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FURTHER STUDY ON THE RESULTS OF SHEREMETA

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Abstract. In this paper we estimate some growth rates of composite entire and meromorphic functions in the light of their relative (p, q)-th order and relative (p, q)-th lower order which considerably extend some results of Sheremeta [14].

1. Introduction, Definitions and Notations

Let us consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [8, 10, 16, 17]. We also use the standard notations and definitions of the theory of entire functions which are available in [18] and therefore we do not explain those in details. Let f be an entire function and $M_f(r) = \max\{|f(z)| : |z| = r\}$. Since $M_f(r)$ is strictly increasing and continuous, therefore there exists its inverse function $M_f^{-1} : (|f(0)|, \infty) \to (0, \infty)$ with $\lim\limits_{s\to\infty} M_f^{-1}(s) = \infty$. In this connection the following definition is relevant:

Definition 1.1. {[2]} A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large r, $[M_f(r)]^2 \le M_f(r^\sigma)$ holds. For examples of functions with or without the Property (A), one may see [2].

However, when f is meromorphic, one may introduce another function $T_f(r)$ defined as $T_f(r) = \frac{1}{2\pi}\int\limits_0^{2\pi} \log^+ \left|f(re^{i\theta})\right| d\theta$ known as Nevanlinna's characteristic function of f (see [8, p.4]), playing the same role as $M_f(r)$. Moreover, if f is non-constant entire then $T_f(r)$ is strictly increasing and continuous functions of r. Also its inverse $T_f^{-1} : (T_f(0), \infty) \to (0, \infty)$ exist and is such that $\lim\limits_{s\to\infty} T_f^{-1}(s) = \infty$.

Now for $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ and $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ where $\mathbb{N}$ be the set of all positive integers. We also denote $\log^{[0]} x =$

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x, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$ and $\exp^{[-1]} x = \log x$. Considering this let us recall that Juneja et al. [9] defined the (p,q)-th order (resp. (p,q)-th lower order) of an entire function f for any two positive integers p, q with $p \geq q$ which is as follows:

$$\rho_f(p,q) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \left(\text{resp. } \lambda_f(p,q) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \right).$$

If f is a meromorphic function, then

$$\begin{matrix} \rho_f(p,q) \\ \lambda_f(p,q) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r}.$$

In this connection we recall the following definition due to Juneja et al. [9]:

Definition 1.2. [9] An entire function f is said to have index-pair (p,q), $p \geq q \geq 1$ if $b < \rho_f(p,q) < \infty$ and $\rho_f(p-1, q-1)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ if $p > q$. Moreover if $0 < \rho_f(p,q) < \infty$, then

$$\begin{cases} \rho_f(p-n, q) = \infty & \text{for} \quad n < p, \\ \rho_f(p, q-n) = 0 & \text{for} \quad n < q, \\ \rho_f(p+n, q+n) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases}.$$

Similarly for $0 < \lambda_f(p,q) < \infty$, one can easily verify that

$$\begin{cases} \lambda_f(p-n, q) = \infty & \text{for} \quad n < p, \\ \lambda_f(p, q-n) = 0 & \text{for} \quad n < q, \\ \lambda_f(p+n, q+n) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases}.$$

Analogously one can easily verify that Definition 1.2 of index-pair can also be applicable to a meromorphic function f.

If $p = l$ and $q = 1$ then we write $\rho_f(l,1) = \rho_f^{[l]}$ and $\lambda_f(l,1) = \lambda_f^{[l]}$ where $\rho_f^{[l]}$ and $\lambda_f^{[l]}$ are respectively known as generalized order and generalized lower order of f. For details about generalized order one may see [13]. Also for $p = 2$ and $q = 1$ we respectively denote $\rho_f(2,1)$ and $\lambda_f(2,1)$ by ρ_f and λ_f. which are classical growth indicators such as order and lower order of f. L. Bernal [1, 2] introduced the relative order (resp. relative lower order) between two entire functions to avoid comparing growth just with $\exp z$ which is as follows:

$$\begin{matrix} \rho_g(f) \\ \lambda_g(f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

These definitions coincide with the classical one [15] if $g = \exp z$.

