Communications in Advanced Mathematical Sciences
Vol. I, No. 2, 137-141, 2018
Research Article
ISSN: 2651-4001

L-Fuzzy Invariant Metric Space

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Abstract
In this paper, we define L-fuzzy invariant metric space, and generalize some well known results in metric and fuzzy metric space including Uniform continuity theorem and Ascoli-Arzela theorem.

Keywords: L-fuzzy invariant metric space, Completeness, Equicontinuity, Compactness
2010 AMS: Primary 46S40, Secondary 54E50

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Received: 17 July 2018, **Accepted:** 02 October 2018, **Available online:** 24 December 2018

1. Introduction

One of the most important problems in fuzzy topology is to obtain an appropriate concept of fuzzy metric. This problem has been investigated by many authors [1]-[10] from different points of views. In particular, Park [8] introduced the notion of intuitionistic fuzzy metric as a generalization of fuzzy metric introduced and studied by George and Veeramani [2].

In this paper, we define L-fuzzy invariant metric space, study completeness and observe that a compact L-fuzzy invariant metric space is separable. Further, we introduce the notion of uniform continuity and equicontinuity. Finally, we prove Uniform continuity theorem and Ascoli-Arzela theorem.

2. L-fuzzy invariant metric space

Lemma 2.1. *[11] Consider the set L^* and operation $\leq_{L^*}$ defined by $L^* = \{(x_1,x_2) : (x_1,x_2) \in [0,1]^2$ and $x_1+x_2 \leq 1\}$ and $(x_1,x_2) \leq_{L^*} (y_1,y_2) \iff x_1 \leq y_1$ and $x_2 \geq y_2$ for every $(x_1,x_2),(y_1,y_2) \in L^*$. Then $(L^*,\leq_{L^*})$ is a complete lattice.*

Definition 2.2. *[9] An intuitionistic fuzzy set $A_{\zeta,\eta}$ in a universe U is an object $A_{\zeta,\eta} = \{(\zeta_A(u),\eta_A(u)) : u \in U\}$ where, for all $u \in U$, $\zeta_A(u) \in [0,1]$ and $\eta_A(u) \in [0,1]$ are called the membership degree and non-membership degree, respectively, of u in $A_{\zeta,\eta}$, and furthermore they satisfy $\zeta_A(u)+\eta_A(u) \leq 1$.*

For every $z_i = (x_i,y_i) \in L^*$, if $c_i \in [0,1]$ such that $\Sigma_{j=1}^n c_j = 1$ then $c_1(x_1,y_1)+c_2(x_2,y_2)+\ldots+c_n(x_n,y_n) = \Sigma_{j=1}^n c_j(x_j,y_j) = \left(\Sigma_{j=1}^n c_jx_j, \Sigma_{j=1}^n c_jy_j\right) \in L^*$.

We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. Classically, a triangular norm (shortly t-norm) $* = T$ on $[0,1]$ is defined as an increasing, commutative and associative mapping $T : [0,1]^2 \to [0,1]$ satisfying $T(1,x) = 1*x = x$ for all $x \in [0,1]$. A triangular conorm (shortly t-conorm) $\Diamond = S$ is defined as an increasing, commutative and associative mapping $S : [0,1]^2 \to [0,1]$ satisfying $S(0,x) = 0\Diamond x = x$ for all $x \in [0,1]$. Using the lattice $(L^*,\leq_{L^*})$, these definitions can be extended.

Definition 2.3. *[12] A triangular norm $\Im$ on L^* is a mapping $\Im : (L^*)^2 \to L^*$ satisfying the following conditions, for every $x,y,z,t \in L^*$:*

(a) $\Im(x, 1_{L^*}) = x$,

(b) $\Im(x,y) = \Im(y,x)$,

(c) $\Im(x,\Im(y,z)) = \Im(\Im(x,y),z)$,

(d) $x \leq_{L^*} z$ and $y \leq_{L^*} t$ imply $\Im(x,y) \leq_{L^*} \Im(z,t)$.

