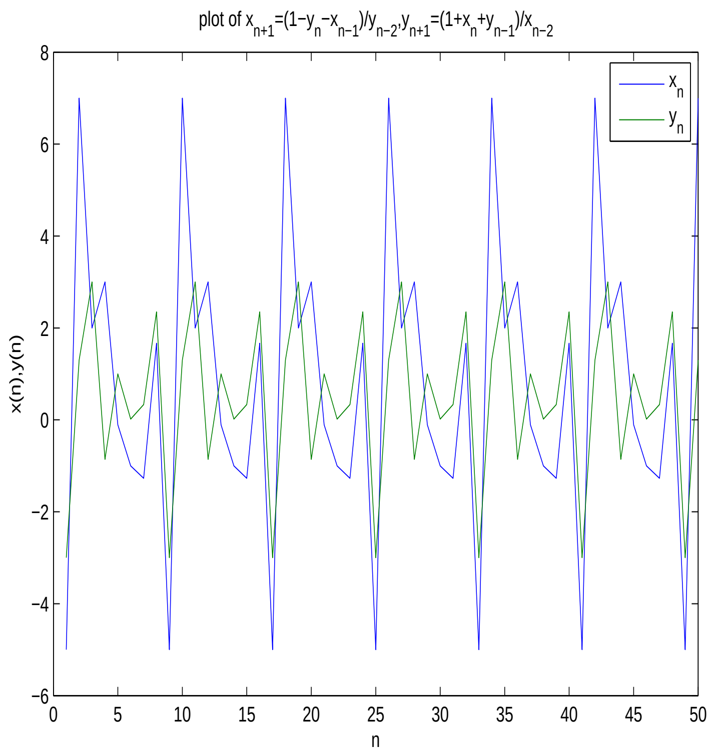

Figure 3.1

4. On the system $x_{n+1}=\frac{1+(y_n+x_{n-1})}{y_{n-2}}, \quad y_{n+1}=\frac{1-(x_n+y_{n-1})}{x_{n-2}}$

In this section, we investigate the solution of the following system of difference equations

$$x_{n+1}=\frac{1+(y_n+x_{n-1})}{y_{n-2}}, \quad y_{n+1}=\frac{1-(x_n+y_{n-1})}{x_{n-2}}, \tag{4.1}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1},\ y_0$, are arbitrary non zero real numbers.
Theorem 4. Suppose that $\{x_n, y_n\}$ are solutions of the system (4.1). Then for $n = 0,1,2,...$

1- $\{x_n\}_{n=-2}^{+\infty}$ and $\{y_n\}_{n=-2}^{+\infty}$ and are periodic with period eight i.e.,

$$x_{n+8}=x_n, \quad y_{n+8}=y_n,$$

for $n \geq -2$.

2- We have the following formulas

$$\begin{aligned}
x_{8n-2} &= c, \quad x_{8n-1}=b, \quad x_{8n}=a, \quad x_{8n+1}=\frac{1+d+b}{f},\\
x_{8n+2} &= \frac{ac-a+c-e+1}{ec}, \quad x_{8n+3}=\frac{b^2+bd+bf-df-d+f-1}{fbd},\\
x_{8n+4} &= -\frac{ac-a+c+e+1}{ea}, \quad x_{8n+5}=-\frac{f+b-1}{d},\\
y_{8n-2} &= f, \quad y_{8n-1}=e, \quad y_{8n}=d, \quad y_{8n+1}=-\frac{a+e-1}{c},\\
y_{8n+2} &= -\frac{df+b+d-f+1}{fb}, \quad y_{8n+3}=-\frac{-e^2+ac-ae-ce-a+c+1}{cea},\\
y_{8n+4} &= -\frac{-df+b-d+f-1}{bd}, \quad y_{8n+5}=\frac{1+c+e}{a}.
\end{aligned}$$

