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Some Properties of Proper UP-Filters of UP-Algebras

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Abstract

The concept of UP-algebras was introduced and analyses by A. Iampan. In our recently published article we introduced the concept of proper UP-filter in UP-algebras in a somewhat different way than it is usual in literature. In this paper we analyse some fundamental properties of such determined proper UP-filters in UP-algebras.

1. Introduction

The concepts of UP-algebra are introduced and analyzed in [1]. The author in his article has introduced and analyzed the concepts of UP-algebra, UP-subalgebra and UP-ideal and their mutual connections. This author introduced in [2] the concept of proper UP-filter in UP-algebras on something different way then it is common in the available literature. In addition, in [2] he established the connection between UP-ideals and proper UP-filters.

In this article, the author further develops the idea of a proper UP-filter by identifying some of the fundamental features of this concept. First, we have shown two criteria (Theorem 3.1 and Theorem 3.2) that allow us to estimate whether a certain subset of UP-algebra is proper UP-filter or not. Other claims relate to a link between the proper UP-filters and UP-homomorphisms. Theorem 3.5 can be viewed as the first isomorphism theorem. For more details, see [3, 4].

The notations and notions appearing in this text are not predefined, the reader can find in the articles [1, 2, 3, 4].

2. Preliminaries

Let us recall the definition of UP-algebra.

Definition 2.1. *[[1], Definition 1.3] An algebra* $A = (A, \cdot, 0)$ *of type* $(2,0)$ *is called a* UP- algebra *if it satisfies the following axioms:*
(UP - 1): $(\forall x,y,z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(UP - 2): $(\forall x \in A)(0 \cdot x = x)$,
(UP - 3): $(\forall x \in A)(x \cdot 0 = 0)$,
(UP - 4): $(\forall x,y \in A)((x \cdot y = 0 \wedge y \cdot x = 0) \Longrightarrow x = y)$.

In the following we give definition of the concept of UP-ideals of UP-algebra.

Definition 2.2. *[[1], Definition 2.1] Let A be a UP-algebra. A subset J of A is called a* UP-ideal *of A if it satisfies the following properties:*

1. $0 \in J$, *and*
2. $(\forall x,y,z \in A)(x \cdot (y \cdot z) \in J \wedge y \in J \Longrightarrow x \cdot z \in J)$.

One of fundamental properties of UP-ideals is given in statement (1) of Proposition 2.7 in the article [1]:

Proposition 2.3. *Let A be a UP-algebra and B a UP-ideal of A. Then*

$$\forall x,y \in A)((x \in B \wedge x \leq y) \Longrightarrow y \in B).$$

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Our intention in short notice [2] was to construct a substructure G in UP-algebras that will have the following property

$$(\forall x,y\in A)((y\in G\wedge x\leq y)\Longrightarrow x\in G)$$

and has a standard attitude toward the UP-ideal. This was done by introducing the concept of a proper UP-filter by the following way.

Definition 2.4 ([2], Definition 3.1)**.** *Let A be a UP-algebra. A subset G of A is called a* proper UP-filter *of A if it satisfies the following properties:*

3. $\neg(0\in G)$*, and*
4. $(\forall x,y,z\in A)((\neg(x\cdot(y\cdot z)\in G)\wedge x\cdot z\in G)\Longrightarrow y\in G)$

In the mentioned article it was shown

Proposition 2.5. *Let A be a UP-algebra and G a proper UP-filter of A. Then*

5. $(\forall x,y\in A)((\neg(x\cdot y\in G)\wedge y\in G)\Longrightarrow x\in G)$.
6. $(\forall x,y\in A)(x\cdot y\in G\Longrightarrow y\in G)$.
7. $(\forall x,y\in A)((x\leqslant y\wedge y\in G)\Longrightarrow x\in G)$.

Proposition 2.6. *A subset G of a UP-algebra A is a proper UP-filter of A if and only if the set $A\backslash G$ is a UP-ideal of A.*

Proposition 2.7. *The family $\mathfrak{G}_A$ of all proper UP-filters in a UP-algebra A forms a completely lattice.*

Finally, the concept of UP-homomorphisms is defined by the following

Definition 2.8 ([1], Definition 4.1)**.** *Let $(A,\cdot,0_A)$ and $(B,\circ,0_B)$ be UP-algebras. A mapping f from A to B is called a UP-homomorphism if holds*

$$(\forall x,y\in A)(f(x\cdot y)=f(x)\circ f(y)).$$

In [1] it was shown that $f(A)$ is a subalgebra of algebra B (Theorem 4.5 (3)) and that $Ker f$ is an UP-ideal in A (Theorem 4.5 (6)).

