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PSEUDO PROJECTIVE CURVATURE TENSOR SATISFYING SOME PROPERTIES ON A NORMAL PARACONTACT METRIC MANIFOLD

ÜMIT YILDIRIM, MEHMET ATÇEKEN, AND SÜLEYMAN DIRIK

Abstract. In the present paper we have studied the curvature tensor of a normal paracontact metric manifold satisfying the conditions $R(\xi, X)\widetilde{P} = 0$, $\widetilde{P}(\xi, X)R = 0$, $\widetilde{P}(\xi, X)\widetilde{P} = 0$, $\widetilde{P}(\xi, X)S = 0$, $\widetilde{P}(\xi, X)\widetilde{Z} = 0$ and pseudo projective flatness, where R, $\widetilde{P}$, S and $\widetilde{Z}$ denote the Riemannian curvature, pseudo projective curvature, Ricci and concircular curvature tensors, respectively.

1. Introduction

The study of paracontact geometry was initiated by Kenayuki and Williams [10]. Zamkovoy studied paracontact metric manifolds and their subclasses [11]. Recently, Welyczko studied curvature and torsion of Frenet Legendre curves in 3-dimensional normal almost paracontact metric manifolds [5]. In the recent years, (para) contact metric manifolds and their curvature properties have been studied by many authors. [6, 9]

In [7, 8], we studied the curvature tensors satisfying some conditions on a $C(\alpha)$-manifold and induced cases were discussed.

In 2002, Prasad [3] defined pseudo projective curvature tensor $\widetilde{P}$ on a Riemannian manifold (M^n, g) $(n > 2)$ of type $(1, 3)$ as follows

$$\begin{aligned}\widetilde{P}(X,Y)Z &= aR(X,Y)Z + b\big[S(Y,Z)X - S(X,Z)Y\big] \\ &- \frac{r}{n}\big[\frac{a}{n-1} + b\big]\big[g(Y,Z)X - g(X,Z)Y\big], \qquad (1)\end{aligned}$$

where R is the Riemann curvature, S is the Ricci tensor, respectively, and a, b are constants such that $a, b \neq 0$. If $a = 1$ and $b = -\frac{1}{n-1}$, then (1) takes the form

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$$\begin{aligned}\widetilde{P}(X,Y)Z &= R(X,Y)Z - \frac{1}{n-1}\big[S(Y,Z)X - S(X,Z)Y\big],\\ &= P(X,Y)Z \qquad (2)\end{aligned}$$

where P is the projective curvature tensor [9]. Hence the projective curvature tensor P can be seen as a particular case of the tensor $\widetilde{P}$.

Narain et. al. studied pseudo projective curvature tensor in Lorentzian para-Sasakian manifolds[4].

Let M be n-dimensional Riemannian manifold. Then the concircular curvature tensor field is defined by

$$\widetilde{Z}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}\big[g(Y,Z)X - g(X,Z)Y\big], \qquad (3)$$

for any $X, Y, Z \in \chi(M)$.

2. Preliminaries

An $n-$dimensional differentiable manifold (M, g) is said to be an almost para-contact metric manifold if there exist on M a $(1,1)$ tensor field ϕ, a contravariant vector ξ and a 1-form η-such that

$$\phi^2 X = X - \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1 \qquad (4)$$

and

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X,\xi), \qquad (5)$$

for any $X, Y \in \chi(M)$.

If in addition to the above relations, we have

$$(\nabla_X \phi)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \qquad (6)$$

then M is called a normal paracontact metric manifold, where ∇ is Levi-Civita connection.
We have also on a normal paracontact metric manifold M

$$\phi X = \nabla_X \xi, \qquad (7)$$

for any $X \in \chi(M)$.

Moreover, if such a manifold has constant sectional curvature equal to c, then the Riemannian curvature tensor is given by

$$\begin{aligned} R(X,Y)Z &= \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ &+ \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\}, \end{aligned} \tag{8}$$

for any vector fields $X, Y, Z \in \chi(M)$.

In a normal paracontact metric manifold by direct calculations, we can easily to see that

$$S(X,Y) = \Big(\frac{c(n-5)+3n+1}{4}\Big)g(X,Y) + \Big(\frac{(c-1)(5-n)}{4}\Big)\eta(X)\eta(Y), \tag{9}$$

and

$$QX = \Big(\frac{c(n-5)+3n+1}{4}\Big)X + \Big(\frac{(c-1)(5-n)}{4}\Big)\eta(X)\xi, \tag{10}$$

for any $X, Y \in \chi(M)$, where Q is the Ricci operator of M such that $g(QX,Y) = S(X,Y)$. Thus we have the following statement.