Extending this notion, Lahiri and Banerjee [11] introduced the definition of relative order of a meromorphic function with respect to an entire function in the following way :

Definition 1.3. [11] Let f be any meromorphic function and g be any entire function. The relative order of f with respect to g is defined as

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r\} \\ &= \overline{\lim_{r\to\infty}} \frac{\log T_g^{-1} T_f(r)}{\log r}.\end{aligned}$$

It is known {cf. [11] } that if $g(z) = \exp z$ then Definition 1.3 coincides with the classical definition of order of a meromorphic function f .

In the case of relative order, Sánchez Ruiz et al. [12] gave the definition of relative (p,q)-th order and relative (p,q)-th lower order of an entire function in the light of index-pair which is as follows:

Definition 1.4. [12] Let f and g be any two entire functions with index-pairs (m,q) and (m,p) respectively where $p,q,m \in \mathbb{N}$ such that $m \geq p$ and $m \geq q$. Then the relative (p,q)-th order and relative (p,q)-th lower order of f with respect to g are defined as

$$\rho_g^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r} \text{ and } \lambda_g^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r}.$$

Further, Debnath et al. [6] introduced the definition of relative (p,q)-th order and relative (p,q)-th lower order of a meromorphic function with respect to an entire function in the following manner:

Definition 1.5. [6] Let f be any meromorphic function and g be any entire function with index-pairs (m,q) and (m,p) respectively where $p,q,m \in \mathbb{N}$ such that $m \geq p$ and $m \geq q$. Then the relative (p,q)-th order and relative (p,q)-th lower order of f with respect to g are defined as

$$\rho_g^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_g^{-1} T_f(r)}{\log^{[q]} r} \text{ and } \lambda_g^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} T_g^{-1} T_f(r)}{\log^{[q]} r}.$$

If f and g have got index-pair $(m,1)$ and (m,k) , respectively, then Definition 1.4 and Definition 1.5 reduce to generalized relative order of f with respect to g and in this we write $\rho_g^{(k,1)}(f) = \rho_g^{[k]}(f)$ and $\lambda_g^{(k,1)}(f) = \lambda_g^{[k]}(f)$. If f and g have the same index-pair $(p,1)$ where $p \in \mathbb{N}$, we get the definition of relative order introduced by Bernal [1, 2] and Lahiri et al. [11]. When $g = \exp^{[m-1]} z$, then $\rho_g(f) = \rho_f^{[m]}$ and $\rho_g^{(p,q)}(f) = \rho_f(m,q)$. Moreover if f have index-pair $(2,1)$ and $g = \exp z$, then Definition 1.4 and Definition 1.5 become the classical one.

Taking into account all these above, in this paper we estimate some growth rates of composite entire and meromorphic functions in the light of their relative (p,q)-th order and relative (p,q)-th lower order which considerably extend some results of Sheremeta [14].

2. Known Results

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [5] *Let f and g are any two entire functions with $g(0) = 0$. Also let β satisfy $0 < \beta < 1$ and $c(\beta) = \frac{(1-\beta)^2}{4\beta}$. Then for all sufficiently large values of r,*

$$M_f(c(\beta) M_g(\beta r)) \leq M_{f \circ g}(r) \leq M_f(M_g(r)) \ .$$

In addition if $\beta = \frac{1}{2}$, then for all sufficiently large values of r,

$$M_{f \circ g}(r) \geq M_f\left(\frac{1}{8} M_g\left(\frac{r}{2}\right)\right) \ .$$

Lemma 2.2. [3] *Let f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T_{f \circ g}(r) \leqslant \{1 + o(1)\} \frac{T_g(r)}{\log M_g(r)} T_f(M_g(r)) .$$

