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The concept of Hukuhara derivative and Aumann integral for intuitionistic fuzzy number valued functions

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Abstract: *In this paper we have rstly dened a metric based on the Hausdor metric in intuitionistic fuzzy environment and studied its properties. Then we have proved that the metric space of intuitionistic fuzzy number valued functions is complete under this metric. Besides, we have studied the concept of Aumann integral for intuitionistic fuzzy number valued functions in terms of α and β cuts. Finally, we have given the relation between the Hukuhara derivative and Aumann integral for intuitionistic fuzzy valued functions by using the fundamental theorem of calculus.*

Keywords: *Intuitionistic fuzzy sets, intuitionistic fuzzy number valued functions, hukuhara derivative, Aumann integral, intuitionistic Hausdor metric.*

1. INTRODUCTION

In science and technology, vagueness or ambiguity is an inevitable phenomena. Hence to understand and interpret the models containing elements of uncertainty, probabilistic (stochastic, random) or possibilistic approaches are developed. Generally, the possibilistic approaches are based on fuzzy set theory.

Fuzzy set theory was firstly introduced by L. A. Zadeh in 1965 [1]. In fuzzy sets, every element in the set is accompanied with a function $\mu : X \to [0,1]$, called membership function. The membership function may have uncertainty in some applications because of the subjectivity of the expert or the missing information in the model, as well. Hence some extensions of fuzzy set theory were proposed [2-4]. One of these extensions is Atanassov's intuitionistic fuzzy set (IFS) theory [2].

In 1986, Atanassov [2] introduced the concept of intuitionistic fuzzy sets and carried out rigorous researches to develop the theory [5]. In this set concept, he introduced a new degree $\nu : X \to [0,1]$, called non-membership function, such that the sum $\mu + \nu$ is less than or equal to 1. Hence the difference $1 - (\mu + \nu)$ is regarded as degree of hesitation. Since intuitionistic fuzzy set theory contains membership function, non-membership function and the degree of hesitation, it can be regarded as a tool which is more flexible and closer to human reasoning in handling uncertainty due to imprecise knowledge or data.

Intuitionistic fuzzy set and fuzzy set theory have flourishing interesting applications in different fields of science and engineering such as population dynamics [6], decision-making problems [7, 8] image processing and pattern recognition [9], medicine [10, 11], fault analysis [12]. More detailed information about applicability of fuzzy sets and intuitionistic fuzzy sets can be found in [13-21].

The space of compact and convex sets has a linear structure with respect to Minkowski sum and scalar multiplication [22]. This linear structure is that of a cone rather than a vector space since the inverse element of a set with respect to Minkowski sum does not always exists. That is, if $A = \{a\}$ is not a singleton set, the Minkowski sum of A and $-A$ is not always the identity element $\{0\}$ i.e., $A + (-1)A \neq \{0\}$ [22]. This is a drawback not only in theory of compact and convex sets but also in theory of fuzzy sets and IFS. That is why Hukuhara tried to handle the inverse element problem. He defined a new difference called Hukuhara difference (H-difference) for compact and convex sets [23]. Later Hukuhara difference of fuzzy sets and Hukuhara derivative (H-derivative) of fuzzy number valued functions were introduced and studied [24–26].

The concept of fuzzy integral was firstly defined by Sugeno [27]. Later Ralescu and Adams defined the fuzzy integral of positive measurable functions [28]. They studied some properties such as monotone convergence theorem and Fatou's lemma. Later Dubois and Parade generalized the Riemann integral over compact and convex sets to fuzzy valued functions [29]. This approach is mainly related with the concept of Aumann integral. Aumann integral is defined for set valued functions by Aumann [30]. Since a fuzzy-valued function is essentially a family of set-valued functions, fuzzy Aumann integration concept is employed in the concept of fuzzy integral and fuzzy differential equations [22, 25].

In this paper we have firstly defined a new metric in intuitionistic fuzzy environment and study its properties. Then we have shown that the metric space of fuzzy number valued functions is complete under this metric. Moreover, we have studied the concepts of the Aumann integration for intuitionistic fuzzy number valued functions in terms of α and β cuts. And we have proved the relation between Hukuhara derivative and Aumann integral for intuitionistic fuzzy valued functions by using the fundamental theorem of calculus.

This paper is organized as follows. In Section 2 some preliminaries are given. In Section 3 we introduce a metric on the set of intuitionistic fuzzy numbers and study its properties. In Section 4, we study some fundamental theorems on Aumann integration and Hukuhara differentiability for intuitionistic fuzzy valued functions. Finally we give summary and conclusions in Section 5.

2. PRELIMINARIES

Definition 2.0.1 *[5] Let $\mu_A, \nu_A : \mathbb{R}^n \to [0,1]$ be two functions such that for each $x \in \mathbb{R}^n$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ holds. The set*

$$\tilde{A}^i = \{(x, \mu_A(x), \nu_A(x)) : x \in \mathbb{R}^n; \mu_A, \nu_A : \mathbb{R}^n \to [0,1]\}$$

is called an intuitionistic fuzzy set in $\mathbb{R}^n$. Here μ_A and ν_A are called membership and non-membership functions, respectively.

We will denote set of all intuitionistic fuzzy sets in $\mathbb{R}^n$ by $IF(\mathbb{R}^n)$.

Definition 2.0.2 *[5] Let $\tilde{A}^i \in IF(\mathbb{R}^n)$. The α-cut of $\tilde{A}^i$ is defined as follows:*

For $\alpha \in (0,1]$

$$A(\alpha) = \{x \in \mathbb{R}^n : \mu_A(x) \geq \alpha\},$$

and for $\alpha = 0$

$$A(0) = cl\left(\bigcup_{\alpha \in (0,1]} A(\alpha)\right).$$

Definition 2.0.3 *[5] Let $\tilde{A}^i \in IF(\mathbb{R}^n)$. The β-cut of $\tilde{A}^i$ is defined as follows:*

For $\beta \in [0,1)$

$$A^*(\beta) = \{x \in \mathbb{R}^n : \nu_A(x) \leq \beta\},$$

and for $\beta = 1$

$$A^*(1) = cl\left(\bigcup_{\beta \in [0,1)} A^*(\beta)\right).$$

Definition 2.0.4 *[5] Let $\tilde{A}^i \in IF(\mathbb{R}^n)$. For α and $\beta \in [0,1]$ with $0 \leq \alpha + \beta \leq 1$, the set*

$$A(\alpha, \beta) = \{x \in \mathbb{R}^n : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$$

is called (α, β)-cut of $\tilde{A}^i$

Theorem 2.0.1 *[5] Let $\tilde{A}^i \in IF(\mathbb{R}^n)$. Then*

$$A(\alpha, \beta) = A(\alpha) \cap A^*(\beta)$$

holds.

