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# $\mathscr{I}$-Cesàro Summability of a Sequence of Order $\alpha$ of Random Variables in Probability

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**Abstract**

In this paper, we define four types of convergence of a sequence of random variables, namely, $\mathscr{I}$-statistical convergence of order $\alpha$, $\mathscr{I}$-lacunary statistical convergence of order $\alpha$, strongly $\mathscr{I}$-lacunary convergence of order $\alpha$ and strongly $\mathscr{I}$-Cesàro summability of order $\alpha$ in probability where $0 < \alpha < 1$. We establish the connection between these notions.

## 1. Introduction and background

Theory of statistical convergence was firstly originated by Fast [1]. After Fridy [2] and Šalát [3] statistical convergence became a notable topic in summability theory. Lacunary statistical convergence was defined by using lacunary sequences in [4]. $\mathscr{I}$-convergence was fistly considered by Kostyrko et al. [5]. Also, Das et al. [6] gave new definitions by using ideal, such as $\mathscr{I}$-statistical convergence, $\mathscr{I}$-lacunary statistical convergence. Ulusu et al. [7] also studied asymptotically $\mathscr{I}$-Cesaro equivalence of sequences of sets.
Statistical convergence of order $\alpha$ $(0 < \alpha < 1)$ was introduced using the notion of natural density of order $\alpha$ where $n$ is replaced by $n^{\alpha}$ in [8]. This new type convergence was different in many ways from statistical convergence. Lacunary statistical convergence of order $\alpha$ is studied by Sengöl and M. Et [9], $\mathscr{I}$-statistical and $\mathscr{I}$-lacunary statistical convergence of order $\alpha$ is studied by Das and Savas [10].
In probability theory, if for $n > 0$, a random variable $X_n$ given on space $S$, a probability function $P : X \to \mathbb{R}$, then we say that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of random variables and it is demonstrated by $\{X_n\}_{n\in\mathbb{N}}$.
It is important that if there exists $c \in \mathbb{R}$ for which $P(|X-c| < \varepsilon) = 1$, where $\varepsilon > 0$ is sufficiently small, that is, it is means that values of $X$ lie in a very small neighbourhood of $c$.
New concepts have begun to be studied in probability theory by Das et al. [6], and others ([11]-[15]).

## 2. Main results

**Definition 2.1.** *$\{X_k\}_{k\in\mathbb{N}}$ is said to be $\mathscr{I}$-statistically convergent of order $\alpha$ in probability to a random variable $X$ if for any $\varepsilon$, $\delta$, $\gamma > 0$*

$$\left\{ n \in \mathbb{N} : \frac{1}{n^{\alpha}} \left| \{ k \leq n : P(|X_k - X| \geq \varepsilon) \geq \delta \} \right| \geq \gamma \right\} \in \mathscr{I},$$

*and demonstrated by $X_k \overset{PS(\mathscr{I})^{\alpha}}{\to} X$.*

**Definition 2.2.** *$\{X_n\}_{n\in\mathbb{N}}$ is said to be $\mathscr{I}$-lacunary statistically convergent of order $\alpha$ in probability to a random variable $X$ if for any $\varepsilon, \delta, \gamma > 0$*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^{\alpha}} \left| \{ k \in I_r : P(|X_k - X| \geq \varepsilon) \geq \delta \} \right| \geq \gamma \right\} \in \mathscr{I},$$

*and it is demonstrated by $X_k \overset{PS_{\theta}(\mathscr{I})^{\alpha}}{\to} X$.*

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**Definition 2.3.** $\{X_k\}_{k\in\mathbb{N}}$ *is said to be strongly $\mathscr{I}$-lacunary convergent or $PV_\theta(\mathscr{I})$-convergent of order $\alpha$ in probability to a random variable X if for every $\varepsilon$, $\delta>0$,*

$$\left\{r\in\mathbb{N}:\frac{1}{h_r^\alpha}\sum_{k\in I_r}P(|X_k-X|\geq\varepsilon)\geq\delta\right\}\in\mathscr{I},$$

*and it is demonstrated by* $X_k\overset{PV_\theta(\mathscr{I})^\alpha}{\to}X$.