Lemma 2.3. [4] *Suppose that f is a meromorphic function and g be an entire function and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f \circ g}(r) \geq T_f(\exp(r^\mu)) \ .$$

Lemma 2.4. [7] *Let f be an entire function which satisfies the Property (A), $\beta > 0$, $\delta > 1$ and $\alpha > 2$. Then*

$$\beta T_f(r) < T_f\left(\alpha r^\delta\right) .$$

Lemma 2.5. [2] *Suppose f is an entire function and $\alpha > 1$, $0 < \beta < \alpha$. Then for all sufficiently large r,*

$$M_f(\alpha r) \geq \beta M_f(r).$$

3. Main Results

In this section we present the main results of the paper.

Theorem 3.1. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < +\infty$. and $\lambda_g(m,n) > 0$ where $p, q, m, n \in \mathbb{N}$ with $m \geq n$. Also let γ be a positive continuous on $[0, +\infty)$ function increasing to $+\infty$. Then for any number $\alpha \geq 0$,*

$$\lim_{r \to +\infty} \frac{\log^{[p]} M_h^{-1} M_{f \circ g}(\exp^{[n-1]} r)}{\left\{\log^{[p]} M_h^{-1} M_f(\exp^{[q]} \gamma(r))\right\}^{1+\alpha}} = \infty,$$

$$\text{when } q < m \text{ and } \lim_{r \to +\infty} \frac{\log \gamma(r)}{\log r} = 0$$

and

$$\lim_{r\to+\infty}\frac{\log^{[p-1]}M_h^{-1}M_{f\circ g}(\exp^{[n-1]}r)}{\left\{\log^{[p]}M_h^{-1}M_f(\exp^{[q]}\gamma(r))\right\}^{1+\alpha}} = \infty,$$

$$\textit{when } q \geqslant m \textit{ and } \lim_{r\to+\infty}\frac{\log\gamma(r)}{\log^{[q-m+1]}r}=0\ .$$

Proof. From the definition of $\rho_h^{(p,q)}(f)$, it follows for all sufficiently large values of r that

$$\log^{[p]}M_h^{-1}M_f(\exp^{[q]}\gamma(r)) \le \left(\rho_h^{(p,q)}(f)+\varepsilon\right)\gamma(r)\,. \tag{3.1}$$

Since $M_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2.1 and Lemma 2.5 for any $\beta > 2$ and for all sufficiently large values r that

$$\log^{[p]}M_h^{-1}M_{f\circ g}(\exp^{[n-1]}\beta r) \ge \log^{[p]}M_h^{-1}M_f\left(\frac{1}{8}M_g\left(\exp^{[n-1]}r\right)\right)$$

$$i.e.,\ \log^{[p]}M_h^{-1}M_{f\circ g}(\exp^{[n-1]}\beta r) \ge \left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\log^{[q]}M_g\left(\exp^{[n-1]}r\right)+O(1). \tag{3.2}$$

Case I. Let $q < m$. Then from (3.2) it follows for all sufficiently large values of r that

$$\log^{[p]}M_h^{-1}M_{f\circ g}(\exp^{[n-1]}\beta r) \ge \left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\exp^{[m-q-1]}\log^{[m-1]}M_g\left(\exp^{[n-1]}r\right)+O(1) \tag{3.3}$$

$$i.e.,\ \log^{[p]}M_h^{-1}M_{f\circ g}(\exp^{[n-1]}r) \ge \left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\beta^{-(\lambda_g(m,n)-\varepsilon)}\exp^{[m-q-1]}r^{(\lambda_g(m,n)-\varepsilon)}+O(1)\ . \tag{3.4}$$