Definition 2.4. *[11, 12]*
A continuous t-norm $\Im$ on L^* is called continuous t-representable if and only if there exist a continuous t-norm $*$ and a continuous t-conorm $\lozenge$ on $[0,1]$ such that $\Im(x,y) = (x_1 * y_1, x_2 \lozenge y_2)$ for all $x = (x_1,x_2), y = (y_1,y_2) \in L^*$.

Now define a sequence $\Im^n$ recursively by $\Im^1 = \Im$ and

$$\Im^n(x_1,...,x_{n+1}) = \Im(\Im^{n-1}(x_1,...,x_n),x_{n+1})$$

for $n \geq 2$ and $x_i \in L^*$.

Definition 2.5. *[11, 12]*
A negator $\mathscr{N}$ on L^* is any decreasing mapping $\mathscr{N} : L^* \to L^*$ satisfying $\mathscr{N}(0_{L^*}) = 1_{L^*}$ and $\mathscr{N}(1_{L^*}) = 0_{L^*}$. If $\mathscr{N}(\mathscr{N}(x)) = x$ for all $x \in L^*$ then $\mathscr{N}$ is called an involutive negator. A negator N on $[0,1]$ is a decreasing mapping N:$[0,1] \to [0,1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_S denotes the standard negator on $[0,1]$ defined as $N_S(x) = 1-x$ for all $x \in [0,1]$.

Next, using fundamental notions above, we give a metric generalization on vector space in the sense of George and Veeramani [2].

Definition 2.6. *Let μ and ν are fuzzy sets from $X \times (0,\infty)$ to $[0,1]$ such that $\mu(x,t) + \nu(x,t) \leq 1$ for all $x \in X$ and $t > 0$. The 3-tuble $(X, M_{\mu,\nu}, \Im)$ is said to be an L-fuzzy invariant metric space if X is a vector space, $\Im$ is a continuous t-representable and $M_{\mu,\nu}$ is a mapping from $X \times (0,\infty)$ to L^* satisfying the following conditions, for every $x,y \in X$ and $t,s > 0$*

(a) $M_{\mu,\nu}(x,t) >_{L^*} 0_{L^*}$,

(b) $M_{\mu,\nu}(x,t) = 1_{L^*}$ if and only if $x = 0$,

(c) $M_{\mu,\nu}(x-y,t) = M_{\mu,\nu}(y-x,t)$,

(d) $M_{\mu,\nu}(x+y,t+s) \geq_{L^*} \Im(M_{\mu,\nu}(x,t), M_{\mu,\nu}(y,s))$,

(e) $M_{\mu,\nu}(x,.) : (0,\infty) \to L^*$ is continuous.

In this case, $M_{\mu,\nu}$ is said to be an L-fuzzy invariant metric on X. Here $M_{\mu,\nu}(x,t) = (\mu(x,t), \nu(x,t))$.

Example 2.7. *Let $(X, \|.\|)$ be a normed space. Denote $\Im(a,b) = (a_1b_1, \min(a_2+b_2,1))$ for all $a = (a_1,a_2), b = (b_1,b_2) \in L^*$ and let μ,ν be fuzzy sets on $X \times (0,\infty)$ defined as follows:*

$$M_{\mu,\nu}(x,t) = (\mu(x,t), \nu(x,t)) = \left(\frac{ht^n}{ht^n + m\|x\|}, \frac{m\|x\|}{ht^n + m\|x\|} \right)$$

for all $t,h,m,n \in \mathbb{R}^+$. Then $(X, M_{\mu,\nu}, \Im)$ is an L-fuzzy invariant metric space. If $h = m = n = 1$ then $(X, M_{\mu,\nu}, \Im)$ is a standard L-fuzzy invariant metric space. Also, if we define

$$M_{\mu,\nu}(x,t) = (\mu(x,t), \nu(x,t)) = \left(\frac{t}{t + m\|x\|}, \frac{\|x\|}{t + \|x\|} \right)$$

in which $m > 1$, then $(X, M_{\mu,\nu}, \Im)$ is an L-fuzzy invariant metric space in which $M_{\mu,\nu}(x,t) <_{L^*} 1_{L^*}$ for all $x \in X$.