We will denote the set of all intuitionistic fuzzy numbers in $\mathbb{R}^n$ by $IF_N(\mathbb{R}^n)$.

Definition 2.0.5 *[31] Let X be a topological space. Let f be a function from X to $\mathbb{R} \cup \{-\infty, \infty\}$.*

1. f is called an upper semi-continuous function if for all $k \in \mathbb{R}\{x \in X : f(x) < k\}$ is an open set.
2. f is called an lower semi-continuous function if for all $k \in \mathbb{R}\{x \in X : f(x) > k\}$ is an open set.

Definition 2.0.6 *[32] Let f be a function defined on a convex subset K of $\mathbb{R}^n$.*

1. f is called a quasi-concave function on K if for all $x, y \in K$ and $t \in [0, 1]$,
$$f(tx + (1 - ty)) \geq \min\{f(x), f(y)\}$$
holds.
2. f is called a quasi-convex function on K if for all x, $y \in K$ and $t \in [0, 1]$,
$$f(tx + (1 - ty)) \leq \max\{f(x), f(y)\}$$
holds.

Definition 2.0.7 *An intuitionistic fuzzy set $\tilde{A}^i \in IF(\mathbb{R}^n)$ satisfying the following properties is called an intuitionistic fuzzy number in $\mathbb{R}^n$:*

1. $\tilde{A}^i$ is a normal set, i.e., $A(1) \neq \varnothing$ and $A^*(0) \neq \varnothing$.
2. $A(0)$ and $A^*(1)$ are bounded sets in $\mathbb{R}^n$.
3. $\mu_A : \mathbb{R}^n \to [0, 1]$ is an upper semi-continuous function; i.e., $\forall k \in \mathbb{R}^+ \cup \{0\}$, $\{x \in A : \mu_A(x) < k\}$ is an open set.
4. $\nu_A : \mathbb{R}^n \to [0, 1]$ is a lower semi-continuous function; i.e., $\forall k \in \mathbb{R}\{x \in A : \nu_A(x) > k\}$ is an open set.
5. The membership function μ_A is quasi-concave; i.e., $\forall \lambda \in [0, 1], \forall x,\ y \in \mathbb{R}^n$
$$\mu_A(\lambda x + (1 - \lambda)y) \geq \min(\mu_A(x), \mu_A(y))$$

6. The non-membership function ν_A is quasi-convex; i.e.,$\forall\lambda \in [0,1], \forall x,\ y \in \mathbb{R}^n$

$$\nu_A(\lambda x + (1-\lambda)y) \le \max(\nu_A(x), \nu_A(y)); \forall\lambda \in [0,1],$$

Definition 2.0.8 *[22] Let $x \in \mathbb{R}^n$ and A be a non-empty subset of $\mathbb{R}^n$. The distance from x to A is determined by*

$$d(x, A) = \inf\{\|x - a\| : a \in A\}$$

Definition 2.0.9 *[22] Let A and B be non-empty subsets of $\mathbb{R}^n$. Let S_1^n denote the closed unit ball in $\mathbb{R}^n$.*

1. The Hausdorff separation of A to B is defined by

$$\rho(A, B) = \sup\{d(a, B) : a \in A\}$$

or, equivalently,

$$\rho(A, B) = \inf\{r > 0 : A \subseteq B + rS_1^n\}$$

2. The Hausdorff separation of A to B is defined by

$$\rho(B, A) = \sup\{d(b, A) : b \in B\}$$

or, equivalently,

$$\rho(B, A) = \inf\{r > 0 : B \subseteq A + rS_1^n\}$$

Note that $\rho(A, B) \neq \rho(B, A)$ in general.

Definition 2.0.10 *[22] Let A and B be non-empty subsets of $\mathbb{R}^n$. The Hausdorff distance of A and B is defined as*

$$d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\}$$

Note that the Hausdorff distance function is a metric on the family of non-empty compact subsets of $\mathbb{R}^n$.

Theorem 2.0.2 *[22] Let A, B, C and D be compact sets in $\mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then the followings hold:*

1.

$$d_H(A + C, B + C) = d_H(A, B)$$

2.

$$d_H(\lambda A, \lambda B) = |\lambda|\, d_H(A, B)$$

3.

$$d_H(A + B, C + D) \le d_H(A, C) + d_H(B, D)$$

We will denote set of all compact and convex subsets of $\mathbb{R}^n$ by $K_C(\mathbb{R}^n)$.

Theorem 2.0.3 *[22]$(K_C(\mathbb{R}^n), d_H)$ is a complete metric space.*

Theorem 2.0.4 *Let $\{C_\beta : \beta \in [0,1]\}$ be a family of subsets of $\mathbb{R}^n$ satisfying the followings:*

1. C_β is a non-empty compact and convex subset of $\mathbb{R}^n$

2. If $0 \leq \beta_1 \leq \beta_2 \leq 1$ then $C_{\beta_1} \subseteq C_{\beta_2}$

3. If (β_n) is a non-increasing converging sequence to β then $C_\beta = \bigcap_{n=1}^{\infty} C_{\beta_n}$

Then $\{\beta \in [0,1] : x \in C_\beta\}$ is a closed and bounded interval.

Proof For $x \in C_1$, define $I_x = \{\beta \in [0,1] : x \in C_\beta\}$. Since I_x is bounded, its infimum exists, say $\beta^* = \inf I_x$. If we show $I_x = [\beta^*, 1]$ then the proof is done.

If $\beta^* = 1$ then $I_x = \{1\}$ is and there is nothing to prove..

Assume that $\beta^* < 1$ and $\beta \in (\beta^*, 1)$. By the definition of infimum there exists a real number $\beta_1 \in I_x$ such that $\beta^* < \beta_1 \leq \beta$. Hence by (2) we can write that $C_{\beta_1} \subseteq C_\beta$. Since $\beta_1 \in I_x$ we get $x \in C_{\beta_1} \Rightarrow x \in C_\beta \Rightarrow \beta \in I_x$. And this means that every β larger than β^* is an element of I_x. Hence $(\beta^*, 1] \subseteq I_x$.

Assume (β_n) is a non-increasing sequence converging to β^*. Since $\beta^* \leq \beta_n$ and $C_{\beta^*} \subseteq C_{\beta_n}$ we have

$$C_{\beta^*} = \bigcap_{n=1}^{\infty} C_{\beta_n}$$

And since $(\beta_n) \subset I_x$, for every $n \in \mathbb{N}$, $x \in C_{\beta_n} \Rightarrow x \in C_{\beta^*} \Rightarrow \beta^* \in I_x$. Hence $[\beta^*, 1] \subseteq I_x$.

Now let us show that $I_x \subseteq [\beta^*, 1]$.