**Definition 2.4.** $\{X_k\}_{k\in\mathbb{N}}$ *is said to be strongly $\mathscr{I}$-Cesàro summable of order $\alpha$ in probability to a random variable X if for every $\varepsilon$, $\delta>0$,*

$$\left\{n\in\mathbb{N}:\frac{1}{n^\alpha}\sum_{k=1}^{n}P(|X_k-X|\geq\varepsilon)\geq\delta\right\}\in\mathscr{I},$$

*and it is demonstrated by* $X_k\overset{PC_1[\mathscr{I}]^\alpha}{\to}X$.

**Theorem 2.5.** *If $0<\alpha\leq\beta\leq1$ then $PS(\mathscr{I})^\alpha\subseteq PS(\mathscr{I})^\beta$.*

*Proof.* From the assumption, we say that

$$\frac{1}{n^\beta}\left|\{k\leq n:P(|X_k-X|\geq\varepsilon)\geq\delta\}\right|\leq\frac{1}{n^\alpha}\left|\{k\leq n:P(|X_k-X|\geq\varepsilon)\geq\delta\}\right|$$

Hence,

$$\left\{n\in\mathbb{N}:\tfrac{1}{n^\beta}\left|\{k\leq n:P(|X_k-X|\geq\varepsilon)\geq\delta\}\right|\geq\gamma\right\}$$
$$\left\{n\in\mathbb{N}:\tfrac{1}{n^\alpha}\left|\{k\leq n:P(|X_k-X|\geq\varepsilon)\geq\delta\}\right|\geq\gamma\right\}$$

for $\gamma>0$. Therefore, we obtain $PS(\mathscr{I})^\alpha\subseteq PS(\mathscr{I})^\beta$. □

**Theorem 2.6.** *If* $\liminf_r q_r>1$, *then*

$$X_k\overset{PC_1[\mathscr{I}]^\alpha}{\to}X\Rightarrow X_k\overset{PV_\theta(\mathscr{I})^\alpha}{\to}X.$$

*Proof.* If $\liminf_r q_r>1$, there exists $\gamma>0$ such that $q_r\geq1+\gamma$ for all $r\geq1$. Since $h_r=k_r-k_{r-1}$, we have $\frac{k_r^\alpha}{h_r^\alpha}\leq\left(\frac{1+\gamma}{\gamma}\right)^\alpha$ and $\frac{k_{r-1}^\alpha}{h_r^\alpha}\leq\left(\frac{1}{\gamma}\right)^\alpha$. Let $\varepsilon>0$ and we define set by

$$S=\left\{k_r\in\mathbb{N}:\frac{1}{k_r^\alpha}\sum_{k=1}^{k_r}P(|X_k-X|\geq\varepsilon)<\delta\right\}.$$

Therefore, $S\in\mathscr{F}(\mathscr{I})$.

$$\begin{aligned}\tfrac{1}{h_r^\alpha}\sum_{k\in I_r}P(|X_k-X|\geq\varepsilon) &= \tfrac{1}{h_r^\alpha}\sum_{k=1}^{k_r}P(|X_k-X|\geq\varepsilon)-\tfrac{1}{h_r^\alpha}\sum_{k=1}^{k_{r-1}}P(|X_k-X|\geq\varepsilon)\\ &= \frac{k_r^\alpha}{h_r^\alpha}\cdot\tfrac{1}{k_r^\alpha}\sum_{k=1}^{k_r}P(|X_k-X|\geq\varepsilon)-\tfrac{k_{r-1}^\alpha}{h_r^\alpha}\cdot\tfrac{1}{k_{r-1}^\alpha}\sum_{k=1}^{k_{r-1}}P(|X_k-X|\geq\varepsilon)\\ &\leq \left(\frac{1+\gamma}{\gamma}\right)^\alpha\delta-\left(\frac{1}{\delta\gamma}\right)^\alpha\delta'\end{aligned}$$

for each $k_r\in S$. Choose $\eta=\left(\frac{1+\gamma}{\gamma}\right)^\alpha\delta-\left(\frac{1}{\delta\gamma}\right)^\alpha\delta'$. Therefore,

$$\left\{r\in\mathbb{N}:\frac{1}{h_r^\alpha}\sum_{k\in I_r}P(|X_k-X|\geq\varepsilon)<\eta\right\}\in\mathscr{F}(\mathscr{I}).$$