Corollary 2.1. *A normal paracontact metric manifold is always an η-Einstein manifold.*

From (9), we can easily see

$$S(X,\xi) = (n-1)\eta(X), \tag{11}$$

$$Q\xi = (n-1)\xi \tag{12}$$

and

$$r = \frac{n-1}{4}[c(n-5)+3n+5]. \tag{13}$$

Let M be an n-dimensional normal paracontact metric manifold and we denote the Riemannian curvature tensor of M by R, then we have from (8), for $X = \xi$

$$R(\xi,Y)Z = g(Y,Z)\xi - \eta(Z)Y, \tag{14}$$

for $Z = \xi$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y. \tag{15}$$

In (15) choosing $Y = \xi$, we get

$$R(X,\xi)\xi = X - \eta(X)\xi. \tag{16}$$

Taking the inner product both of the sides (8) with $\xi \in \chi(M)$, we obtain

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y). \tag{17}$$

In the same way we obtain from (1) and (3),

$$\widetilde{P}(X,Y)\xi = \big[a + b(n-1) - \frac{r}{n}\big[\frac{a}{n-1} + b\big]\big]\big[\eta(Y)X - \eta(X)Y\big], \tag{18}$$

$$\begin{aligned}\widetilde{P}(\xi,Y)Z &= \big[a - \frac{r}{n}\big[\frac{a}{n-1} + b\big]\big]\big[g(Y,Z)\xi - \eta(Z)Y\big] \\ &+ b\big[S(Y,Z)\xi - (n-1)\eta(Z)Y\big],\end{aligned} \tag{19}$$

$$\widetilde{Z}(\xi,Y)Z = \big[1 - \frac{r}{n(n-1)}\big]\big[g(Y,Z)\xi - \eta(Z)Y\big], \tag{20}$$

and

$$\widetilde{Z}(\xi,Y)\xi = \big[1 - \frac{r}{n(n-1)}\big]\big[\eta(Y)\xi - Y\big], \tag{21}$$

for any $X, Y, Z \in \chi(M)$.

3. Pseudo Projective Curvature Tensor of a Normal Paracontact Metric Manifold

Theorem 3.1. *Let $M(c)$ be an $n-$dimensional normal paracontact metric space form. Then $M(c)$ is pseudo projective semi-symmetric if and only if either $M(c)$ reduces an Einstein manifold or pseudo projective curvature tensor $\widetilde{P}$ reduces projective curvature tensor.*

Proof: Suppose that $n-$dimensional normal paracontact metric manifold $M(c)$ is pseudo projective semi symmetric. Then we have

$$R(X,Y)\widetilde{P} = 0, \tag{22}$$

for any $X, Y \in \chi(M)$. (22) implies that

$$\begin{aligned}(R(X,Y)\widetilde{P})(Z,U,W) &= R(X,Y)\widetilde{P}(Z,U)W - \widetilde{P}(R(X,Y)Z,U)W \\ &- \widetilde{P}(Z,R(X,Y)U)W - \widetilde{P}(Z,U)R(X,Y)W \\ &= 0,\end{aligned} \tag{23}$$

for any $U, Z, W \in \chi(M)$. Substituting $X = \xi$ in (23), we have

$$\begin{aligned}0 &= R(\xi,Y)\widetilde{P}(Z,U)W - \widetilde{P}(R(\xi,Y)Z,U)W \\ &- \widetilde{P}(Z,R(\xi,Y)U)W - \widetilde{P}(Z,U)R(\xi,Y)W.\end{aligned} \tag{24}$$

Using (14) in (24), we obtain

$$\begin{aligned}0 &= g(Y,\widetilde{P}(Z,U)W)\xi - \eta(\widetilde{P}(Z,U)W)Y \\ &- g(Y,Z)\widetilde{P}(\xi,U)W + \eta(Z)\widetilde{P}(Y,U)W \\ &- g(Y,U)\widetilde{P}(Z,\xi)W + \eta(U)\widetilde{P}(Z,Y)W \\ &- g(Y,W)\widetilde{P}(Z,U)\xi + \eta(W)\widetilde{P}(Z,U)Y.\end{aligned} \tag{25}$$