Let $\beta \in I_x$. Then $\beta^* \leq \beta \Rightarrow \beta \in [\beta^*, 1] \Rightarrow I_x \subseteq [\beta^*, 1]$.

Therefore we obtain that $I_x = [\beta^*, 1]$. Namely, I_x is a closed and bounded interval. □

Theorem 2.0.5 *[34] Let $\tilde{A}^i \in IF_N(\mathbb{R}^n)$ and $\alpha, \beta \in [0,1]$ such that its α and β cuts given by $A(\alpha) = \{x \in \mathbb{R}^n : \mu_A(x) \geq \alpha\}$ and $A^*(\beta) = \{x \in \mathbb{R}^n : \nu_A(x) \leq \beta\}$. Then the followings hold:*

1. For every $\alpha \in [0,1]$, $A(\alpha)$ are non-empty compact and convex sets in $\mathbb{R}^n$.

2. If $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ then $A(\alpha_2) \subseteq A(\alpha_1)$.

3. If (α_n) is a non-decreasing sequence in $[0,1]$ converging to α then

$$\bigcap_{n=1}^{\infty} A(\alpha_n) = A(\alpha).$$

4. If (α_n) is a non-increasing sequence in $[0,1]$ converging to 0 then

$$cl\left(\bigcup_{n=1}^{\infty} A(\alpha_n)\right) = A(0).$$

5. For every $\beta \in [0,1]$, $A^*(\beta)$ are non-empty compact and convex sets in $\mathbb{R}^n$.

6. If $0 \leq \beta_1 \leq \beta_2 \leq 1$ then $A^*(\beta_1) \subseteq A^*(\beta_2)$.

7. If (β_n) is a non-increasing sequence in $[0,1]$ converging to β then

$$\bigcap_{n=1}^{\infty} A^*(\beta_n) = A^*(\beta).$$

8. If (β_n) is a non-decreasing sequence in $[0,1]$ converging to 1 then

$$cl\left(\bigcup_{n=1}^{\infty} A^*(\beta_n)\right) = A^*(1).$$

Theorem 2.0.6 *[33] Let $\{C_\alpha \subseteq \mathbb{R}^n : \alpha \in [0,1]\}$ be a family of sets in $\mathbb{R}^n$ satisfying (1.)-(4.) in Theorem 2.0.5 and $\{C_\beta \subseteq \mathbb{R}^n : \beta \in [0,1]\}$ be a family of set in $\mathbb{R}^n$ satisfying (5.)-(8.) in Theorem 2.0.5 Let us define the functions $\mu : \mathbb{R}^n \to [0,1]$ and $\nu : \mathbb{R}^n \to [0,1]$ by*

$$\mu(x) = \begin{cases} \sup\{\alpha \in [0,1] : x \in M_\alpha\}, & x \in C_0 \\ 0, & x \notin C_0 \end{cases}$$

$$\nu(x) = \begin{cases} \inf\{\beta \in [0,1] : x \in M_\beta\}, & x \in C_1 \\ 1, & x \notin C_1 \end{cases}$$

Then there exists an intuitionistic fuzzy number $\tilde{A}^i \in IF_N(\mathbb{R}^n)$ with its α and β cuts $A(\alpha)$ and $A^*(\beta)$ satisfying the followings:

1. For all $\alpha \in [0,1]$, $A(\alpha) = C_\alpha$.

2. For all $\beta \in [0,1]$, $A^*(\beta) = C_\beta$.

Definition 2.0.11 *[22] Let A and B be two nonempty subsets of $\mathbb{R}^n$ and $c \in \mathbb{R}$. The Minkowski addition and scalar multiplication are defined as follows:*

$$\begin{aligned} A + B &= \{a + b : a \in A \text{ and } b \in B\} \\ cA &= \{ca : a \in A\} \end{aligned}$$

Definition 2.0.12 *[33]Let $\tilde{A}^i$, $\tilde{B}^i \in IF_N(\mathbb{R}^n)$ and $c \in \mathbb{R} - \{0\}$. Minkowski addition and scalar multiplication of fuzzy numbers in $IF_N(\mathbb{R}^n)$ is defined level wise as follows:*

$$\begin{aligned} \tilde{A}^i + \tilde{B}^i &= \tilde{C}^i \Leftrightarrow C(\alpha) = A(\alpha) + B(\alpha) \text{ and } C^*(\beta) = A^*(\beta) + B^*(\beta), \\ c(\tilde{A}^i) &= \tilde{D}^i \Leftrightarrow D(\alpha) = cA(\alpha) \text{ and } D^*(\beta) = cA^*(\beta). \end{aligned}$$

Theorem 2.0.7 *[33] $IF_N(\mathbb{R}^n)$ is closed under Minkowski addition and scalar multiplication.*

Definition 2.0.13 *Let $\tilde{A}^i$, $\tilde{B}^i \in IF_N(\mathbb{R}^n)$. The Hukuhara difference of $\tilde{A}^i$ and $\tilde{B}^i$ is $\tilde{C}^i$, if it exists, such that*

$$\tilde{A}^i \ominus_H \tilde{B}^i = \tilde{C}^i \Longleftrightarrow \tilde{A}^i = \tilde{B}^i + \tilde{C}^i$$

Definition 2.0.14 *Let $\tilde{A}^i$, $\tilde{B}^i \in IF_N(\mathbb{R}^n)$. The generalized Hukuhara difference of $\tilde{A}^i$ and $\tilde{B}^i$ is $\tilde{C}^i$, if it exists, such that*

$$\tilde{A}^i \ominus_{gH} \tilde{B}^i = \tilde{C}^i \Longleftrightarrow \tilde{A}^i = \tilde{B}^i + \tilde{C}^i \text{ or } \tilde{B}^i = \tilde{A}^i + (-1)\tilde{C}^i$$

3. AN INTUITIONISTIC FUZZY METRIC AND ITS PROPERTIES

Theorem 3.0.1 *Let $\tilde{A}^i$, $\tilde{B}^i \in IF_N(\mathbb{R}^n)$. Let*

$$\begin{aligned} D_1(\tilde{A}^i, \tilde{B}^i) &= \sup\{d_H(A(\alpha), B(\alpha)) : \alpha \in [0,1]\} \\ D_2(\tilde{A}^i, \tilde{B}^i) &= \sup\{d_H(A(\beta), B(\beta)) : \beta \in [0,1]\} \end{aligned}$$

The function

$$D_\infty(\tilde{A}^i, \tilde{B}^i) = \max\{D_1(\tilde{A}^i, \tilde{B}^i), D_2(\tilde{A}^i, \tilde{B}^i)\}$$

defines a metric on $IF_N(\mathbb{R}^n)$. Hence $(IF_N(\mathbb{R}^n), D)$ is a metric space.