Case II. Let $q \geqslant m$. Then from (3.2) we obtain for all sufficiently large values of r that

$$\log^{[p]}M_h^{-1}M_{f\circ g}(\exp^{[n-1]}\beta r) \ge \left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\log^{[q-m]}\log^{[m]}M_g\left(\exp^{[n-1]}r\right)+O(1). \tag{3.5}$$

$$i.e.,\ \log^{[p]}M_h^{-1}M_{f\circ g}(\exp^{[n-1]}r) \geqslant \left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\log^{[q-m+1]}r+O(1)$$

$$i.e.,\ \log^{[p-1]}M_h^{-1}M_{f\circ g}(\exp^{[n-1]}r) \geqslant \left(\log^{[q-m]}r\right)^{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)}+O(1). \tag{3.6}$$

Now combining (3.1) and (3.4) of Case I it follows for all sufficiently large values of r that

$$\frac{\log^{[p]}M_h^{-1}M_{f\circ g}(\exp^{[n-1]}r)}{\left\{\log^{[p]}M_h^{-1}M_f(\exp^{[q]}\gamma(r))\right\}^{1+\alpha}} \ge$$

$$\frac{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\beta^{(\lambda_g(m,n)-\varepsilon)}\exp^{[m-q-1]}r^{(\lambda_g(m,n)-\varepsilon)}+O(1)}{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)^{1+\alpha}\{\gamma(r)\}^{1+\alpha}}.$$

Since $\lim_{r\to+\infty}\frac{\log\gamma(r)}{\log r}=0$, therefore $\frac{\exp^{[m-q-1]}r^{(\lambda_g(m,n)-\varepsilon)}}{\{\gamma(r)\}^{1+\alpha}}\to+\infty$ as $r\to+\infty$, then from above it follows that

$$\varliminf_{r\to+\infty}\frac{\log^{[p]}M_h^{-1}M_{f\circ g}(\exp^{[n-1]}r)}{\left\{\log^{[p]}M_h^{-1}M_f(\exp^{[q]}\gamma(r))\right\}^{1+\alpha}}=\infty,$$

from which the first part of the theorem follows.

Again combining (3.1) and (3.6) of Case II it follows for all sufficiently large values of r that

$$\frac{\log^{[p-1]}M_h^{-1}M_{f\circ g}(\exp^{[n-1]}r)}{\left\{\log^{[p]}M_h^{-1}M_f(\exp^{[q]}\gamma(r))\right\}^{1+\alpha}}\geq\frac{\left(\log^{[q-m]}r\right)^{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)}+O(1)}{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)^{1+\alpha}\{\gamma(r)\}^{1+\alpha}}.$$

As $\lim_{r\to+\infty}\frac{\log\gamma(r)}{\log^{[q-m+1]}r}=0$, so $\frac{\left(\log^{[q-m]}r\right)^{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)}}{\{\gamma(r)\}^{1+\alpha}}\to+\infty$ as $r\to+\infty$. Thus it follows from above that

$$\lim_{r\to+\infty}\frac{\log^{[p-1]}M_h^{-1}M_{f\circ g}(\exp^{[n-1]}r)}{\left\{\log^{[p]}M_h^{-1}M_f(\exp^{[q]}\gamma(r))\right\}^{1+\alpha}}=\infty\ .$$

This proves the second part of the theorem. Thus the theorem follows . □

Remark 3.2. Theorem 3.1 is still valid with "limit superior" instead of "limit" if we replace the condition " $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<+\infty$" by " $0<\lambda_h^{(p,q)}(f)<+\infty$".

In the line of Theorem 3.1 one may state the following theorem without proof:

