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Qualitative Behavior of Two Rational Difference Equations

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Abstract

Obtaining the exact solutions of most rational recursive equations is sophisticated sometimes. Therefore, a considerable number of nonlinear difference equations is often investigated by studying the qualitative behavior of the governing forms of these equations. The prime purpose of this work is to analyse the equilibria, local stability, global stability character, boundedness character and the solution behavior of the following fourth order fractional difference equations:

$$x_{n+1} = \frac{\alpha x_n x_{n-3}}{\beta x_{n-3} - \gamma x_{n-2}}, \quad x_{n+1} = \frac{\alpha x_n x_{n-3}}{-\beta x_{n-3} + \gamma x_{n-2}}, \quad n = 0, 1, ...,$$

where the constants $\alpha,\ \beta,\ \gamma \in \mathbb{R}^+$ and the initial values $x_{-3},\ x_{-2},\ x_{-1}$ and x_0 are required to be arbitrary non zero real numbers. Furthermore, some numerical figures will be obviously shown in this paper.

1. Introduction

The present paper aims to offer a significant analysis about local asymptotic stability, global attractivity and periodicity of the following rational recursive equations:

$$x_{n+1} = \frac{\alpha x_n x_{n-3}}{\beta x_{n-3} - \gamma x_{n-2}}, \quad x_{n+1} = \frac{\alpha x_n x_{n-3}}{-\beta x_{n-3} + \gamma x_{n-2}}, \quad n = 0, 1, ...,$$

where the initial data $x_{-3},\ x_{-2},\ x_{-1}$ and x_0 are required to be arbitrary non zero real numbers. Moreover, the parameters $\alpha,\ \beta$ and γ are required to be positive arbitrary values.

The theory of nonlinear difference equations has been extraordinarily developed in recent decades. Obviously, this development can be evidently seen in the studies which have been published on difference equations. Take, for instance, the following ones. Avotina [1] investigated the periodicity of three special cases from the fractional difference equation given by

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + B x_n + C x_{n-1}}.$$

Bajo et al. [2] analyzed the global character of the following second order recursive equation:

$$x_{n+1} = \frac{x_{n-1}}{a + b x_n x_{n-1}}.$$

Çınar [3] provided the solution of the next fractional recursive relation

$$x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}.$$

Din [4] explored some qualitative behaviors such as the stability and the periodicity of the following system:

$$x_{n+1} = \frac{a y_n}{b + c y_n}, \quad y_{n+1} = \frac{d y_n}{e + f x_n}.$$

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El-Moneam et al. [5] explored the qualitative behavior of the difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + Dx_{n-\sigma} + \frac{bx_{n-k} + hx_{n-l}}{dx_{n-k} + ex_{n-l}}.$$

Elsayed [6] obtained the forms of the solutions of the recursive relations given on the form:

$$x_{n+1} = \frac{x_n}{x_{n-1}(x_n \pm 1)}.$$

Ibrahim [7] examined the global and local stability of the second order recursive relation on the form:

$$x_{n+1} = \frac{ax_{n-1}}{-1 + bx_n x_{n-1}}.$$

More details on this aspect can be simply found in refs. [8], [9]-[14], [15].

2. On the recursive relation $x_{n+1} = \frac{\alpha x_n x_{n-3}}{\beta x_{n-3} - \gamma x_{n-2}}$

This section underlines widely some aspects and properties of the recursive equation

$$x_{n+1} = \frac{\alpha x_n x_{n-3}}{\beta x_{n-3} - \gamma x_{n-2}}, \quad n = 0, 1, 2, \dots, \tag{2.1}$$

where the initial values are required to be arbitrary constants. The parameters α, β and γ are as mentioned above.

2.1. Local stability analysis

The local behaviour of the fixed point of our equation will be proved under an intrinsic hypothesis in this subsection. The equilibrium point of Eq.(2.1) can be evaluated from the following equation:

$$\bar{x} = \frac{\alpha \bar{x}\bar{x}}{\beta \bar{x} - \gamma \bar{x}} = \frac{\alpha \bar{x}}{\beta - \gamma}.$$

This implies that

$$\bar{x} = 0.$$

Assume that a function $h : (0, \infty)^3 \longrightarrow (0, \infty)$ is described by the following form:

$$h(t, s, z) = \frac{\alpha t z}{\beta z - \gamma s}, \tag{2.2}$$

from which we can obtain that

$$\begin{aligned} \frac{\partial h(t,s,z)}{\partial t} &= \frac{\alpha z}{\beta z - \gamma s}, \\ \frac{\partial h(t,s,z)}{\partial s} &= \frac{\alpha \gamma t z}{(\beta z - \gamma s)^2}, \\ \frac{\partial h(t,s,z)}{\partial z} &= -\frac{\alpha \gamma t s}{(\beta z - \gamma s)^2}. \end{aligned} \tag{2.3}$$