Proof Let us show that the metric axioms are satisfied by D_∞

M1.

$$\begin{aligned} D_\infty(\tilde{A}^i, \tilde{B}^i) &= 0 \Leftrightarrow D_1(\tilde{A}^i, \tilde{B}^i) = 0 \text{ and } D_2(\tilde{A}^i, \tilde{B}^i) = 0 \\ &\Leftrightarrow A(\alpha) = B(\alpha) \text{ and } A^*(\beta) = B^*(\beta) \\ &\Leftrightarrow \tilde{A}^i = \tilde{B}^i \end{aligned}$$

M2.

$$\begin{aligned} D_\infty(\tilde{A}^i, \tilde{B}^i) &= \max\{D_1(\tilde{A}^i, \tilde{B}^i), D_2(\tilde{B}^i, \tilde{A}^i)\} \\ &= \max\{D_1(\tilde{B}^i, \tilde{A}^i), D_2(\tilde{A}^i, \tilde{B}^i)\} \\ &= D_\infty(\tilde{B}^i, \tilde{A}^i) \end{aligned}$$

M3. Let $\tilde{A}^i$, $\tilde{B}^i$ and $\tilde{C}^i \in IF_N$. Since

$$\begin{aligned} D_1(\tilde{A}^i, \tilde{B}^i) &\leq D_1(\tilde{A}^i, \tilde{C}^i) + D_1(\tilde{C}^i, \tilde{B}^i) \\ D_2(\tilde{A}^i, \tilde{B}^i) &\leq D_2(\tilde{A}^i, \tilde{C}^i) + D_2(\tilde{C}^i, \tilde{B}^i) \end{aligned}$$

Then

$$\max\{D_1(\tilde{A}^i, \tilde{B}^i), D_2(\tilde{B}^i, \tilde{A}^i)\} \leq \max\{D_1(\tilde{A}^i, \tilde{C}^i) + D_1(\tilde{C}^i, \tilde{B}^i), D_2(\tilde{A}^i, \tilde{C}^i) + D_2(\tilde{C}^i, \tilde{B}^i)\}$$

Since for any reel numbers a and b

$$\max\{a, b\} = \frac{a + b + |a - b|}{2}$$

holds. So one can easily obtain that

$$\begin{aligned} \max\{D_1(\tilde{A}^i, \tilde{C}^i) + D_1(\tilde{C}^i, \tilde{B}^i), D_2(\tilde{A}^i, \tilde{C}^i) + D_2(\tilde{C}^i, \tilde{B}^i)\} &\leq \max\{D_1(\tilde{A}^i, \tilde{C}^i), D_2(\tilde{A}^i, \tilde{C}^i)\} \\ &+ \max\{D_1(\tilde{C}^i, \tilde{B}^i), D_2(C, B)\} \end{aligned}$$

Hence the triangle inequality

$$D_\infty(\tilde{A}^i, B) \leq D_\infty(\tilde{A}^i, \tilde{C}^i) + D_\infty(\tilde{C}^i, \tilde{B}^i)$$

is satisfied. □

Theorem 3.0.2 *Let $\tilde{A}^i$, $\tilde{B}^i$, $\tilde{C}^i$ and $\tilde{D}^i \in IF_N(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$. Then the followings hold:*

1.

$$D_\infty(\tilde{A}^i + \tilde{C}^i, \tilde{B}^i + \tilde{C}^i) = D_\infty(\tilde{A}^i, \tilde{B}^i)$$

2.

$$D_\infty(\lambda\tilde{A}^i, \lambda\tilde{B}^i) = |\lambda|\, D_\infty(\tilde{A}^i, \tilde{B}^i)$$

3.

$$D_\infty(\tilde{A}^i + \tilde{B}^i, \tilde{C}^i + \tilde{D}^i) \leq D_\infty(\tilde{A}^i, \tilde{C}^i) + D_\infty(\tilde{B}^i, \tilde{D}^i)$$

Proof

it is easy to see (1.) and (2.) are satisfied. Let us show (3.). By (3.) in Theorem 2.0.2 we have

$$\begin{aligned} D_1(\tilde{A}^i + \tilde{B}^i, \tilde{C}^i + \tilde{D}^i) &\leq D_1(\tilde{A}^i, \tilde{C}^i) + D_1(\tilde{B}^i, \tilde{D}^i) \\ D_2(\tilde{A}^i + \tilde{B}^i, \tilde{C}^i + \tilde{D}^i) &\leq D_2(\tilde{A}^i, \tilde{C}^i) + D_2(\tilde{B}^i, \tilde{D}^i) \end{aligned}$$

Since for any reel numbers a and b

$$\max\{a,b\} = \frac{a+b+|a-b|}{2}$$

holds then we can write that

$$\begin{aligned}\max\{D_1(\tilde{A}^i+\tilde{B}^i,\tilde{C}^i+\tilde{D}^i), D_2(\tilde{A}^i+\tilde{B}^i,\tilde{C}^i+\tilde{D}^i)\} &\leq \max\{D_1(\tilde{A}^i,\tilde{C}^i), D_2(\tilde{A}^i,\tilde{C}^i)\} \\ &+ \max\{D_1(\tilde{B}^i,\tilde{D}^i), D_2(\tilde{B}^i,\tilde{D}^i)\}\end{aligned}$$

Hence

$$D_\infty(\tilde{A}^i+\tilde{B}^i,\tilde{C}^i+\tilde{D}^i) \leq D_\infty(\tilde{A}^i,\tilde{C}^i) + D_\infty(\tilde{B}^i,\tilde{D}^i)$$

holds. □

Theorem 3.0.3 $(IF_N(\mathbb{R}^n), D_\infty)$ *is a complete metric space.*

Proof Let $(\tilde{A}^i_k)$ be a Cauchy sequence in $IF_N(\mathbb{R}^n)$. Let $A_k(\alpha)$ and $A^*_k(\beta)$ be the α and β cuts of $\tilde{A}^i_k$ for each $k \in \mathbb{N}$. Since $(\tilde{A}^i_k)$ is a Cauchy sequence then for each α and $\beta \in [0,1]$, $A_k(\alpha)$ and $A^*_k(\beta)$ are Cauchy sequences in $(K_C(\mathbb{R}^n), d_H)$ as well. Since $(K_C(\mathbb{R}^n), d_H)$ is a complete metric space then there exist $C(\alpha)$ and $C^*(\beta)$ in $K_C(\mathbb{R}^n)$ such that

$$\begin{aligned} d_H(A_k(\alpha), C(\alpha)) &\to 0 \\ d_H(A^*_k(\beta), C^*(\beta)) &\to 0 \end{aligned}$$

If the family of sets $\{A(\alpha) : \alpha \in [0,1]\}$ satisfies (1.)-(4.) in Theorem 2.0.5 and $\{A^*(\beta) : \beta \in [0,1]\}$ satisfies (5.)-(8.) in Theorem 2.0.5 then there exist an intuitionistic fuzzy number $\tilde{A}^i$ such that its α and β cuts are $A(\alpha) = C(\alpha)$ and $A^*(\beta) = C^*(\beta)$ by Theorem 2.0.6. Now let us prove this $\tilde{A}^i$ exists. By [22] $\{A(\alpha) : \alpha \in [0,1]\}$ satisfies (1.)-(4.) in Theorem 2.0.5. So let us show that $\{A^*(\beta) : \beta \in [0,1]\}$ satisfies (5.)-(8.) in Theorem 2.0.5.