Definition 2.8. *Let $(X, M_{\mu,\nu}, \Im)$ be an L-fuzzy invariant metric space.*
For $t > 0$, define the open ball $B(x,r,t)$ with center $x \in X$ and radius $r \in (0,1)$ as

$$B(x,r,t) = \{y \in X : M_{\mu,\nu}(x-y,t) >_{L^*} (N_S(r), r)\}.$$

A subset $A \subseteq X$ is called open if for each $x \in A$, there exist $r \in (0,1)$ and $t > 0$ such that $B(x,r,t) \subseteq A$. Let $\tau_{M_{\mu,\nu}}$ denote the family of all open subsets of X. $\tau_{M_{\mu,\nu}}$ is called the topology induced by L-fuzzy invariant metric $M_{\mu,\nu}$.

Definition 2.9. *A sequence $\{x_n\}$ in an L-fuzzy invariant metric space $(X, M_{\mu,\nu}, \Im)$ is said to be Cauchy if for each $\varepsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$M_{\mu,\nu}(x_n - x_m, t) >_{L^*} (N_S(\varepsilon), \varepsilon)$$

for each $n, m \geq n_0$. The sequence $\{x_n\}$ is said to be convergent to $x \in X$ in X and denoted by $x_n \xrightarrow{M_{\mu,\nu}} x$ if $M_{\mu,\nu}(x_n - x, t) \to 1_{L^*}$ whenever $n \to \infty$ for every $t > 0$. An L-fuzzy invariant metric space is said to be complete if and only if every Cauchy sequence is convergent.

The proofs of following two lemmas are similar from classical cases and omitted [2, 3].

Lemma 2.10. *Let $M_{\mu,\nu}$ be an L-fuzzy invariant metric. Then, for any $t > 0$, $M_{\mu,\nu}(x,t)$ is non-decreasing with respect to t in $(L^*, \leq_{L^*})$ for all $x \in X$.*

Lemma 2.11. *Let $(X, M_{\mu,\nu}, \Im)$ be an L-fuzzy invariant metric space. Then $M_{\mu,\nu}$ is continuous function on $X \times (0,\infty)$.*

Theorem 2.12. *Every L-fuzzy invariant metric space is normal.*

Proof. Let $(X, M_{\mu,\nu}, \Im)$ be an L-fuzzy invariant metric space and F, G be two disjoint closed subsets of X. Let $x \in X$. Then $x \in G^c$ since G^c is open there exist $t_x > 0$ and $r_x \in (0,1)$ such that $B(x, r_x, t_x) \cap G = \varnothing$ for all $x \in F$. Similarly, there exist $t_y > 0$ and $r_y \in (0,1)$ such that $B(x, r_y, t_y) \cap F = \varnothing$ for all $y \in G$. Let $s = \min\{r_x, t_x, r_y, t_y\}$. Then we can find a $s_0 \in (0,s)$ such that $\Im((N_S(s_0), s_0), (N_S(s_0), s_0)) >_{L^*} (N_S(s), s)$. Define $U = \cup_{x\in F} B(x, s_0, s/2)$ and $V = \cup_{y\in G} B(y, s_0, s/2)$. Clearly U and V are open sets such that $F \subset U$ and $G \subset V$. Now, we claim that $U \cap V = \varnothing$. Let $z \in U \cap V$. Then there exist $x \in F$ and $y \in G$ such that $z \in B(x, s_0, s/2)$ and $z \in B(y, s_0, s/2)$. Therefore, we have

$$\begin{aligned} M_{\mu,\nu}(x-y, s) &\geq_{L^*} \Im(M_{\mu,\nu}(x-z, s/2), M_{\mu,\nu}(z-y, s/2) \\ &\geq_{L^*} \Im((N_S(s_0), s_0), (N_S(s_0), s_0)) >_{L^*} (N_S(s), s). \end{aligned}$$