These partial derivatives can be obviously calculated at $\bar{x} = 0$, as follows:

$$\begin{aligned} \frac{\partial h(\bar{x},\bar{x},\bar{x})}{\partial t} &= \frac{\alpha \bar{x}}{\beta \bar{x} - \gamma \bar{x}} = \frac{\alpha}{\beta - \gamma} = -p_2, \\ \frac{\partial h(\bar{x},\bar{x},\bar{x})}{\partial s} &= \frac{\alpha \gamma \bar{x}\bar{x}}{(\beta \bar{x} - \gamma \bar{x})^2} = \frac{\alpha \gamma}{(\beta - \gamma)^2} = -p_1, \\ \frac{\partial h(\bar{x},\bar{x},\bar{x})}{\partial z} &= -\frac{\alpha \gamma \bar{x}\bar{x}}{(\beta \bar{x} - \gamma \bar{x})^2} = -\frac{\alpha \gamma}{(\beta - \gamma)^2} = -p_0. \end{aligned}$$

Now, the corresponding linearized form of Eq.(2.1) about $\bar{x} = 0$, is given by

$$y_{n+1} + p_2 y_n + p_1 y_{n-2} + p_0 y_{n-3} = 0.$$

Theorem 2.1. *Let*

$$(\beta - \gamma)^2 > \max\{\alpha(\beta + \gamma),\ \alpha(3\gamma - \beta)\}.$$

Then, the fixed point of Eq.(2.1) is locally asymptotically stable.

Proof. According to Theorem A in **[16]**, Eq.(2.1) is said to be asymptotically stable if

$$|p_0|+|p_1|+|p_2|<1.$$

This expression leads to

$$\left|-\frac{\alpha\gamma}{(\beta-\gamma)^2}\right|+\left|\frac{\alpha\gamma}{(\beta-\gamma)^2}\right|+\left|\frac{\alpha}{\beta-\gamma}\right|<1.$$

- If $\beta>\gamma$, then

$$\frac{2\alpha\gamma}{(\beta-\gamma)^2}+\frac{\alpha}{\beta-\gamma}<1,$$

which can be easily simplified as

$$\alpha(\beta+\gamma)<(\beta-\gamma)^2. \tag{2.4}$$

- If $\beta<\gamma$, then

$$\frac{2\alpha\gamma}{(\beta-\gamma)^2}+\frac{\alpha}{\gamma-\beta}<1.$$

Therefore,

$$\alpha(3\gamma-\beta)<(\beta-\gamma)^2. \tag{2.5}$$

Combining condition (2.4) with condition (2.5) gives us

$$(\beta-\gamma)^2>\max\left\{\alpha(\beta+\gamma),\ \alpha(3\gamma-\beta)\right\}.$$

This achieves the proof completely. □

2.2. Global stability analysis

Here, we will present an approach to determine the global behavior of Eq.(2.1). In this equation, two different cases will emerge as illustrated in the following fundamental theorem.

Theorem 2.2. *The fixed point of Eq.(2.1) is said to be a global attractor if* $\alpha\neq\gamma$.

Proof. Suppose that $r_1,\ r_2\in\mathbb{R}$ and let $h:[r_1,r_2]^3\longrightarrow[r_1,r_2]$ be a function defined by Eq.(2.2). Then, we take into consideration the following situations.
Case 1: Let $\beta z<\gamma s$ be true. Then, equations (2.3) tell us that Eq.(2.2) is nondecreasing in s and nonincreasing in t and z. Next, let (φ,χ) be a solution of the following system:

$$\begin{aligned}\varphi &= h(\chi,\varphi,\chi)=\frac{\alpha\chi^2}{\beta\chi-\gamma\varphi},\\ \chi &= h(\varphi,\chi,\varphi)=\frac{\alpha\varphi^2}{\beta\varphi-\gamma\chi}.\end{aligned}$$

Or,

$$\beta\varphi\chi-\gamma\varphi^2=\alpha\chi^2, \tag{2.6}$$

$$\beta\varphi\chi-\gamma\chi^2=\alpha\varphi^2. \tag{2.7}$$

Subtracting Eq.(2.6) from Eq.(2.7) gives

$$\gamma(\chi^2-\varphi^2)=\alpha\left(\chi^2-\varphi^2\right).$$

Now, if $\gamma\neq\alpha$, we have

$$\varphi=\chi.$$

As claimed by Theorem B in **[17]**, the fixed point of Eq.(2.1) is a global attractor.
Case 2: This case shows the global behaviour when $\beta z>\gamma s$. The proof of this case is similar to the previous one. □

Remark 2.3. *Eq.(2.1) is not prime period two.*

2.3. Special case of eq.(2.1)

In the following paragraph, we will specify an effective theorem to verify the periodicity of the solution of the following fourth order recursive relation:

$$x_{n+1}=\frac{x_n x_{n-3}}{x_{n-3}-x_{n-2}}, \tag{2.8}$$

where the initial values are as illustrated above.