5. Since for every $\beta \in [0,1]$, $C^*(\beta) \in K_C(\mathbb{R}^n)$, $C^*(\beta)$ is a compact and convex set.

6. Let $0 \leq \beta_1 \leq \beta_2 \leq 1$.

$$\rho(C^*(\beta_1), C^*(\beta_2)) \leq d^*_H(C^*(\beta_1), A^*_k(\beta_1)) + \rho(A^*_k(\beta_1), A^*_k(\beta_2)) + \rho(A^*_k(\beta_2), C^*(\beta_2))$$

Since $\beta_1 \leq \beta_2$ we can write that $A^*_k(\beta_1) \subseteq A^*_k(\beta_2)$. Hence the Hausdorff separation $d^*_H(A^*_k(\beta_1), A^*_k(\beta_2)) = 0$

And as $k \to \infty$ we have $\rho(C^*(\beta_1), A^*_k(\beta_1)) \to 0$ and $d^*_H(A^*_k(\beta_2), C^*(\beta_2)) \to 0$

So the Hausdorff separation $\rho(C^*(\beta_1), C^*(\beta_2)) = 0$ and this implies that $C^*(\beta_1) \subseteq C^*(\beta_2)$.

7. Let (β_n) be a non-increasing sequence converging to β in $[0,1]$. So by the result above $C^*(\beta) \subseteq C^*(\beta_n)$ for $n = 1, 2, 3, ...$, so

$$C^*(\beta) \subseteq \bigcap_{n=1}^{\infty} C^*(\beta_n)$$

Let $x \in \cap_{n=1}^{\infty} C^*(\beta_n)$ then for every $n \in \mathbb{N}$, $x \in C^*(\beta_n)$. Since $\{x\} \subseteq C^*(\beta_n)$, we can write that

$$\begin{aligned} \rho(\{x\}, C^*(\beta)) &\leq d_H^*(C^*(\beta_n), C^*(\beta)) \\ &\leq \rho(C^*(\beta_n), A_k^*(\beta_n)) + \rho(A_k^*(\beta_n), A_k^*(\beta)) + \rho(A_k^*(\beta), C^*(\beta)) \end{aligned}$$

Since $d_H(A_k^*(\beta), C^*(\beta)) \to 0$ we have $\rho(C^*(\beta_n), A_k^*(\beta_n)) \to 0$ and $\rho(A_k^*(\beta), C^*(\beta)) \to 0$. As $n \to \infty$ we have $\rho(A_k^*(\beta_n), A_k^*(\beta)) \to 0$. Thus $x \in C^*(\beta)$. So $\bigcap C^*(\beta_n) \subseteq C^*(\beta)$.

As a result

$$C^*(\beta) = \bigcap_{n=1}^{\infty} C^*(\beta_n)$$

8. Let (β_n) be a non-increasing sequence converging to 1 in $[0,1]$. Since for any $n \in \mathbb{N}$, $C^*(\beta_n) \subseteq C^*(1)$ then we can obtain that

$$kap\left(\bigcup_{n=1}^{\infty} C^*(\beta_n)\right) \subseteq C^*(1).$$

Let $x \in C^*(1)$. Then by the Hausdorff separation we can write that

$$\rho(\{x\}, C^*(\beta_n)) \leq \rho(\{x\}, C^*(1)) + \rho(C^*(1), C^*(\beta_n)).$$

Since $\{x\} \subseteq C^*(1)$ we obtain that $\rho(\{x\}, C^*(1)) = 0$. As $n \to \infty$,

$$\rho(C^*(1), C^*(\beta_n)) \to 0$$

is satisfied. Hence we obtain that $\rho(\{x\}, C^*(\beta_n)) \to 0$ and $\{x\} \subseteq C^*(\beta_n)$. So

$$C^*(\beta_n) \subseteq kap(\bigcup_{n=1}^{\infty} C^*(\beta_n))$$

and

$$C^*(1) \subseteq kap\left(\bigcup_{n=1}^{\infty} C^*(\beta_n)\right)$$

are satisfied. As a result, we obtain

$$C^*(1) = kap\left(\bigcup_{n=1}^{\infty} C^*(\beta_n)\right).$$

So, by Theorem 2.0.6 there exists an intuitionistic fuzzy number $\tilde{A}^i$ such that $A(\alpha) = C(\alpha)$ and $A^*(\beta) = C^*(\beta)$ Moreover,

$$\begin{aligned} d_H(A_k^*(\beta), A^*(\beta)) &\leq d_H(A_k^*(\beta), A_j^*(\beta)) + d_H(A_j^*(\beta), A^*(\beta)) \\ &\leq d_\infty(A_k^*(\beta), A_j^*(\beta)) + d_\infty(A_j^*(\beta), A^*(\beta)) \\ &\leq \varepsilon + d_\infty(A_j^*(\beta), A^*(\beta)) \end{aligned}$$

Taking the limit as $j \to \infty$ we obtain $d_H(A_k^*(\beta), A^*(\beta)) \leq \epsilon$ for all $k \geq N(\epsilon)$ uniformly in $\beta \in [0,1]$ so that $d_\infty(A_k^*(\beta), A^*(\beta)) \leq \epsilon$ for all $k \geq N(\epsilon)$.

Hence $\tilde{A}_k^i \to \tilde{A}^i$ in $IF_N(\mathbb{R}^n)$, which completes the proof. □

4. INTUITIONISTIC FUZZY NUMBER VALUED FUNCTIONS

Definition 4.0.1 *A mapping f from a domain $T \subseteq \mathbb{R}$ into $IF(\mathbb{R}^n)$ is called an intuitionistic fuzzy function (mapping).*

Definition 4.0.2 *An intuitionistic fuzzy function $f : [a,b] \subseteq \mathbb{R} \to IF_N(\mathbb{R}^n)$ is called continuous at $x_0 \in [a,b]$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$D_\infty(f(x), f(x_0)) < \epsilon$$

holds for all $x \in [a,b]$ with $|x - x_0| < \delta$.