Using (18) and (19) in (25), choosing $Z=\xi$ in (25) it follows that

$$\begin{aligned} 0 &= \big[a-\frac{r}{n}\big[\frac{a}{n-1}+b\big]\big]\big[g(Y,W)\eta(U)\xi-g(U,W)Y\big] \\ &+ \big[a+b(n-1)-\frac{r}{n}\big[\frac{a}{n-1}+b\big]\big]\big[g(Y,W)U-g(Y,W)\eta(U)\xi\big] \\ &+ b\big[S(U,Y)\eta(W)\xi+S(Y,W)\eta(U)\xi-(n-1)g(Y,U)\eta(W)\xi-S(U,W)Y\big] \\ &+ \widetilde{P}(Y,U)W \end{aligned} \tag{26}$$

By choosing $W=\xi$ and taking the inner product on both sides of (26) with $\xi\in\chi(M)$, we find

$$b\big[S(U,Y)-(n-1)g(Y,U)\big]=0. \tag{27}$$

This proves our assertion.

Theorem 3.2. *Let $M(c)$ be an $n-$dimensional normal paracontact metric space form. Then $\widetilde{P}(\xi,Y)R=0$ if and only if either $M(c)$ reduces an Einstein manifold or pseudo projective curvature tensor $\widetilde{P}$ reduces concircular curvature tensor.*

Proof: Suppose that $\widetilde{P}(\xi,Y)R=0$, then we have

$$\begin{aligned} 0 &= \widetilde{P}(\xi,Y)R(Z,U)W-R(\widetilde{P}(\xi,Y)Z,U)W \\ &- R(Z,\widetilde{P}(\xi,Y)U)W-R(Z,U)\widetilde{P}(\xi,Y)W, \end{aligned} \tag{28}$$

for any $Y,U,Z,W\in\chi(M)$. Using (19) in (28), choosing $Z=\xi$, we obtain

$$\begin{aligned} 0 &= \big[a-\frac{r}{n}\big[\frac{a}{n-1}+b\big]\big]\big[g(Y,R(\xi,U)W)\xi-\eta(R(\xi,U)W)Y \\ &- \eta(Y)R(\xi,U)W+R(Y,U)W+\eta(U)R(\xi,Y)W+\eta(W)R(\xi,U)Y\big] \\ &+ b\big[S(Y,R(\xi,U)W)\xi-(n-1)\eta(R(\xi,U)W)Y-(n-1)\eta(Y)R(\xi,U)W \\ &+ (n-1)R(Y,U)W+(n-1)\eta(U)R(\xi,Y)W-S(Y,W)R(\xi,U)\xi \\ &+ (n-1)\eta(W)R(\xi,U)Y\big] \end{aligned} \tag{29}$$

In (29) using (14) and (15), choosing $W=\xi$, we find

$$-bS(Y,U)\xi+b(n-1)g(Y,U)\xi=0. \tag{30}$$

Taking the inner product on both sides of (30) with $\xi\in\chi(M)$, we obtain

$$b\big[S(Y,U)-(n-1)g(Y,U)\big]=0. \tag{31}$$

The proof is completed.

Theorem 3.3. *Let $M(c)$ be an $n-$dimensional normal paracontact metric space form. Then, $\widetilde{P}(\xi,Y)\widetilde{P}$ is always identically zero, for any $Y\in\chi(M)$.*

Proof: Let $M(c)$ be $n-$ dimensional a normal paracontact metric space form. Then, we have

$$\begin{aligned}(\widetilde{P}(\xi,Y)\widetilde{P})(U,W,Z) &= \widetilde{P}(\xi,Y)\widetilde{P}(U,W)Z-\widetilde{P}(\widetilde{P}(\xi,Y)U,W)Z\\ &- \widetilde{P}(U,\widetilde{P}(\xi,Y)W)Z-\widetilde{P}(U,W)\widetilde{P}(\xi,Y)Z \end{aligned}\tag{32}$$