Hence, we get $X_k\overset{PV_\theta(\mathscr{I})^\alpha}{\to}X$. □

**Theorem 2.7.** *If $\{X_k\}$ is strongly $\mathscr{I}$-Cesàro summable of order $\alpha$ then, it is $\mathscr{I}$-statistical convergent of order $\alpha$ in probability to a random variable $X$.*

*Proof.* Let $X_k \overset{PC_1[\mathscr{I}]^\alpha}{\rightarrow} X$, and $\varepsilon > 0$ given. Then

$$\begin{aligned}\frac{1}{n^\alpha}\sum_{k=1}^{n} P\left(|X_k - X| \geq \varepsilon\right) &\geq \frac{1}{n^\alpha}\sum_{\substack{k=1\\ P(|X_k-X|\geq\varepsilon)}}^{n} P\left(|X_k - X| \geq \varepsilon\right)\\ &\geq \frac{\delta}{n^\alpha}.\left|\left\{k \leq n : P\left(|X_k - X| \geq \varepsilon\right) \geq \delta\right\}\right|\end{aligned}$$

and so

$$\frac{1}{\delta.n^\alpha}\sum_{k=1}^{n} P\left(|X_k - X| \geq \varepsilon\right) \geq \frac{1}{n^\alpha}\left|\left\{k \leq n : P\left(|X_k - X| \geq \varepsilon\right) \geq \delta\right\}\right|.$$

So for a given $\tau > 0$,

$$\begin{aligned}&\left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\left|\left\{k \leq n : P\left(|X_k - X| \geq \varepsilon\right) \geq \delta\right\}\right| \geq \tau\right\}\\ &\subseteq \left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\sum_{k=1}^{n} P\left(|X_k - X| \geq \varepsilon\right) \geq \delta.\tau\right\} \in \mathscr{I}.\end{aligned}$$

Therefore, $X_k \overset{PS(\mathscr{I})^\alpha}{\rightarrow} X$. □

**Theorem 2.8.** *Let a bounded $\{X_k\}$ is $\mathscr{I}$-statistical convergent of order $\alpha$ to $X$. Hence, it is strongly $\mathscr{I}$-Cesàro summable of order $\alpha$ to $X$.*

*Proof.* Assume that $\{X_k\}$ is bounded and $X_k \overset{PS(\mathscr{I})^\alpha}{\rightarrow} X$. Since $\{X_k\}$ is bounded, we get $P\left(|X_k - X| > \varepsilon\right) \leq M$ for all $k$. For $\varepsilon > 0$, we have

$$\begin{aligned}\frac{1}{n^\alpha}\sum_{k=1}^{n} P\left(|X_k - X| \geq \varepsilon\right) &= \frac{1}{n^\alpha}\sum_{\substack{k=1\\ P(|X_k-X|\geq\varepsilon)\geq\delta}}^{n} P\left(|X_k - X| \geq \varepsilon\right)\\ &+ \frac{1}{n^\alpha}\sum_{\substack{k=1\\ P(|X_k-X|\geq\varepsilon)<\delta}}^{n} P\left(|X_k - X| \geq \varepsilon\right)\\ &\leq \frac{1}{n^\alpha}M\left|\left\{k \leq n : P\left(|X_k - X| \geq \varepsilon\right) \geq \delta\right\}\right|\\ &\quad+\frac{1}{n^\alpha}n^\alpha\delta\end{aligned}$$