Corollary 4.0.1 *Let $C([a,b], IF_N(\mathbb{R}^n))$ be the space of continuous functions from $[a,b]$ to $IF_N(\mathbb{R}^n)$. $C([a,b], IF_N(\mathbb{R}^n))$ is complete under the following metric:*

$$D_s(f,g) = \sup\{D_\infty(f(x), g(x)) : x \in [a,b]\}.$$

Proof Let $(f_n) \subseteq C([a,b], IF_N(\mathbb{R}^n))$ be a Cauchy sequence of functions. So for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ so that when m, $n > n_0$, we have

$$D_s(f_m, f_n) = \sup\{D_\infty(f_m(x), f_n(x)) : x \in [a,b]\} < \epsilon.$$

Hence for every $x \in [a,b]$

$$D_\infty(f_m(x), f_n(x)) < \epsilon$$

holds.. That is why, $(f_n(x))$, $IF_N(\mathbb{R})$ is a Cauchy sequence in $IF_N(\mathbb{R}^n)$ too. Since the space of intuitionistic numbers is complete with respect to D_∞,

$$f_n(x) \to f(x) \in IF_N(\mathbb{R}^n)$$

holds as $n \to \infty$. As the convergence under supremum metric is uniform and f_n is continuous for any $n \in \mathbb{N}$ then f is continuous, as well. That is why, $f \in C([a,b], IF_N(\mathbb{R}^n))$ and so $C([a,b], IF_N(\mathbb{R}))$ is complete with respect to the metric D_s. □

4.1. Hukuhara Differentiability

Definition 4.1.1 *[35] Let $f : (a, b) \to IF_N(\mathbb{R})$ be an intuitionistic fuzzy number valued function and $t_0, t_0 + h \in (a, b)$. f is called Hukuhara differentiable at t_0 if there exists an element $f'_H(t_0) \in IF_N(\mathbb{R})$ such that for all $h > 0$ the followings is satisfied*

$$\lim_{h\to 0^+} \frac{f(t_0 + h) \ominus_H f(t_0)}{h} = \lim_{h\to 0^+} \frac{f(t_0) \ominus_H f(t_0 - h)}{h} = f'_H(t_0$$

Theorem 4.1.1 *[35] Let $f : (a, b) \to IF(\mathbb{R})$ be an intuitionistic fuzzy function. Let $f(t, \alpha) = [f_1(t, \alpha), f_2(t, \alpha)]$ and $f(t, \beta) = [f_1(t, \beta), f_2(t, \beta)]$ be its α and β cuts, respectively. If $f : (a, b) \to IF(\mathbb{R})$ is Hukuhara differentiable then the functions $f_1(t, \alpha)$, $f_2(t, \alpha)$, $f_1^*(t, \beta)$ and $f_2^*(t, \beta)$ are differentiable in classical sense for each $\alpha, \beta \in [0, 1]$ such that*

$$\begin{aligned} f'(t, \alpha) &= [f'_1(t, \alpha), f'_2(t, \alpha)] \\ f^{*\prime}(t, \beta) &= [f_1^{*\prime}(t, \beta), f_2^{*\prime}(t, \beta)] \end{aligned}$$

4.2. Aumann Integration

Definition 4.2.1 *[22] A selector of a set valued function $f : T \subseteq \mathbb{R} \to K_C(\mathbb{R}^n)$ is a single valued function $g : T \to \mathbb{R}^n$ such that for all $t \in T$, $g(t) \in f(t)$.*

We will denote the set of integrable selectors of f by $\mathbf{S}(f)$.

Definition 4.2.2 *[22] Let $f : [a, b] \to K_C(\mathbb{R}^n)$. Then the Aumann integral of f over $[a, b]$ is defined as*

$$\int_a^b f(t)dt = \{\int_a^b k(t)dt : k \in \mathbf{S}(f)\}$$

If $\mathbf{S}(f) \neq \emptyset$ then we say that f is Aumann integrable over $[a, b]$.

Definition 4.2.3 *An intuitionistic fuzzy function $f : T \subseteq \mathbb{R} \to IF(\mathbb{R}^n)$ is continuous at $t_0 \in T$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$D_\infty(f(t), f(t_0)) < \varepsilon$$

for all $t \in T$ with $|t - t_0| < \delta$.

Definition 4.2.4 *$f : [a, b] \to IF(\mathbb{R}^n)$ is called integrably bounded if there exists an integrable real valued function $h : [0, 1] \to \mathbb{R}$ such that*

$$D(f(t), 0) \le h(t).$$

Definition 4.2.5 *Let $f : [a, b] \to IF(\mathbb{R}^n)$ be an intuitionistic fuzzy function. If for each α and $\beta \in [0, 1]$, the α and β cuts of f are (Lebesgue) measurable then f is called strongly measurable.*

Definition 4.2.6 *Let $f : [a,b] \to IF(\mathbb{R}^n)$ and denote its α and β cuts by $f(t,\alpha)$ and $f^*(t,\beta)$ respectively. If there exists an intuitionistic fuzzy number $\tilde{A}^i \in IF(\mathbb{R}^n)$ such that for each α and $\beta \in [0,1]$*

$$A(\alpha) = \int_a^b f(t,\alpha)dt$$
$$A^*(\beta) = \int_a^b f^*(t,\beta)dt$$

then f is said to be Aumann integrable over $[a,b]$ and $\tilde{A}^i$, which is denoted by $\int_a^b f(t)dt$, is called its (intuitionistic) Aumann integral over $[a,b]$.

Theorem 4.2.1 *[22] If $f : [a,b] \to K_C(\mathbb{R}^n)$ is measurable and integrably bounded then it is Aumann integrable over $[a,b]$. Moreover, the integral of f is a compact and convex set in $K_C(\mathbb{R}^n)$.*

Theorem 4.2.2 *[22]Let f_k $(k = 1,2,....)$and $f : [a,b] \subseteq \mathbb{R} \to K_C(\mathbb{R}^n)$ be measurable and uniformly integrably bounded set valued functions. If for all $x \in [a,b]$, $f_k(x) \to f(x)$ as $k \to \infty$ then*

$$\int_a^b f_k(x)dx \to \int_a^b f(x)dx$$

Theorem 4.2.3 *Let $f : [a,b] \to IF_N(\mathbb{R}^n)$ and denote its α and β cut by $f(t,\alpha)$ and $f^*(t,\beta)$ respectively.*

If f is strongly measurable and integrably bounded, then it is Aumann integrable over $[a,b]$.

Proof

Let f be strongly measurable and integrably bounded then $f(t,\alpha)$ and $f^*(t,\beta)$ are integrable for each α and β in $[0,1]$.