Or equivalently

$$\{x_n\}_{n=-2}^{+\infty} = \left\{ \begin{array}{c} c, b, a, \frac{1+d+b}{f}, \frac{ac-a+c-e+1}{ec}, \frac{b^2+bd+bf-df-d+f-1}{fbd}, \\ -\frac{ac-a+c+e+1}{ea}, -\frac{f+b-1}{d}, c, b, a, \ldots \end{array} \right\},$$

$$\{y_n\}_{n=-2}^{+\infty} = \left\{ \begin{array}{c} f, e, d, -\frac{a+e-1}{c}, -\frac{df+b+d-f+1}{fb}, -\frac{-e^2+ac-ae-ce-a+c+1}{cea}, \\ -\frac{-df+b-d+f-1}{bd}, \frac{1+c+e}{a}, f, e, d, , \ldots \end{array} \right\}.$$

Example 3. We assume $x_{-2} = 1.3,\ x_{-1} = -7,\ x_0 = 2,\ y_{-2} = -3,\ y_{-1} = 9$ *and* $y_0 = -4$ for system (3.1) see Fig. 4.1.

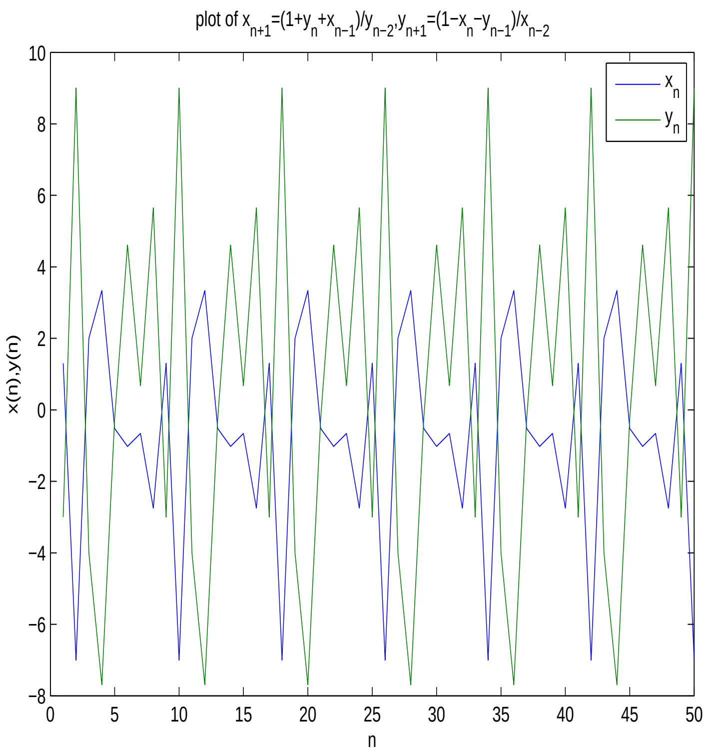

Figure 4.1

5. On the system $x_{n+1} = \frac{1-(y_n+x_{n-1})}{y_{n-2}}, \quad y_{n+1} = \frac{1-(x_n+y_{n-1})}{x_{n-2}}$

In this section we study the solution of the following system of difference equations

$$x_{n+1} = \frac{1-(y_n+x_{n-1})}{y_{n-2}}, \quad y_{n+1} = \frac{1-(x_n+y_{n-1})}{x_{n-2}}, \tag{5.1}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1},\ y_0$, are arbitrary non zero real numbers.