Then for any $\gamma > 0$,

$$\begin{aligned}&\left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\sum_{k=1}^{n} P\left(|X_k - X| \geq \varepsilon\right) \geq \gamma\right\}\\ &\subseteq \left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\left|\left\{k \leq n : P\left(|X_k - X| \geq \varepsilon\right) \geq \delta\right\}\right| \geq \frac{\gamma}{M}\right\} \in \mathscr{I}.\end{aligned}$$

Therefore $X_k \overset{PC_1[\mathscr{I}]^\alpha}{\rightarrow} X$. □

**Theorem 2.9.** *For $\theta = \{k_r\}$,*

$(i)$ If $\{X_k\} \overset{PV_\theta(\mathscr{I})^\alpha}{\rightarrow} X$ then $\{X_k\} \overset{PS_\theta(\mathscr{I})^\alpha}{\rightarrow} X$, and
$(ii)$ $PV_\theta\left(\mathscr{I}\right)^\alpha$ is proper subset of $PS_\theta\left(\mathscr{I}\right)^\alpha$.

*Proof.* $(i)$ Let $\varepsilon, \delta > 0$ and $\{X_k\} \overset{PV_\theta(\mathscr{I})^\alpha}{\rightarrow} X$. Then, we can write

$$\begin{aligned}\frac{1}{h_r^\alpha}\sum_{k\in I_r} P\left(|X_k - X| \geq \varepsilon\right) &\geq \frac{1}{h_r^\alpha}\sum_{\substack{k\in I_r\\ P(|X_k-X|\geq\varepsilon)\geq\delta}} P\left(|X_k - X| \geq \varepsilon\right)\\ &\geq \frac{\delta}{h_r^\alpha}.\left|\left\{k \in I_r : P\left(|X_k - X| \geq \varepsilon\right) \geq \delta\right\}\right|.\end{aligned}$$

Therefore

$$\frac{1}{\delta h_r^\alpha}\sum_{k\in I_r} P\left(|X_k - X| \geq \varepsilon\right) \geq \frac{1}{h_r^\alpha}.\left|\left\{k \in I_r : P\left(|X_k - X| \geq \varepsilon\right) \geq \delta\right\}\right|.$$

which implies that for any $\gamma > 0$,

$$\begin{aligned}&\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\left|\left\{k \in I_r : P\left(|X_k - X| \geq \varepsilon\right) \geq \delta\right\}\right| \geq \gamma\right\}\\ &\subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\sum_{k\in I_r} P\left(|X_k - X| \geq \varepsilon\right) \geq \delta\gamma\right\} \in \mathscr{I}.\end{aligned}$$

Hence we get $X_k\} \overset{PS_\theta(\mathscr{I})^\alpha}{\rightarrow} X$.

$(ii)$ Let $\{X_k\}$ be defined by

$$X_k = \begin{cases} \{-1,1\} & , \text{ with probability } \frac{1}{2}, \text{ if } n \text{ is the first } \left[\sqrt{h_r^\alpha}\right] \text{ integers in the interval } I_r, \\ \{0,1\} & , \text{ with probability } P(X_n = 0) = \left(1-\frac{1}{n}\right) \text{ and } P(X_n = 1) = \frac{1}{n}, \\ & \text{if } n \text{ is other than the first} \\ \left[\sqrt{h_r^\alpha}\right] & \text{integers in the interval } I_r. \end{cases}$$

Let $0 < \varepsilon < 1$ and $\delta < 1$. Then, we obtain

$$P(|X_k - 0| \geq \varepsilon) = \begin{cases} 1 & , \text{ if } n \text{ is the first } \left[\sqrt{h_r^\alpha}\right] \text{ integers in the interval } I_r, \\ \frac{1}{n} & \text{if } n \text{ is other than the first } \left[\sqrt{h_r^\alpha}\right] \text{ integers in the interval } I_r. \end{cases}$$