Let $C_\alpha = \int_a^b f(t,\alpha)dt$ and $C_\beta = \int_a^b f(t,\beta)dt$. We need to show whether the family of sets $\{C_\alpha : \alpha \in [0,1]\}$ satisfies (1.)-(4.) in Theorem 2.0.5 and $\{C_\beta : \beta \in [0,1]\}$ satisfies (5.)-(8.) in Theorem 2.0.5. By [22], $\{C_\alpha : \alpha \in [0,1]\}$ satisfies (1.)-(4.) in Theorem 2.0.5. So let us show that $\{C_\beta : \beta \in [0,1]\}$ satisfies (5.)-(8.) in Theorem 2.0.5.

Since $f^*(t,\beta)$ is integrable for each $\beta \in [0,1]$, $C_\beta \neq \varnothing$ and since for each $0 \le \beta_1 \le \beta_2 \le 1$ we have $f^*(t,\beta_1) \subseteq f^*(t,\beta_2)$. So we obtain $C_{\beta_1} \subseteq C_{\beta_2}$.

Let $(\beta_k) \subseteq [0,1]$ be a non-increasing sequence converging to β. Since f is an intuitionistic fuzzy number valued function, we obtain that $f^*(t,\beta_k) \to f^*(t,\beta)$. by Theorem 2.0.5. Moreover, since for all $\beta \in [0,1]$ and $t \in [a,b]$ we have $d_H(f(t,\beta),\{0\}) \le d_H(f(t,1),\{0\})$, $f^*(t,\beta)$ is uniformly integrably bounded on $[a,b]$. So by Theorem 4.2.2

$$C_{\beta_k} \to C_\beta$$

in $(K_C(\mathbb{R}^n), d_H)$.

Let $(\beta_k) \subseteq [0,1]$ be a non-decreasing sequence converging to 1 then by the similar reasoning above we obtain

$$C_{\beta_k} \to C_1$$

Hence by Theorem 2.0.6, there exists an intuitionistic fuzzy number $\tilde{A}^i \in IF(\mathbb{R}^n)$ with $A(\alpha) = \int_a^b f(t,\alpha)dt$ and $A^*(\beta) = \int_a^b f^*(t,\beta)dt$. So f is Aumann integrable. □

Lemma 4.2.1 *If $f : [a,b] \to IF_N(\mathbb{R})$ is continuous then it is strongly measurable.*

Proof Let $\epsilon > 0$ be given. Since $f : [a,b] \to IF_N(\mathbb{R})$ is continuous at an arbitrary point $x_0 \in [a,b]$ then there exists a $\delta > 0$ such that for $x \in [a,b]$ and $|x - x_0| < \delta$ we have

$$D_\infty(f(x), f(x_0)) < \epsilon.$$

Since

$$D_\infty(f(x), f(x_0)) = \max\{D_1(f(x), f(x_0)), D_2(f(x), f(x_0))\} < \epsilon$$

for $x \in [a,b]$ and $|x - x_0| < \delta$ we have

$$d_H(f(x;\alpha), f(x_0;\alpha)) < \epsilon$$

and

$$d_H(f^*(x;\beta), f^*(x_0;\beta)) < \epsilon.$$

So $f(x;\alpha)$ and $f^*(x;\beta)$ are continuous an arbitrary point $x_0 \in [a,b]$. Since α and β cuts of intuitionistic fuzzy number valued functions are continuous they are measurable set valued functions [22]. Hence $f : [a,b] \to IF_N(\mathbb{R})$ is a strongly measurable intuitionistic fuzzy number valued function. □

Theorem 4.2.4 *If $f : [a,b] \to IF_N(\mathbb{R})$ is a continuous fuzzy number valued function then it is Aumann integrable.*

Proof Since f is continuous then it is strongly measurable by Lemma 4.2.1. Since

$$d_H(f(x;\alpha), 0) \leq d_H(f(x;0), 0)$$

and

$$d_H(f^*(x;\beta), 0) \leq d_H(f^*(x;1), 0)$$

then we have

$$D_\infty(f, 0) \leq \max\{d_H(f(x;0), 0), d_H(f^*(x;1), 0)\}.$$

Let us define a function $h : [a,b] \to \mathbb{R}$ such that

$$h(x) = \max\{d_H(f(x;0), 0), d_H(f^*(x;1), 0)\}.$$

So for every $x \in [a,b]$

$$D_\infty(f, 0) \leq h(x)$$

is satisfied. Since h is integrable f is integrably bounded.

Therefore, since $f : [a,b] \to IF_N(\mathbb{R})$ is strongly measurable and integrably bounded, f is Aumann integrable □

Theorem 4.2.5 *Let $f, g : [a, b] \to IF_N(\mathbb{R})$ and $\lambda \in \mathbb{R}$ be given. If f and g are strongly measurable and integrably bounded then the followings are satisfied.*

1. $f + g$ is Aumann integrable on $[a, b]$ and

$$\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

holds.

2. $\lambda f$is Aumann integrable on $[a, b]$ and

$$\int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx$$

holds.

3. For any $c \in (a, b)$,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

holds.

Proof

1. Let f and $g \in IF_N(\mathbb{R})$ such that their α and β cuts denoted by $f(x;\alpha) = [f_1(x;\alpha), f_2(x;\alpha)]$, $f^*(x;\beta) = [f_1^*(x;\beta), f_2^*(x;\beta)]$ and $g(x;\alpha) = [g_1(x;\alpha), g_2(x;\alpha)]$, $g^*(x;\beta) = [g_1^*(x;\beta), g_2^*(x;\beta)]$ respectively. Since f and g are are strongly measurable and integrably bounded then $f + g$ is also strongly measurable and integrably bounded. Hence $f + g$ is Aumann integrable. By the definition of Aumann integration we can obtain the followings.

$$\begin{aligned}
\left(\int_a^b (f(x) + g(x))dx\right)(\alpha) &= \int_a^b (f(x) + g(x))(\alpha)dx \\
&= \int_a^b (f(x;\alpha) + g(x;\alpha))dx \\
&= \int_a^b [f_1(x;\alpha) + g_1(x;\alpha), f_2(x;\alpha) + g_2(x;\alpha)]dx \\
&= \int_a^b [f_1(x;\alpha), f_2(x;\alpha)]dx + \int_a^b [g_1(x;\alpha), g_2(x;\alpha)]dx.
\end{aligned}$$

and

$$\begin{aligned}
\left(\int_a^b (f(x) + g(x))dx\right)(\beta) &= \int_a^b [f_1^*(x;\beta) + g_1^*(x;\beta), f_2^*(x;\beta) + g_2^*(x;\beta)]dx \\
&= \int_a^b [f_1^*(x;\beta), f_2^*(x;\beta)]dx + \int_a^b [g_1^*(x;\beta), g_2^*(x;\beta)]dx.
\end{aligned}$$

Hence

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

is satisfied.