Hence $y \in B(x,s,s)$. Since $s < t_x, r_x$ we have $B(x,s,s) \subset B(x, r_x, t_x)$. Thus $B(x, r_x, t_x) \cap G$ is nonempty which is a contradiction. Therefore $U \cap V = \varnothing$. Hence X is normal. □

Remark 2.13. *From the above theorem, we can easily deduce that every metrizable space is normal. Since every L-fuzzy invariant metric space is normal, Urysohn's lemma and Tietze extension theorem are true in the case of L-fuzzy invariant metric space.*

Definition 2.14. *A function f from an L-fuzzy invariant metric space X to an other L-fuzzy invariant metric space Y is said to be uniformly continuous if for given $t > 0$ and $r \in (0,1)$, there exist $t_0 > 0$ and $r_0 \in (0,1)$ such that $M_{\mu,\nu}(x-y, t_0) >_{L^*} (N_S(r_0), r_0)$ implies $M_{\mu,\nu}(f(x) - f(y), t) >_{L^*} (N_S(r), r)$.*

As usual by a compact L-fuzzy invariant metric space we mean an L-fuzzy invariant metric space $(X, M_{\mu,\nu}, \Im)$ such that $(X, \tau_{M_{\mu,\nu}})$ is a compact topological space.

Theorem 2.15 (Uniform continuity theorem)**.** *If f is a continuous function from a compact L-fuzzy invariant metric space X to an other L-fuzzy invariant metric space Y, then f is uniformly continuous.*

Proof. Let $t > 0$ and $s \in (0,1)$. Then we can find $r \in (0,1)$ such that $\Im((N_S(r), r), (N_S(r), r)) >_{L^*} (N_S(s), s)$. Since $f : X \to Y$ is continuous, for each $x \in X$ we can find $t_x > 0$ and $r_x \in (0,1)$ such that $M_{\mu,\nu}(x-y, t) >_{L^*} (N_S(r_x), r_x)$ implies $M_{\mu,\nu}(f(x) - f(y), \frac{t}{2}) >_{L^*} (N_S(r), r)$. But $r_x \in (0,1)$ and then we can find $s_x \in (0, r_x)$ such that $\Im((N_S(s_x), s_x), (N_S(s_x), s_x)) >_{L^*} (N_S(r_x), r_x)$. Since X is compact and $\{B(x, s_x, \frac{t_x}{2} : x \in X\}$ is an open covering of X, there exist $x_1, x_2, \ldots, x_k$ in X such that $X = \cup_{i=1}^{k} B(x_i, s_{x_i}, \frac{t_{x_i}}{2})$. Put $s_0 = \min s_{x_i}$ and $t_0 = \min \frac{t_{x_i}}{2}$, $i = 1, 2, \ldots, k$. For any $x, y \in X$, if $M_{\mu,\nu}(x-y, t_0) >_{L^*} (N_S(s_0), s_0)$, then $M_{\mu,\nu}(f(x) - f(y), \frac{t_{x_i}}{2}) >_{L^*} (N_S(s_{x_i}), s_{x_i})$. Since $x \in X$, there exists a x_i such that $M_{\mu,\nu}(x - x_i, \frac{t_{x_i}}{2}) >_{L^*} (N_S(s_{x_i}), s_{x_i})$. Hence we have $M_{\mu,\nu}(f(x) - f(x_i), \frac{t}{2}) >_{L^*} (N_S(r), r)$. Now

$$\begin{aligned} M_{\mu,\nu}(x_i - y, t_{x_i}) &\geq_{L^*} \Im(M_{\mu,\nu}(x_i - x, \frac{t_{x_i}}{2}), M_{\mu,\nu}(x-y, \frac{t_{x_i}}{2})) \\ &\geq_{L^*} \Im((N_S(s_{x_i}), s_{x_i}), (N_S(s_{x_i}), s_{x_i})) >_{L^*} (N_S(r_{x_i}), r_{x_i}). \end{aligned}$$