for any $Y,U,W,Z\in\chi(M)$. Using (19) in (32), we obtain

$$\begin{aligned}(\widetilde{P}(\xi,Y)\widetilde{P})(U,W,Z) &= \big[a-\frac{r}{n}\big[\frac{a}{n-1}+b\big]\big]\big[g(Y,\widetilde{P}(U,W)Z)\xi-\eta(\widetilde{P}(U,W)Z)Y\\ &- g(Y,U)\widetilde{P}(\xi,W)Z+\eta(U)\widetilde{P}(Y,W)Z-g(Y,W)\widetilde{P}(\xi,U)Z\\ &+ \eta(W)\widetilde{P}(U,Y)Z-g(Y,Z)\widetilde{P}(U,W)\xi+\eta(Z)\widetilde{P}(U,W)Y\big]\\ &+ b\big[S(Y,\widetilde{P}(U,W)Z)\xi-(n-1)\eta(\widetilde{P}(U,W)Z)Y\\ &- S(Y,U)\widetilde{P}(\xi,W)Z+(n-1)\eta(U)\widetilde{P}(Y,W)Z\\ &- S(Y,W)\widetilde{P}(U,\xi)Z+(n-1)\eta(W)\widetilde{P}(U,Y)Z\\ &- S(Y,Z)\widetilde{P}(U,W)\xi+(n-1)\eta(Z)\widetilde{P}(U,W)Y\big].\end{aligned}\tag{33}$$

Substituting $U=\xi$ and using (18) and (19) in (33), we obtain

$$\begin{aligned}(\widetilde{P}(\xi,Y)\widetilde{P})(U,W,Z) &= \big[a-\frac{r}{n}\big[\frac{a}{n-1}+b\big]\big]^2\big[g(Y,Z)W-g(W,Z)Y\big]\\ &+ \big[a-\frac{r}{n}\big[\frac{a}{n-1}+b\big]\big]\widetilde{P}(Y,W)Z+b(n-1)\widetilde{P}(Y,W)Z\\ &+ \big[a-\frac{r}{n}\big[\frac{a}{n-1}+b\big]\big]\big[S(W,Z)Y-(n-1)g(W,Z)Y\\ &+ S(Y,Z)W+(n-1)g(Y,Z)W\big]\\ &+ b^2(n-1)\big[S(Y,Z)W-S(W,Z)Y\big].\end{aligned}\tag{34}$$

In(34), choosing $Z=\xi$, we obtain

$$\widetilde{P}(\xi,Y)\widetilde{P}=0.$$

This proves our assertion.

Theorem 3.4. *Let $M(c)$ be an $n-$dimensional normal paracontact metric space form. Then $\widetilde{P}(\xi,Y)S=0$ if and only if $M(c)$ either reduces an Einstein manifold or the scalar curvature*

$$r=\frac{an(n-1)}{a+b(n-1)},$$

provided that $(a+b(n-1))\neq 0$.

Proof: Assume that $\widetilde{P}(\xi,Y)S=0$. This implies that

$$S(\widetilde{P}(\xi,Y)Z,W)+S(Z,\widetilde{P}(\xi,Y)W)=0,\tag{35}$$

for any $Y, Z, W \in \chi(M)$. In (35) using (19), we obtain

$$\begin{aligned} 0 &= \big[a - \frac{r}{n}\big[\frac{a}{n-1} + b\big]\big]\big[g(Y,Z)S(\xi,W) - \eta(Z)S(Y,W) \\ &+ g(Y,W)S(\xi,Z) - \eta(W)S(Y,Z)\big] \\ &+ b\big[S(Y,Z)S(\xi,W) - (n-1)\eta(Z)S(\xi,W) \\ &+ S(Y,W)S(\xi,Z) - (n-1)\eta(W)S(Y,Z)\big]. \end{aligned} \tag{36}$$

Substituting $Z = \xi$ and using (11) in (36), we can infer

$$\big[a - \frac{r}{n}\big[\frac{a}{n-1} + b\big]\big]\big[S(Y,W) - (n-1)g(Y,W)\big] = 0.$$

So either $M(c)$ reduces an Einstein manifold or the scalar curvature

$$r = \frac{an(n-1)}{a + b(n-1)}.$$

On the other hand, if $a + b(n-1) = 0$ then, one can easily to see that the pseudo projective curvature tensor reduces projective curvature tensor.

Theorem 3.5. *Let $M(c)$ be an $n-$dimensional normal paracontact metric space form. Then $\widetilde{P}(\xi, Y)\widetilde{Z} = 0$ if and only if $M(c)$ satisfies one of the least following conditions*

i) $M(c)$ is an Einstein Manifold,
ii) $\widetilde{P}$ pseudo projective curvature tensor reduces the concircular curvature tensor,
iii) The scalar curvature r of $M(c)$ is $r = n(n-1)$.