The proof of **2.** and **3.** can be done in a similar way. □

Theorem 4.2.6 *[22] If $f,g : [a,b] \to K_C(\mathbb{R}^n)$ are Aumann integrable, then $d_H(f,g) : [a,b] \to \mathbb{R}$ is integrable and*

$$d_H(\int_a^b f(t)dt, \int_a^b g(t)dt) \leq \int_a^b d_H(f(t),g(t))dt$$

Theorem 4.2.7 *If $f,g : [a,b] \to IF(\mathbb{R}^n)$ are Aumann integrable, then $D(f,g) : [a,b] \to \mathbb{R}$ is integrable and*

$$D_\infty(\int_a^b f(t)dt, \int_a^b g(t)dt) \leq \int_a^b D_\infty(f(t),g(t))dt$$

Proof We need to show that

$$\begin{aligned} D_1(\int_a^b f(t)dt, \int_a^b g(t)dt) &\leq \int_a^b D_1(f(t),g(t))dt \\ D_2(\int_a^b f(t)dt, \int_a^b g(t)dt) &\leq \int_a^b D_2(f(t),g(t))dt \end{aligned}$$

Since α and β cuts of an intuitionistic fuzzy number are non-empty compact and convex sets in $\mathbb{R}^n$,

$$D_\infty(\int_a^b f(t)dt, \int_a^b g(t)dt) \leq \int_a^b D_\infty(f(t),g(t))dt$$

holds by Theorem 4.2.6. □

Theorem 4.2.8 *Let $f : [a,b] \to IF(\mathbb{R})$ be a continuous function. Assume its α and β cuts by $f(t,\alpha) = [f_1(t,\alpha), f_2(t,\alpha)]$ and $f^*(t,\beta) = [f_1^*(t,\beta), f_2^*(t,\beta)]$ respectively. Then*

1. If $u(x) = \int_a^x f(t)dt$ is H-differentiable then $u'_H = f(x)$

2. If $u(x) = \int_x^b f(t)dt$ is H-differentiable then $u'_H = -f(x)$

Proof

1. Since f is a continuous function then it is Aumann integrable. So there exists $u \in IF(\mathbb{R})$ such that $u(x) = \int_a^x f(t)dt$. Then

$$\begin{aligned} u(x,\alpha) &= \int_a^x f(t,\alpha)dt \\ u^*(x,\beta) &= \int_a^x f^*(t,\alpha)dt. \end{aligned}$$

So

$$\begin{aligned}[u_1(x,\alpha),u_2(x,\alpha)] &= \int_a^x [f_1(t,\alpha),f_2(t,\alpha)]dt = [\int_a^x f_1(t,\alpha), \int_a^x f_2(t,\alpha)] \\ [u_1^*(x,\beta),u_2^*(x,\beta)] &= \int_a^x [f_1^*(t,\beta),f_2(t,\beta)]dt = [\int_a^x f_1^*(t,\beta), \int_a^x f_2(t,\beta)]dt\end{aligned}$$

Since $u(x)$ is H-differentiable then for each α and β in $[0,1]$, $u_1(x,\alpha)$, $u_2(x,\alpha)$, $u_1^*(x,\beta)$ and $u_2^*(x,\beta)$ are differentiable and by the fundamental theorem of calculus we obtain

$$\begin{aligned}[u_1'(x,\alpha),u_2'(x,\alpha)] &= [f_1(t,\alpha),f_2(t,\alpha)] \\ [(u_1^*)'(x,\beta),(u_2^*)'(x,\beta)] &= [f_1^*(t,\beta),f_2(t,\beta)]\end{aligned}$$

Hence, $u_H'(x) = f(x)$

2. This can be proved in a similar manner.

□

Theorem 4.2.9 *If $f : [a,b] \to IF(\mathbb{R})$ be an H-differentiable function on $[a,b]$ and $f_H'(t)$ is Aumann integrable then*

$$\int_a^b f_H'(t)dt = f(b) \ominus_H f(a)$$

holds.

Proof Assume α and β cuts of f are $f(t,\alpha) = [f_1(t,\alpha), f_2(t,\alpha)]$ and $f^*(t,\beta) = [f_1^*(t,\beta), f_2^*(t,\beta)]$, respectively. Since f is differentiable the following classical derivatives $f_1'(t,\alpha)$, $f_2'(t,\alpha)$, $(f_1^*)'(t,\beta)$ and $(f_2^*)'(t,\beta)]$ exist such that
$f'(t,\alpha) = [f_1'(t,\alpha), f_2'(t,\alpha)]$ and $(f^*)'(t,\beta) = [(f_1^*)'(t,\beta), (f_2^*)'(t,\beta)]$.

$$[\int_a^b f_H'(t)dt]^\alpha = f(b,\alpha) \ominus_H f(a,\alpha)$$

is proven in [22]. Similarly we can obtain that

$$\begin{aligned}[\int_a^b f_H'(t)dt]^\beta &= \int_a^b [(f_1^*)'(t,\beta),(f_2^*)'(t,\beta)]dt \\ &= [\int_a^b (f_1^*)'(t,\beta)dt, \int_a^b (f_2^*)'(t,\beta)dt] \\ &= [f_1^*(b,\beta) - f_1^*(a,\beta), f_2^*(b,\beta) - f_2^*(a,\beta)] \\ &= f^*(b,\beta) \ominus_H f^*(a,\beta)\end{aligned}$$

Since

$$f(b) = \int_a^b f_H'(t)dt + f(a)$$

holds if and only if

$$f(b,\alpha) = [\int_a^b f'_H(t)dt]^\alpha + f(a,\alpha) \text{ and } f^*(b,\beta) = [\int_a^b (f'_H)^*(t)dt]^\beta + f^*(a,\beta)$$

then we obtain

$$\int_a^b f'_H(t)dt = f(b) \ominus_H f(a)$$

□

5. SUMMARY AND RESULTS

In this paper, we have firstly defined a new distance function based on the well known Hausdorff metric in order to study the metric properties of intuitionistic fuzzy numbers by using α and β cuts. Then we have proved that this function satisfies the metric axioms and the space of intuitionistic fuzzy number valued functions is complete under the given metric. Besides, as a corollary we have proved that the space of continuous intuitionistic fuzzy number valued functions is complete with respect to the supremum metric D_s. And then, we have extended some fundamental theorems from fuzzy Aumann integration to intuitionistic fuzzy environment with the help of α and β cuts. Finally we have proved the relation between Hukuhara derivative and Aumann integral for intuitionistic fuzzy valued functions by using the fundamental theorem of calculus.

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