Theorem 3.3. *Let f, g, h and k be any four entire functions such that g is of finite (m,n)-th lower order, $\lambda_h^{(p,q)}(f)>0$ and $\rho_k^{(l,n)}(g)<+\infty$ where $p,q,m,n,l\in\mathbb{N}$ with $m\geq\min\{l,n\}$. Also let γ be a positive continuous on $[0,+\infty)$ function increasing to $+\infty$. Then for any number $\alpha\geq0$,*

$$\lim_{r\to+\infty}\frac{\log^{[p]}M_h^{-1}M_{f\circ g}(\exp^{[n-1]}r)}{\left\{\log^{[l]}M_k^{-1}M_g(\exp^{[n]}\gamma(r))\right\}^{1+\alpha}}=\infty,$$

$$\text{when } q<m \text{ and } \lim_{r\to+\infty}\frac{\log\gamma(r)}{\log r}=0$$

and

$$\lim_{r\to+\infty}\frac{\log^{[p-1]}M_h^{-1}M_{f\circ g}(\exp^{[n-1]}r)}{\left\{\log^{[l]}M_k^{-1}M_g(\exp^{[n]}\gamma(r))\right\}^{1+\alpha}} = \infty,$$

$$\text{when } q \geqslant m \text{ and } \lim_{r\to+\infty}\frac{\log\gamma(r)}{\log^{[q-m+1]}r}=0\ .$$

Remark 3.4. In Theorem 3.3 if we take the condition $\lambda_h^{(p,n)}(g) < +\infty$ instead of $\rho_h^{(p,n)}(g) < +\infty$, then also Theorem 3.3 remains true with "limit superior" in place of " limit ".

Next we prove our theorem for composite entire and meromorphic function.

Theorem 3.5. *Let f be a meromorphic function and g, h be an entire function with $0 < \lambda_h^{[p]}(f) \le \rho_h^{[p]}(f) < +\infty$ and $\rho_g > 0$ where $p \in \mathbb{N}$. Also let γ be a positive continuous on $[0,+\infty)$ function increasing to $+\infty$. Then for any number $\alpha \ge 0$,*

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p]}T_h^{-1}T_{f\circ g}(r)}{\left\{\log^{[p]}T_h^{-1}T_f(\exp\{\gamma(r)\})\right\}^{1+\alpha}}=\infty$$

where

$$\lim_{r\to+\infty}\frac{\log\gamma(r)}{\log r}=0\ .$$

Proof. Let $0 < \mu < \rho_g$. As $T_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2.3 for a sequence of values of r tending to infinity that

$$\begin{aligned}\log^{[p]}T_h^{-1}T_{f\circ g}(r) &\ge \log^{[p]}T_h^{-1}T_f(\exp(r^{\mu}))\\ \text{i.e., }\log^{[p]}T_h^{-1}T_{f\circ g}(r) &\ge \left(\lambda_h^{[p]}(f)-\varepsilon\right)r^{\mu}\ . \qquad (3.7)\end{aligned}$$

Again for all sufficiently large values of r we get that

$$\log^{[p]}T_h^{-1}T_f(\exp\{\gamma(r)\}) \le \left(\rho_h^{[p]}(f)+\varepsilon\right)\gamma(r)\ .$$

So combining (3.7) and above, we obtain for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]}T_h^{-1}T_{f\circ g}(r)}{\left\{\log^{[p]}T_h^{-1}T_f(\exp\{\gamma(r)\})\right\}^{1+\alpha}} \ge \frac{\left(\lambda_h^{[p]}(f)-\varepsilon\right)r^{\mu}}{\left(\rho_h^{[p]}(f)+\varepsilon\right)^{1+\alpha}\{\gamma(r)\}^{1+\alpha}}\ .$$

Since $\lim\limits_{r\to+\infty}\frac{\log\gamma(r)}{\log r}=0$, therefore $\frac{r^{\mu}}{\{\gamma(r)\}^{1+\alpha}} \to +\infty$ as $r\to+\infty$, then from above it follows that

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p]}T_h^{-1}T_{f\circ g}(r)}{\left\{\log^{[p]}T_h^{-1}T_f(\exp\{\gamma(r)\})\right\}^{1+\alpha}}=\infty\ .$$

Hence the theorem follows. □

Similarly one may state the following theorem without proof as it can be carried out in the line of Theorem 3.5.