3. The main results

First, for a subset G of UP-algebra A we show that from (5) and (6) follows (3) and (4) if we assume that $G\neq A$.

Theorem 3.1. *For a subset G of a UP-algebra A (5) and (6) implies (3) and (4) if we assume that $G\neq A$*

Proof. Let formulas (5) and (6) be valid for the proper subset G in A. Suppose that $\neg(x\cdot(y\cdot z)\in G)$ and $x\cdot z\in G$ is valid for arbitrary elements $x,y,z\in A$.
If we put $y=x$ in (6) we get that $0=x\cdot x\in G$ implies $x\in G$ for any element $x\in A$. This is in a contradiction with $G\neq A$. The resulting contradiction yields $\neg(0\in G)$. Thus, (3) is proven. From here it follows immediately that the subset G satisfies the formula (7). Indeed, if $x\leqslant y$ and $y\in G$, then we have$\neg(x\cdot y=0\in G)$ and $y\in G$. From here follows $x\in G$ according to (5).
First, from $x\cdot z\in G$ we have $z\in G$ by (6). Second, suppose it is $\neg(y\in G)$ holds. Thus, from $\neg(y\in G)$ and $z\in G$ follows $y\cdot z\in G$ by the contraposition of (5). Third, we have $y\cdot z\leqslant x\cdot(y\cdot z)$ by statement (6) in Theorem 1.8 in the article [1]. Now, from this and $y\cdot z\in G$ we conclude $x\cdot(y\cdot z)\in G$ by (7). This is in a contradiction with the first hypothesis. So, it has to be $y\in G$. Therefore, (4) is proven. □

Our second proposition is one more criterion for determining whether a subset G of A is a proper UP-filter or not.

Theorem 3.2. *Let A be a UP-algebra and $G\subseteq A$ such that $\neg(0\in G)$. Then G is a proper UP-filter in A if and only if*

8. $(\forall x,y,z\in A)((\neg(y\in G)\wedge x\cdot z\in G)\Longrightarrow x\cdot(y\cdot z)\in G)$.

Proof. Let G be a proper UP-filter in a UP-algebra A and x,y,z be arbitrary elements of A. Suppose $\neg(y\in G)$ and $x\cdot z\in G$. If there were $\neg(x\cdot(y\cdot z)\in G)$ then from this and $x\cdot z\in G$ would have $y\in G$. The resulting result is in contradiction with the first hypothesis. Therefore, it must be $x\cdot(y\cdot z)\in G$.
Opposite, let for subset G of A (3) and (8) be hold for any $x,y,z\in A$. Suppose $\neg(x\cdot(y\cdot z)\in G)$ and $x\cdot z\in G$ are valid. If there were $\neg(y\in G)$ then from this and the second hypothesis would have $x\cdot(y\cdot z)\in G$ by (8). The resulting result is in contradiction with the first hypothesis. Therefore, it must be $y\in G$. □

Corollary 3.3. *Let G be a proper UP-filter in a UP-algebra A. Then*

9. $(\forall x,y\in A)((\neg(x\in G)\wedge y\in G)\Longrightarrow x\cdot y\in G)$.

Proof. If we put $x=0$, $y=x$ and $z=y$ in (8) we will got (9). □

Theorem 3.4. *Let $(A,\cdot,0_A)$ and $(B,\circ,0_B)$ be UP-algebras and let $f:A\longrightarrow B$ be a UP-homomorphism. Then the following statements hold:*
(a) *If F is a proper UP-filter in a UP-algebra A, then $f(F)$ is a proper UP-filter in a UP-algebra $f(A)$.*
(b) *If G is a proper UP-filter of B, then $f^{-1}(G)$ is a proper UP-filter of A.*