Therefore $M_{\mu,\nu}(f(x_i) - f(y), \frac{t}{2}) >_{L^*} (N_S(r), r)$. Now we have

$$\begin{aligned} M_{\mu,\nu}(f(x) - f(y), t) &\geq_{L^*} \Im(M_{\mu,\nu}(f(x) - f(x_i), \frac{t}{2}), M_{\mu,\nu}(f(x_i) - f(y), \frac{t}{2}) \\ &\geq_{L^*} \Im((N_S(r), r), (N_S(r), r)) >_{L^*} (N_S(s), s). \end{aligned}$$

Hence f is uniformly continuous. □

Remark 2.16. *Let f be a uniformly continuous function from the L-fuzzy invariant metric space X to an other L-fuzzy invariant metric space Y. If $\{x_n\}$ is a Cauchy sequence in X, then $\{f(x_n)\}$ is also a Cauchy sequence in Y.*

Theorem 2.17. *Every compact L-fuzzy invariant metric space is separable.*

Proof. Let $(X, M_{\mu,\nu}, \mathfrak{I})$ be the given compact L-fuzzy invariant metric space and $t > 0$, $r \in (0,1)$. Since X is compact, there exist $x_1, x_2, ..., x_n$ in X such that $X = \cup_{i=1}^{n} B(x_i, r, t)$. In particular, for each $n \in \mathbb{N}$, we can find a finite subset A_n such that $X = \cup_{a \in A} B(a, r_n, \frac{1}{n})$ in which $r_n \to 0_{L^*}$. Let $A = \cup_{n \in \mathbb{N}} A_n$. Then A is countable. Now, we claim that $X \subset \overline{A}$. For that let $x \in X$, then, for each n, there exists $a_n \in A_n$ such that $x \in B(a_n, r_n, \frac{1}{n})$. Thus a_n is converges to x. Since $a_n \in A_n$ for all n then $x \in \overline{A}$. Therefore A is dense in X, thus X is separable. □

Definition 2.18. *Let X be any nonempty set and $(Y, M_{\mu,\nu}, \mathfrak{I})$ be an L-fuzzy invariant metric space. Then a sequence $\{f_n\}$ of functions from X to Y is said to be converge uniformly to a function f from X to Y if for given $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M_{\mu,\nu}(f_n(x) - f(x), t) >_{L^*} (N_S(r), r)$ for all $n \geq n_0$ and $x \in X$.*

Definition 2.19. *A family F of functions from an L-fuzzy invariant metric space X to a complete L-fuzzy invariant metric space Y is said to be equicontinuous if for given $r \in (0,1)$ and $t > 0$, there exists $r_0 \in (0,1)$ and $t_0 > 0$ such that $M_{\mu,\nu}(x - y, t_0) >_{L^*} (N_S(r_0), r_0)$ implies $M_{\mu,\nu}(f(x) - f(y), t) >_{L^*} (N_S(r), r)$ for all $f \in F$.*

Lemma 2.20. *Let $\{f_n\}$ be an equicontinuous sequence of functions from an L-fuzzy invariant metric space X to a complete L-fuzzy invariant metric space Y. If $\{f_n\}$ converges for each point of a dense subset D of X, then $\{f_n\}$ converges for each point of X and the limit function is continuous.*

Proof. Let $s \in (0,1)$ and $t > 0$ be given. Then we can find $r \in (0,1)$ such that $\mathfrak{I}^2((N_S(r), r), (N_S(r), r), (N_S(r), r)) >_{L^*} (N_S(s), s)$. Since $F = \{f_n\}$ is an equicontinuous family, for given $r \in (0,1)$ and $t > 0$, there exist $r_1 \in (0,1)$ and $t_1 > 0$ such that for each $x, y \in X$, $M_{\mu,\nu}(x - y, t_1) >_{L^*} (N_S(r_1), r_1)$ implies $M_{\mu,\nu}(f_n(x) - f_n(y), \frac{t}{3}) >_{L^*} (N_S(r), r)$ for all $f_n \in F$. Since D is dense in X, there exists $y \in B(a, r_1, t_1) \cap D$ and $\{f_n(y)\}$ converges for that y. Since $\{f_n(y)\}$ is a Cauchy sequence, for given $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M_{\mu,\nu}(f_n(y) - f_m(y), \frac{t}{3}) >_{L^*} (N_S(r), r)$ for all $m, n \geq n_0$. Now for any $x \in X$, we have