Proof: Suppose that $\widetilde{P}(\xi, Y)\widetilde{Z} = 0$, then we have

$$\begin{aligned} (\widetilde{P}(\xi,Y)\widetilde{Z})(U,W,Z) &= \widetilde{P}(\xi,Y)\widetilde{Z}(U,W)Z - \widetilde{Z}(\widetilde{P}(\xi,Y)U,W)Z \\ &- \widetilde{Z}(U,\widetilde{P}(\xi,Y)W)Z - \widetilde{Z}(U,W)\widetilde{P}(\xi,Y)Z \\ &= 0, \end{aligned} \tag{37}$$

for any $Y, U, W, Z \in \chi(M)$. Using (20) in (37), we obtain

$$\begin{aligned} 0 &= \big[a - \frac{r}{n}\big[\frac{a}{n-1} + b\big]\big]\big[g(Y,\widetilde{Z}(U,W)Z)\xi - \eta(\widetilde{Z}(U,W)Z)Y \\ &- g(Y,U)\widetilde{Z}(\xi,W)Z + \eta(U)\widetilde{Z}(Y,W)Z - g(Y,W)\widetilde{Z}(\xi,U)Z \\ &+ \eta(W)\widetilde{Z}(U,Y)Z - g(Y,Z)\widetilde{Z}(U,W)\xi + \eta(Z)\widetilde{Z}(U,W)Y\big] \\ &+ b\big[S(Y,\widetilde{Z}(U,W)Z)\xi - (n-1)\eta(\widetilde{Z}(U,W)Z)Y \\ &- S(Y,U)\widetilde{Z}(\xi,W)Z + (n-1)\eta(U)\widetilde{Z}(Y,W)Z \\ &- S(Y,W)\widetilde{Z}(U,\xi)Z + (n-1)\eta(W)\widetilde{Z}(U,Y)Z \\ &- S(Y,Z)\widetilde{Z}(U,W)\xi + (n-1)\eta(Z)\widetilde{Z}(U,W)Y\big]. \end{aligned} \tag{38}$$

In (38), using (20) and (21) and substituting $U = Z = \xi$, we have

$$b\big[1 - \frac{r}{n(n-1)}\big]\big[S(Y,W) - (n-1)g(Y,W)\big] = 0.$$

This proves our assertion.

Definition 3.1. *An $n-$dimensional normal paracontact metric manifold M is called pseudo projective flat if the condition*

$$\widetilde{P}(X,Y)Z = 0$$

holds on $M(c)$.

Let us consider the space form $M(c)$ under consideration is pseudo projective flat, then we have from Definition 3.1. and relation (1)

$$aR(X,Y)Z = \big[S(X,Z)Y - S(Y,Z)X\big] + \frac{r}{n}\big[\frac{a}{n-1} + b\big]\big[g(Y,Z)X - g(X,Z)Y\big]. \quad (39)$$

In (39), substituting $Z = \xi$ and using (11) and (15), we have

$$a\big[\eta(Y)X - \eta(X)Y\big] = b(n-1)\big[\eta(X)Y - \eta(Y)X\big] + \frac{r}{n}\big[\frac{a}{n-1} + b\big]\big[\eta(Y)X - \eta(X)Y\big]. \quad (40)$$

Taking the inner product on both sides of (40) with $\xi \in \chi(M)$, we obtain

$$r = \frac{n(n-1)\big[a + b(n-1)\big]}{a + b(n-1)}. \quad (41)$$

This leads to the following statement:

Theorem 3.6. *An $n-$dimensional $(n \geq 3)$ normal paracontact metric manifold is pseudo projective flat if and only if the scalar curvature of $M(c)$ is given by*

$$r = \frac{n(n-1)\big[a + b(n-1)\big]}{a + b(n-1)} \quad (42)$$

provided that $(a + b(n-1)) \neq 0$.