Theorem 5. Let $\{x_n, y_n\}_{n=-2}^{+\infty}$ be solutions of system (4.1). Then

1- $\{x_n\}_{n=-2}^{+\infty}$ and $\{y_n\}_{n=-2}^{+\infty}$ and are periodic with period eight i.e.,

$$x_{n+8} = x_n, \quad y_{n+8} = y_n,$$

2- We have the following form

$$\begin{aligned}
x_{8n-2} &= c, \quad x_{8n-1}=b, \quad x_{8n}=a, \quad x_{8n+1}=-\frac{-1+d+b}{f},\\
x_{8n+2} &= -\frac{ac-a-c-e+1}{ec}, \quad x_{8n+3}=\frac{b^2+bd+bf+df-2b-d-f+1}{fbd},\\
x_{8n+4} &= -\frac{ac-a-c-e+1}{ea}, \quad x_{8n+5}=-\frac{f+b-1}{d},\\
y_{8n-2} &= f, \quad y_{8n-1}=e, \quad y_{8n}=d, \quad y_{8n+1}=-\frac{-1+a+e}{c},\\
y_{8n+2} &= \frac{-df+b+d+f-1}{fb}, \quad y_{8n+3}=\frac{e^2+ac+ae+ce-2e-a-c+1}{cea},\\
y_{8n+4} &= \frac{-df+b+d+f-1}{bd}, \quad y_{8n+5}=-\frac{c+e-1}{a}.
\end{aligned}$$

Or equivalently

$$\{x_n\}_{n=-2}^{+\infty}=\left\{\begin{array}{c} c,b,a,-\frac{-1+d+b}{f},-\frac{ac-a-c-e+1}{ec},\frac{b^2+bd+bf+df-2b-d-f+1}{fbd},\\ -\frac{ac-a-c-e+1}{ea},-\frac{f+b-1}{d},c,b,a,\ldots \end{array}\right\},$$

$$\{y_n\}_{n=-2}^{+\infty}=\left\{\begin{array}{c} f,e,d,-\frac{-1+a+e}{c},\frac{-df+b+d+f-1}{fb},\frac{e^2+ac+ae+ce-2e-a-c+1}{cea}\\ \frac{-df+b+d+f-1}{bd},-\frac{c+e-1}{a},f,e,d,\ldots \end{array}\right\}.$$

Example 4. See Fig. 5.1, if we take system (5.1) with $x_{-2}=-8,\ x_{-1}=5,\ x_0=-2.8,\ y_{-2}=-9,\ y_{-1}=3$ *and* $y_0=4$.

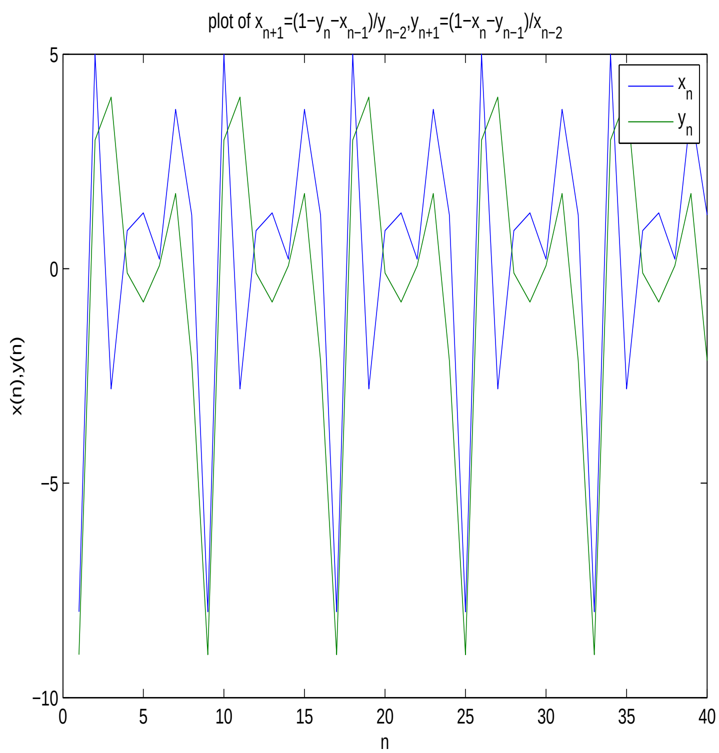

Figure 5.1

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