Theorem 3.6. *Let f be a meromorphic function and g, h, k be any three entire functions with $\lambda_h^{[p]}(f) > 0$, $\rho_k(g) < +\infty$ and $\rho_g > 0$ where $p \in \mathbb{N}$. Also let γ be a positive continuous on $[0, +\infty)$ function increasing to $+\infty$. Then for any number $\alpha \geq 0$,*

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p]} T_h^{-1} T_{f\circ g}(r)}{\left\{\log T_k^{-1} T_g(\exp\{\gamma(r)\})\right\}^{1+\alpha}} = \infty$$

where

$$\lim_{r\to+\infty} \frac{\log \gamma(r)}{\log r} = 0\ .$$

Theorem 3.7. *Let f be a meromorphic function and g, h be any two entire functions such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < +\infty$, $\rho_g(m,n) < +\infty$ where $p, q, m, n \in \mathbb{N}$ with $m > n$. Also let γ be a positive continuous on $[0, +\infty)$ function increasing to $+\infty$. If h satisfy the Property (A), then for any number $\alpha \geq 0$,*

$$\lim_{r\to+\infty} \frac{\left\{\log^{[p]} T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[p]} T_h^{-1} T_f\left(\exp^{[q]}\{\gamma(r)\}\right)} = 0 \text{ if } q \geqslant m$$

and

$$\lim_{r\to+\infty} \frac{\left\{\log^{[p+m-q-1]} T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[p]} T_h^{-1} T_f\left(\exp^{[q]}\{\gamma(r)\}\right)} = 0 \text{ if } q < m,$$

where

$$\lim_{r\to+\infty} \frac{\log \gamma(r)}{\log r} = +\infty\ .$$

Proof. Let us suppose that $\alpha > 2$ and $\delta \to 1^+$ in Lemma 2.4. Since $T_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2.2, Lemma 2.4 and the inequality $T_g(r) \leq \log^+ M_g(r)$ {cf. [8] } for all sufficiently large values of r that

$$\begin{aligned} T_h^{-1} T_{f\circ g}(r) &\leqslant T_h^{-1}\left[\{1 + o(1)\} T_f(M_g(r))\right] \\ i.e.,\ T_h^{-1} T_{f\circ g}(r) &\leqslant \alpha T_h^{-1} T_f(M_g(r)) \\ i.e.,\ \log^{[p]} T_h^{-1} T_{f\circ g}(r) &\leqslant \log^{[p]} T_h^{-1} T_f(M_g(r)) + O(1) \end{aligned}$$

$$i.e.,\ \log^{[p]} T_h^{-1} T_{f\circ g}(r) \leqslant \left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[q]} M_g(r) + O(1)\ . \tag{3.8}$$

Now the following cases may arise :

Case I. Let $q \geqslant m$. Then we have from (3.8) for all sufficiently large values of r that

$$\log^{[p]} T_h^{-1} T_{f\circ g}(r) \leqslant \left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[m-1]} M_g(r) + O(1) \tag{3.9}$$

Now from the definition of (m,n)-th order of g, we get for arbitrary positive ε and for all sufficiently large values of r that

$$\log^{[m]} M_g(r) \leqslant \left(\rho_g(m,n)+\varepsilon\right)\log^{[n]} r$$
$$i.e.,\ \log^{[m]} M_g(r) \leq \left(\rho_g(m,n)+\varepsilon\right)\log r \tag{3.10}$$

$$i.e.,\ \log^{[m-1]} M_g(r) \leq r^{\left(\rho_g(m,n)+\varepsilon\right)} . \tag{3.11}$$

So from (3.9) and (3.11) it follows for all sufficiently large values of r that

$$\log^{[p]} T_h^{-1} T_{f\circ g}(r) \leqslant \left(\rho_h^{(p,q)}(f)+\varepsilon\right) r^{\left(\rho_g(m,n)+\varepsilon\right)} + O(1) . \tag{3.12}$$