Proof. (a) Assume that F is a proper UP-filter of A. Since $\neg(0_A \in F)$ and the statement (1) of Theorem 4.5 in article [1], we have $\neg(0_B = f(0_A) \in f(F))$.
Let $a,b,c \in f(A)$ be arbitrary elements such that $\neg(a\circ(b\circ c) \in f(F))$ and $a\circ c \in f(F)$. Then there exist elements $x,y,z \in A$ such that $f(x) = a$, $f(y) = b$ and $f(z) = c$ and $\neg(f(x)\circ(f(y)\circ f(z)) \in f(F))$ and $f(x)\circ f(z) \in f(F)$. This means $\neg(f(x\cdot(y\cdot z) \in f(F))$ and $f(x\cdot z) \in f(F)$. So, we conclude $\neg(x\cdot(y\cdot z) \in F)$ and $x\cdot z \in F$. Thus $y \in F$ by (4). Therefore, $c = f(y) \in f(F)$.
(b) Assume that G is a proper UP-filter of B. Since $\neg(0_B \in G)$, we have $\neg(f(0_A) = 0_B \in G)$. Thus $\neg(0_A \in f^{-1}(G))$.
Let $x,y,z \in A$ be arbitrary elements of A such that $\neg(x\cdot(y\cdot z) \in f^{-1}(G))$ and $x\cdot z \in f^{-1}(G)$. Then $\neg(f(x\cdot(y\cdot z)) \in G)$ and $f(x\cdot z) \in G$. Since f is a UP-homomorphism, we have $\neg(f(x)\circ(f(y)\circ f(z))) \in G)$ and $f(x)\circ f(z) \in G$. Since G is a proper UP-filter of B, we have $f(y) \in G$. Thus $y \in f^{-1}(G)$. □

Without major difficulties, it can be proved that if J is a UP-ideal in a UP-algebra A and $'\sim'$ the congruence on A determined by the ideal J ([1], Proposition 3.5), then $A/J \equiv A/\sim = \{[x]_\sim : x \in A\}$ is also UP-algebra with the internal operation $'*'$ defined by

$$(\forall x, y \in A)([x]_\sim * [y]_\sim = [x\cdot y]_\sim)$$

and the fixed element J. The following claims is proven by direct verification.

Theorem 3.5. *Let $f : A \longrightarrow B$ be a UP-homomorphism between UP-algebras. Then there exists the UP-isomorphism $g : A/Ker(f) \longrightarrow f(A)$ such that $f = g\circ\pi$ where $\pi : A \longrightarrow A/Ker(f)$ is the canonical UP-epimorphism.*

Theorem 3.6. *Let $f : A \longrightarrow B$ be a UP-homomorphism between UP-algebras and J be a UP-ideal in A.*
If K is a UP-ideal in a UP-algebra A such that $J \subseteq K$, then the set $K/J = \{[x]_J \in A/J : x \in K\}$ is a UP-ideal in UP-algebra A/J.
If G is a proper UP-filter in a UP-algebra A such that $G \subseteq A\setminus J$, then the set $G/J = \{[x]_J : x \in G\}$ is a proper UP-filter in a UP-algebra A/J.

Proof. (a) It is clear that $J = [0]_J \in A/J$ is the fixed element in a UP-algebra A/J. Let $x,y,z \in A$ be arbitrary elements such that $[x]_J * ([y]_J * [z]_J) \in K/J$ and $[y]_J \in H/L$. Since $x\cdot(y\cdot z) \in K$ and $y \in K$ and since K is a UP-ideal in a UP-algebra A we conclude $x\cdot z \in K$. Thus $[x]_J * [z]_J \in K/J$. Therefore, the set K/J is a UP-ideal in a UP-algebra A/J.

(b) If there were $[0]_J \in G/J$, they would have $0 \in G$, which is a contradiction. So, we have $\neg([0]_J \in G/J)$. Let $x,y,z \in A$ be arbitrary elements such that $\neg([x]_J * ([y]_J * [z]_J) \in G/J)$ and $[x]_J * [z]_J \in G/J$. This means that $\neg(x\cdot(y\cdot z) \in G)$ and $x\cdot z \in G$. Since G is a proper UP-filter in a UP-algebra A, we have $y \in G$. Thus $[y]_J \in G/J$. Therefore, the set G/J is a proper UP-filter in a UP-algebra A/J. □

Corollary 3.7. *There is a mutually unambiguous correspondence between the family $F_{A/J}$ of all proper UP-filters in a UP-algebra A/J and the family of all proper UP-filters contained in $A\setminus J$.*

4. Final observation

In the present paper, in order to continue developing the theory of proper UP-filters and UP-algebras, we given some fundamental properties of proper UP-filters in UP-algebra. The author believes that this new properties of proper UP-filters in UP-algebras enrich our knowledge about UP-algebras.

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