Example 3.7. *Let us consider a 7-dimensional manifold $M^7 = \{(x_1, x_2, x_3, y_1, y_2, y_3, z) \in R^7\}$, where $(x_1, x_2, x_3, y_1, y_2, y_3, z)$ are standard coordinates in $\in R^7$. Taking the vector fields*

$$e_i = e^z \frac{\partial}{\partial x_i}, \qquad e_j = e^z \frac{\partial}{\partial y_i}, \qquad 1 \leq i, j \leq 3, \qquad e_7 = \frac{\partial}{\partial z},$$

which are linearly independent at each point of M. Let g be the Riemannian metric ı-on M defined by

$$g = e^{-2z} \sum_{n=1}^{3} \{dx_i \otimes dy_i + dy_i \otimes dy_i\} + dz + dz.$$

We note that $g(e_i, e_j) = \delta_{ij}$. Thus the set e_i, $1 \leq i, j \leq 7$, is an orthonormal basis of M. Let

$$X = \sum_{i=1}^{3}(X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}) + Z \frac{\partial}{\partial z}$$

be a vector field on M. We define the almost paracontact structure ϕ and 1-form η as

$$\phi X = \sum_{i=1}^{3}(-X_i \frac{\partial}{\partial x_i} - Y_i \frac{\partial}{\partial y_i}) \text{ and } \eta(X) = g(X, e_7). \tag{43}$$

Thus we have

$$\phi e_i = -e_i, \quad \phi e_7 = 0, \quad 1 \leq i \leq 6. \tag{44}$$

It is easy to see that $\phi^2 X = X - \eta(X)e_7$, $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$, and $\eta(e_7) = 1$, for any $X, Y \in \Gamma(TM)$. Thus $(\phi, \xi = e_7, \eta, g)$ is an almost paracontact metric structure on M. By direct calculations, we have

$$[e_i, e_7] = -e_i, \quad 1 \leq i \leq 6, \quad [e_i, e_j] = 0, \quad 1 \leq j \leq 6.$$

By using Kozsul formula, we can easily to find that

$$\nabla_{e_i} e_i = e_7, \quad \nabla_{e_i} e_j = 0, \quad i \neq j, 1 \leq i, j \leq 6.$$

$$\nabla_{e_i} e_7 = \phi e_i = -e_i, \quad \nabla_{e_7} e_7 = 0, \quad \nabla_{e_7} e_i = 0, \quad 1 \leq i \leq 6.$$

Using the Kozsul's formula, we get

$$(\nabla_X \phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi \tag{45}$$

for any $X, Y \in \Gamma(TM)$. Thus $M^n(\phi, \xi, \eta, g)$ is a normal paracontact metric manifold. By R we denote the Riemannian curvature tensor of M, it can be easily too seen that

$$R(e_i, e_j)e_j = -e_i, \quad 1 \leq i \neq j \leq 7, R(e_i, e_j)e_k = 0, \quad 1 \leq i, j, k \leq 6, \quad i \neq j \neq k. \tag{46}$$

Let $X = X_i e_i, Y = Y_j e_j$ and $Z = Z_k e_k,$, $1 \leq i, j, k \leq n$, be vector fields on M. By using the properties of R, we get

$$\begin{aligned} R(X, Y)Z &= X_i Y_j Z_k R(e_i, e_j)e_k = Y_j Z_j X_i R(e_i e_j e_j) + X_i Y_j Z_i R(e_i, e_j)e_i \\ &= Y_j Z_j X_i e_i + X_i Z_i Y_j e_j = -\{g(Y, Z)X - g(X, Z)Y\} \end{aligned}$$

that is, M has a constant curvature -1 and

$$S(X, Y) = -(n-1)g(X, Y) = -6g(X, Y), \quad \tau = -42. \tag{47}$$

Conclusion 3.1. *In this paper, the curvature tensors act to each other cases are discussed and normal paracontact metric space form is characterized with respect to these cases.*

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Current address: Ümit Yıldırım :Department of Mathematics, Faculty of Arts and Sciences, Gaziosmanpasa University, Tokat, Turkey

E-mail address: umit.yildirim@gop.edu.tr

ORCID Address: http://orcid.org/0000-0002-7178-4223

Current address: Mehmet Atçeken :Department of Mathematics, Faculty of Arts and Sciences, Gaziosmanpasa University, Tokat, Turkey

E-mail address: mehmet.atceken@gop.edu.tr

ORCID Address: http://orcid.org/0000-0001-8665-5945

Current address: Süleyman Dirik : Department of Statistic, Faculty of Arts and Sciences, Amasya University, Amasya, Turkey

E-mail address: suleyman.dirik@amasya.edu.tr

ORCID Address: http://orcid.org/0000-0001-9093-1607