Case II. Let $q<m$. Then we get from(3.8) for all sufficiently large values of r that

$$\log^{[p]} T_h^{-1} T_{f\circ g}(r) \leqslant \left(\rho_h^{(p,q)}(f)+\varepsilon\right)\exp^{[m-q]}\log^{[m]} M_g(r) + O(1). \tag{3.13}$$

Also we obtain from (3.10) for all sufficiently large values of r that

$$\exp^{[m-q]}\log^{[m]} M_g(r) \leqslant \exp^{[m-q]}\log r^{\left(\rho_g(m,n)+\varepsilon\right)}$$
$$i.e.,\ \exp^{[m-q]}\log^{[m]} M_g(r) \leqslant \exp^{[m-q-1]} r^{\left(\rho_g(m,n)+\varepsilon\right)} . \tag{3.14}$$

Now from (3.13) and (3.14) we obtain for all sufficiently large values of r that

$$\log^{[p]} T_h^{-1} T_{f\circ g}(r) \leqslant \left(\rho_h^{(p,q)}(f)+\varepsilon\right)\exp^{[m-q-1]} r^{\left(\rho_g(m,n)+\varepsilon\right)} + O(1)$$

$$i.e.,\ \log^{[p+m-q-1]} T_h^{-1} T_{f\circ g}(r) \leqslant r^{\left(\rho_g(m,n)+\varepsilon\right)} + O(1) . \tag{3.15}$$

Again for all sufficiently large values of r we get that

$$\log^{[p]} T_h^{-1} T_f\left(\exp^{[q]}\{\gamma(r)\}\right) \geq \left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\gamma(r) . \tag{3.16}$$

Now if $q \geqslant m$, we get from (3.12) and (3.16) for all sufficiently large values of r that

$$\frac{\left\{\log^{[p]} T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[p]} T_h^{-1} T_f\left(\exp^{[q]}\{\gamma(r)\}\right)} \leqslant$$

$$\frac{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)^{1+\alpha} r^{\left(\rho_g(m,n)+\varepsilon\right)(1+\alpha)}\left(1+\frac{O(1)}{\left(\rho_h^{(p,q)}(f)+\varepsilon\right) r^{\left(\rho_g(m,n)+\varepsilon\right)}}\right)^{1+\alpha}}{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\gamma(r)} .$$

Since $\lim\limits_{r\to+\infty}\frac{\log\gamma(r)}{\log r}=+\infty$, therefore $\frac{r^{\left(\rho_g(m,n)+\varepsilon\right)(1+\alpha)}}{\gamma(r)}\to+\infty$ as $r\to+\infty$, then the first part of the theorem follows from above.

Further when $q < m$, we obtain from (3.15)and (3.16) for all sufficiently large values of r that

$$\frac{\left\{\log^{[p+m-q-1]} T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[p]} T_h^{-1} T_f\left(\exp^{[q]}\{\gamma(r)\}\right)} \leqslant \frac{r^{\left(\rho_g(m,n)+\varepsilon\right)(1+\alpha)}\left(1+\frac{O(1)}{r^{\left(\rho_g(m,n)+\varepsilon\right)}}\right)^{1+\alpha}}{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\gamma(r)}$$

$$i.e., \ \overline{\lim_{r\to+\infty}} \frac{\left\{\log^{[p+m-q-1]} T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[p]} T_h^{-1} T_f\left(\exp^{[q]}\{\gamma(r)\}\right)} = 0,$$

This proves the second part of the theorem. □

Remark 3.8. In Theorem 3.7 if we take the condition $\rho_h^{(p,q)}(f) > 0$ instead of $\lambda_h^{(p,q)}(f) > 0$, the theorem remains true with " limit inferior" in place of "limit ".