$$\begin{aligned}
M_{\mu,\nu}(f_n(x) - f_m(x), t) &\geq {}_{L^*}\mathfrak{I}^2(M_{\mu,\nu}(f_n(x) - f_n(y), \frac{t}{3}), \\
&\quad M_{\mu,\nu}(f_n(y) - f_m(y), \frac{t}{3}), M_{\mu,\nu}(f_m(x) - f_m(y), \frac{t}{3})) \\
&\geq {}_{L^*}\mathfrak{I}^2((N_S(r), r), (N_S(r), r), (N_S(r), r)) \\
&> {}_{L^*}(N_S(s), s)
\end{aligned}$$

Hence $\{f_n(x)\}$ is a Cauchy sequence in Y. Since Y is complete, $f_n(x)$ converges. Let $f(x) = \lim_{n\to\infty} f_n(x)$. We claim that f is continuous. Let $s_0 \in (0,1)$ and $t_0 > 0$ be given. Then we can find that $r_0 \in (0,1)$ such that $\mathfrak{I}^2((N_S(r_0), r_0), (N_S(r_0), r_0), (N_S(r_0), r_0)) >_{L^*} (N_S(s_0), s_0)$. Since F is equicontinuous, for given $r_0 \in (0,1)$ and $t_0 > 0$, there exist $r_2 \in (0,1)$ and $t_2 > 0$ such that $M_{\mu,\nu}(x - y, t_2) >_{L^*} (N_S(r_2), r_2)$ implies $M_{\mu,\nu}(f_n(x) - f_n(y), \frac{t_0}{3}) >_{L^*} (N_S(r_0), r_0)$ for all $f_n \in F$. Since $f_n(x)$ converges to $f(x)$, for given $r_0 \in (0,1)$ and $t_0 > 0$, there exists $n_1 \in \mathbb{N}$ such that $M_{\mu,\nu}(f_n(y) - f(x), \frac{t_0}{3}) >_{L^*} (N_S(r_0), r_0)$ for all $n \geq n_1$. Also since $f_n(y)$ converges to $f(y)$, for given $r_0 \in (0,1)$ and $t_0 > 0$, there exists $n_2 \in \mathbb{N}$ such that $M_{\mu,\nu}(f_n(y) - f(y), \frac{t_0}{3}) >_{L^*} (N_S(r_0), r_0)$ for all $n \geq n_2$. Now for all $n \geq \max\{n_1, n_2\}$, we have

$$\begin{aligned}
M_{\mu,\nu}(f(x) - f(y), t_0) &\geq {}_{L^*}\mathfrak{I}^2(M_{\mu,\nu}(f(x) - f_n(x), \frac{t_0}{3}), \\
&\quad M_{\mu,\nu}(f_n(x) - f_n(y), \frac{t_0}{3}), M_{\mu,\nu}(f_n(y) - f(y), \frac{t_0}{3})) \\
&\geq {}_{L^*}\mathfrak{I}^2((N_S(r_0), r_0), (N_S(r_0), r_0), (N_S(r_0), r_0)) \\
&> {}_{L^*}(N_S(s_0), s_0).
\end{aligned}$$

Hence f is continuous. □

Theorem 2.21 (Ascoli-Arzela theorem)**.** *Let X be a compact L-fuzzy invariant metric space and Y be a complete L-fuzzy invariant metric space. Let F be an equicontinuous family of functions from X to Y. If $\{f_n\}$ is a sequence in F such that $\overline{\{f_n(x) : n \in \mathbb{N}\}}$ is a compact subset of Y for each $x \in X$, then there exists a continuous function f from X to Y and a subsequence $\{g_n\}$ of $\{f_n\}$ such that g_n converges uniformly to f on X.*