Now

$$\frac{1}{h_r^\alpha} |\{k \in I_r : P(|X_k - 0| \geq \varepsilon) \geq \delta\}| \leq \frac{\left[\sqrt{h_r^\alpha}\right]}{h_r^\alpha}$$

and for any $\gamma > 0$ we get

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} |\{k \in I_r : P(|X_k - 0| \geq \varepsilon) \geq \delta\}| \geq \gamma\right\} \subseteq \left\{r \in \mathbb{N} : \frac{\left[\sqrt{h_r^\alpha}\right]}{h_r^\alpha} \geq \gamma\right\}.$$

Since the set

$$\left\{r \in \mathbb{N} : \frac{\left[\sqrt{h_r^\alpha}\right]}{h_r^\alpha} \geq \gamma\right\}$$

is finite and so belongs to $\mathscr{I}$, therefore, we obtain

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} |\{k \in I_r : P(|X_k - 0| \geq \varepsilon) \geq \delta\}| \geq \gamma\right\} \in \mathscr{I}$$

which means that $X_k \overset{PS_\theta(\mathscr{I})^\alpha}{\rightarrow} 0$. Also,

$$\frac{1}{h_r^\alpha} \sum_{k \in I_r} P(|X_k - 0| \geq \varepsilon) = \frac{1}{h_r^\alpha} . \frac{\left[\sqrt{h_r^\alpha}\right]\left(\left[\sqrt{h_r^\alpha}\right]+1\right)}{2},$$

then

$$\begin{aligned} \left\{r \in \mathbb{N} : \tfrac{1}{h_r} \sum_{k \in I_r} P(|X_k - 0| \geq \varepsilon) \geq \tfrac{1}{4}\right\} &= \left\{r \in \mathbb{N} : \tfrac{\left[\sqrt{h_r^\alpha}\right]\left(\left[\sqrt{h_r^\alpha}\right]+1\right)}{h_r} \geq \tfrac{1}{2}\right\} \\ &= \{m, m+1, m+2, ...\} \in \mathscr{F}(\mathscr{I}) \end{aligned}$$

for some $m \in \mathbb{N}$. Hence, $X_k \overset{PS_\theta(\mathscr{I})^\alpha}{\nrightarrow} 0$. □

**Theorem 2.10.** *$\mathscr{I}$-statistical convergence in probability of order $\alpha$ implies $\mathscr{I}$-lacunary statistical convergence in probability of order $\alpha$* $\liminf_r q_r > 1$.

*Proof.* By assumption $\liminf_r q_r > 1$, then there exists a $\sigma > 0$ such that $q_r \geq 1 + \sigma$ for sufficiently large $r$, that is,

$$\frac{h_r}{k_r} \geq \frac{\sigma}{1+\sigma} \Rightarrow \frac{1}{h_r^\alpha} \leq \frac{1}{k_r^\alpha}\left(\frac{1+\sigma}{\sigma}\right)^\alpha$$

If $\{X_k\} \overset{PS(\mathscr{I})^\alpha}{\rightarrow} X$, then for $\varepsilon > 0$ and for $r > 0$, we have

$$\frac{1}{h_r^\alpha} |\{k \in I_r : P(|X_k - X| \geq \varepsilon) \geq \delta\}| \leq \frac{1}{k_r^\alpha}\left(\frac{1+\sigma}{\sigma}\right)^\alpha |\{k \leq k_r : P(|X_k - X| \geq \varepsilon) \geq \delta\}|$$

Then for any $\gamma > 0$, we get

$$\begin{aligned} &\left\{r \in \mathbb{N} : \tfrac{1}{h_r^\alpha} |\{k \in I_r : P(|X_k - X| \geq \varepsilon)\} \geq \delta| \geq \gamma\right\} \\ &\subseteq \left\{r \in \mathbb{N} : \frac{1}{k_r^\alpha} |\{k \leq k_r : P(|X_k - X| \geq \varepsilon)\} \geq \delta| \geq \tfrac{\gamma\sigma^\alpha}{(1+\sigma)^\alpha}\right\} \in \mathscr{I}. \end{aligned}$$

□

**Theorem 2.11.** *$\mathscr{I}$-lacunary statistical convergence in probability of order $\alpha$ implies $\mathscr{I}$-statistical convergence in probability of order $\alpha$, $0<\alpha<1$, if* $\sup_r \sum_{i=0}^{r-1} \frac{h_{i+1}^{\alpha}}{(k_{r-1})^{\alpha}} = B < \infty$.