Theorem 3.9. *Let f be a meromorphic function and g, h, k be any three entire functions such that g is of finite (m,n)-th order, $\rho_h^{(p,q)}(f) < +\infty$, $\rho_k^{(l,n)}(g) > 0$ where $p,q,m,n,l \in \mathbb{N}$ with $m \geq \min\{l,n\}$. Also let γ be a positive continuous on $[0,+\infty)$ function increasing to $+\infty$. If h satisfy the Property (A), then for any number $\alpha \geq 0$,*

$$\lim_{r\to+\infty} \frac{\left\{\log^{[p]} T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[l]} T_k^{-1} T_g\left(\exp^{[n]}\{\gamma(r)\}\right)} = 0 \ \textit{if } q \geqslant m$$

and

$$\lim_{r\to+\infty} \frac{\left\{\log^{[p+m-q-1]} T_h^{-1} T_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[l]} T_k^{-1} T_g\left(\exp^{[n]}\{\gamma(r)\}\right)} = 0 \ \textit{if } q < m,$$

where

$$\lim_{r\to+\infty} \frac{\log\gamma(r)}{\log r} = +\infty \ .$$

The proof of Theorem 3.9 would run parallel to that of Theorem 3.7. We omit the details.

Remark 3.10. In Theorem 3.9, if we take the condition $\rho_h^{(p,n)}(g) > 0$ instead of $\lambda_h^{(p,n)}(g) > 0$, the theorem remains true with "limit replaced by limit inferior".

Theorem 3.11. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < +\infty$, $\rho_g(m,n) < +\infty$ where $p,q,m,n \in \mathbb{N}$ with $m > n$. Also let γ be a positive continuous on $[0,+\infty)$ function increasing to $+\infty$. Then for any number $\alpha \geq 0$,*

$$\lim_{r\to+\infty} \frac{\left\{\log^{[p]} M_h^{-1} M_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[p]} M_h^{-1} M_f\left(\exp^{[q]}\{\gamma(r)\}\right)} = 0 \ \textit{if } q \geqslant m$$

and

$$\lim_{r\to+\infty}\frac{\left\{\log^{[p+m-q-1]}M_h^{-1}M_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[p]}M_h^{-1}M_f\left(\exp^{[q]}\{\gamma(r)\}\right)}=0 \text{ if } q<m,$$

where

$$\lim_{r\to+\infty}\frac{\log\gamma(r)}{\log r}=+\infty\ .$$

Theorem 3.12. *Let f, g, h and k be any four entire functions such that g is of finite (m,n)-th order, $\rho_h^{(p,q)}(f)<+\infty$, $\rho_k^{(l,n)}(g)>0$ where $p,q,m,n,l\in\mathbb{N}$ with $m\geq\min\{l,n\}$. Also let γ be a positive continuous on $[0,+\infty)$ function increasing to $+\infty$. Then for any number $\alpha\geq0$,*

$$\lim_{r\to+\infty}\frac{\left\{\log^{[p]}M_h^{-1}M_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[l]}M_k^{-1}M_g\left(\exp^{[n]}\{\gamma(r)\}\right)}=0 \text{ if } q\geqslant m$$

and

$$\lim_{r\to+\infty}\frac{\left\{\log^{[p+m-q-1]}M_h^{-1}M_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[l]}M_k^{-1}M_g\left(\exp^{[n]}\{\gamma(r)\}\right)}=0 \text{ if } q<m,$$

where

$$\lim_{r\to+\infty}\frac{\log\gamma(r)}{\log r}=+\infty\ .$$

A similar arguments in the proofs of Theorem 3.7 and Theorem 3.9 respectively will establish the results in Theorem 3.11 and Theorem 3.12 by the help of Lemma 2.1. Therefore we omit the details.

Remark 3.13. In Theorem 3.11 if we take the condition $\rho_h^{(p,q)}(f)>0$ instead of $\lambda_h^{(p,q)}(f)>0$, the theorem remains true with " limit inferior" in place of "limit ".

Remark 3.14. In Theorem 3.12, if we take the condition $\rho_h^{(p,n)}(g)>0$ instead of $\lambda_h^{(p,n)}(g)>0$, the theorem remains true with " limit replaced by limit inferior".

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