Proof. Since X is compact L-fuzzy invariant metric space, by Theorem 2.17, X is separable. Let $D = \{x_i : i = 1,2,...\}$ be a countable dense subset of X. By hypothesis, for each i, $\overline{\{f_n(x_i) : n \in \mathbb{N}\}}$ is compact subset of Y. Since every L-fuzzy invariant metric space is first countable space, every compact subset of Y is sequentially compact. Thus by standard argument, we have a subsequence $\{g_n\}$ of $\{f_n\}$ such that $\{g_n(x_i)\}$ converges for each $i = 1,2,...$ By Lemma 2.20, there exists a continuous function f from X to Y such that $g_n(x)$ converges to $f(x)$ for all $x \in X$. Now we claim that g_n converges to f on X. Let $s \in (0,1)$ and $t > 0$ be given. Then we can find $r \in (0,1)$ such that $\Im^2((N_S(r),r),(N_S(r),r),(N_S(r),r)) >_{L^*} (N_S(s),s)$. Since F is equicontinuous, there exist $r_1 \in (0,1)$ and $t_1 > 0$ such that $M_{\mu,\nu}(x-y,t_1) >_{L^*} (N_S(r_1),r_1)$ implies $M_{\mu,\nu}(g_n(x)-g_n(y),\frac{t}{3}) >_{L^*} (N_S(r),r)$ for all n. Since X is compact, by Theorem 2.15, f is uniformly continuous. Hence for given $r \in (0,1)$ and $t > 0$, there exists $r_2 \in (0,1)$ and $t_2 > 0$ such that $M_{\mu,\nu}(x-y,t_2) >_{L^*} (N_S(r_2),r_2)$ implies $M_{\mu,\nu}(f(x)-f(y),\frac{t}{3}) >_{L^*} (N_S(r),r)$ for all $x,y \in X$. Let $r_0 = \min\{r_1,r_2\}$ and $t_0 = \min\{t_1,t_2\}$. Since X is compact and D is dense in X, $X = \cup_{i=1}^{k} B(x_i,r_0,t_0)$ for some finite k. Thus for each $x \in X$, there exists $i, 1 \le i \le k$, such that $M_{\mu,\nu}(x-x_i,t_0) >_{L^*} (N_S(r_0),r_0)$. But since $r_0 = \min\{r_1,r_2\}$ and $t_0 = \min\{t_1,t_2\}$, we have, by the equicontinuity of F, $M_{\mu,\nu}(g_n(x)-g_n(x_i),\frac{t}{3}) >_{L^*} (N_S(r),r)$ and also we have, by the uniform continuity of f, $M_{\mu,\nu}(f(x)-f(x_i),\frac{t}{3}) >_{L^*} (N_S(r),r)$. Since $g_n(x_j)$ converges to $f(x_j)$, $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M_{\mu,\nu}(g_n(x_j)-f(x_j),\frac{t}{3}) >_{L^*} (N_S(r),r)$ for all $j = 1,2,...,n$. Now, for each $x \in X$, we have

$$\begin{aligned}
M_{\mu,\nu}(g_n(x)-f(x),t) &\geq_{L^*} \Im^2(M_{\mu,\nu}(g_n(x)-g_n(x_i),\frac{t}{3}),\\
&\quad M_{\mu,\nu}(g_n(x_i)-f(x_i),\frac{t}{3}),M_{\mu,\nu}(f(x_i)-f(x),\frac{t}{3}))\\
&\geq_{L^*} \Im^2((N_S(r),r),(N_S(r),r),(N_S(r),r))\\
&>_{L^*} (N_S(s),s).
\end{aligned}$$

Hence g_n converges uniformly to f on X. □

3. Conclusion

The aim of this paper is to introduce L-fuzzy invariant metric space, and to generalize Uniform continuity theorem and Ascoli-Arzela theorem for this space. Aside from their numerous applications to Partial Differential Equations such as existence theorems in differential and integral equations, and Lorentzian Geometry such as guaranteing convergence to isometry using Lorentzian analogues, these results can be also used as a tool in obtaining Functional Analysis results such as compactness for duals of compact operators, conformal mapping and extremal problems in complex variable theory.

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