*Proof.* Suppose that $\{X_k\} \overset{PS_\theta(\mathscr{I})^\alpha}{\to} X$, and for $\varepsilon$, $\delta$, $\gamma_1$, $\gamma_2 > 0$ define the sets

$$C=\left\{r\in\mathbb{N}:\frac{1}{h_r^{\alpha}}\left|\{k\in I_r:P(|X_k-X|\geq\varepsilon)\geq\delta\}\right|<\gamma_1\right\}$$

and

$$T=\left\{n\in\mathbb{N}:\frac{1}{n^{\alpha}}\left|\{k\leq n:P(|X_k-X|\geq\varepsilon)\geq\delta\}\right|<\gamma_2\right\}.$$

From our assumption we get $C\in\mathscr{F}(\mathscr{I})$. Further observe that

$$K_j=\frac{1}{h_j^{\alpha}}\left|\left\{k\in I_j:P(|X_k-X|\geq\varepsilon)\geq\delta\right\}\right|<\gamma_1$$

for all $j\in C$. Let $n\in\mathbb{N}$ be such that $k_{r-1}<n\leq k_r$ for some $r\in C$. Hence, we obtain

$$\begin{aligned}
&\frac{1}{n^{\alpha}}\left|\{k\leq n:P(|X_k-X|\geq\varepsilon)\geq\delta\}\right|\\
&\leq\frac{1}{k_{r-1}^{\alpha}}\left|\{k\leq k_r:P(|X_k-X|\geq\varepsilon)\geq\delta\}\right|\\
&=\frac{1}{k_{r-1}^{\alpha}}\left|\{k\in I_1:P(|X_k-X|\geq\varepsilon)\geq\delta\}\right|\\
&+\frac{1}{k_{r-1}^{\alpha}}\left|\{k\in I_2:P(|X_k-X|\geq\varepsilon)\geq\delta\}\right|\\
&+\ldots+\frac{1}{k_{r-1}^{\alpha}}\left|\{k\in I_r:P(|X_k-X|\geq\varepsilon)\geq\delta\}\right|\\
&=\frac{k_1^{\alpha}}{k_{r-1}^{\alpha}}\frac{1}{h_1^{\alpha}}\left|\{k\in I_1:P(|X_k-X|\geq\varepsilon)\geq\delta\}\right|\\
&+\frac{(k_2-k_1)^{\alpha}}{k_{r-1}^{\alpha}}\frac{1}{h_2^{\alpha}}\left|\{k\in I_2:P(|X_k-X|\geq\varepsilon)\geq\delta\}\right|\\
&+\ldots+\frac{(k_r-k_{r-1})^{\alpha}}{k_{r-1}^{\alpha}}\frac{1}{h_r^{\alpha}}\left|\{k\in I_r:P(|X_k-X|\geq\varepsilon)\geq\delta\}\right|\\
&=\frac{k_1^{\alpha}}{k_{r-1}^{\alpha}}K_1+\frac{(k_2-k_1)^{\alpha}}{k_{r-1}^{\alpha}}K_2+\ldots+\frac{(k_r-k_{r-1})^{\alpha}}{k_{r-1}^{\alpha}}K_r\\
&\leq\left\{\sup\nolimits_{j\in C}K_j\right\}\sup\nolimits_r\sum_{i=0}^{r-1}\frac{h_{i+1}^{\alpha}}{(k_{r-1})^{\alpha}}\\
&<\gamma_1 B.
\end{aligned}$$

Choosing $\gamma_2=\dfrac{\gamma_1}{B}$ and by $\bigcup\{n:k_{r-1}<n\leq k_r,\, r\in C\}\subset T$ where $C\in\mathscr{F}(\mathscr{I})$ Then the set $T$ belongs to $\mathscr{F}(\mathscr{I})$ and this completes the proof. □

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