Theorem 2.4. *Each solution of Eq.(2.8) is periodic with period eighteen.*

Proof. We assume that $\{x_n\}_{n=-3}^{\infty}$ is a solution of Eq.(2.8), then

$$\begin{aligned}
x_{n+1} &= \frac{x_n x_{n-3}}{x_{n-3}-x_{n-2}},\\
x_{n+2} &= \frac{x_{n+1}x_{n-2}}{x_{n-2}-x_{n-1}} = \frac{\left(\frac{x_n x_{n-3}}{x_{n-3}-x_{n-2}}\right)x_{n-2}}{x_{n-2}-x_{n-1}} = \frac{x_{n-3}x_{n-2}x_n}{\left(x_{n-3}-x_{n-2}\right)\left(x_{n-2}-x_{n-1}\right)},\\
x_{n+3} &= \frac{x_{n+2}x_{n-1}}{x_{n-1}-x_n} = \frac{\left(\frac{x_{n-3}x_{n-2}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})}\right)x_{n-1}}{x_{n-1}-x_n}\\
&= \frac{x_{n-3}x_{n-2}x_{n-1}x_n}{\left(x_{n-3}-x_{n-2}\right)\left(x_{n-2}-x_{n-1}\right)\left(x_{n-1}-x_n\right)},\\
x_{n+4} &= \frac{x_{n+3}x_n}{x_n-x_{n+1}} = \frac{\left(\frac{x_{n-3}x_{n-2}x_{n-1}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}\right)x_n}{x_n-\frac{x_n x_{n-3}}{x_{n-3}-x_{n-2}}}\\
&= -\frac{x_{n-3}x_{n-1}x_n}{\left(x_{n-2}-x_{n-1}\right)\left(x_{n-1}-x_n\right)},\\
x_{n+5} &= \frac{x_{n+4}x_{n+1}}{x_{n+1}-x_{n+2}} = \frac{\left(-\frac{x_{n-3}x_{n-1}x_n}{(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}\right)\left(\frac{x_n x_{n-3}}{x_{n-3}-x_{n-2}}\right)}{\frac{x_n x_{n-3}}{x_{n-3}-x_{n-2}}-\frac{x_{n-3}x_{n-2}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})}} = \frac{x_{n-3}x_n}{\left(x_{n-1}-x_n\right)},\\
x_{n+6} &= \frac{x_{n+5}x_{n+2}}{x_{n+2}-x_{n+3}}\\
&= \frac{\left(\frac{x_{n-3}x_n}{x_{n-1}-x_n}\right)\left(\frac{x_{n-3}x_{n-2}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})}\right)}{\frac{x_{n-3}x_{n-2}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})}-\frac{x_{n-3}x_{n-2}x_{n-1}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}} = -x_{n-3},\\
x_{n+7} &= \frac{x_{n+6}x_{n+3}}{x_{n+3}-x_{n+4}}\\
&= \frac{-x_{n-3}\left(\frac{x_{n-3}x_{n-2}x_{n-1}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}\right)}{\frac{x_{n-3}x_{n-2}x_{n-1}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}+\frac{x_{n-3}x_{n-1}x_n}{(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}} = -x_{n-2},\\
x_{n+8} &= \frac{x_{n+7}x_{n+4}}{x_{n+4}-x_{n+5}} = \frac{-x_{n-2}\left(-\frac{x_{n-3}x_{n-1}x_n}{(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}\right)}{-\frac{x_{n-3}x_{n-1}x_n}{(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}-\frac{x_{n-3}x_n}{(x_{n-1}-x_n)}} = -x_{n-1},\\
x_{n+9} &= \frac{x_{n+8}x_{n+5}}{x_{n+5}-x_{n+6}} = \frac{-x_{n-1}\left(\frac{x_{n-3}x_n}{x_{n-1}-x_n}\right)}{\frac{x_{n-3}x_n}{x_{n-1}-x_n}+x_{n-3}} = -x_n,\\
x_{n+10} &= \frac{x_{n+9}x_{n+6}}{x_{n+6}-x_{n+7}} = \frac{-x_n\left(-x_{n-3}\right)}{-x_{n-3}+x_{n-2}} = -\frac{x_n x_{n-3}}{x_{n-3}-x_{n-2}},\\
x_{n+11} &= \frac{x_{n+10}x_{n+7}}{x_{n+7}-x_{n+8}} = \frac{\left(-\frac{x_n x_{n-3}}{x_{n-3}-x_{n-2}}\right)\left(-x_{n-2}\right)}{-x_{n-2}+x_{n-1}}\\
&= -\frac{x_{n-3}x_{n-2}x_n}{\left(x_{n-3}-x_{n-2}\right)\left(x_{n-2}-x_{n-1}\right)},
\end{aligned}$$

$$
\begin{aligned}
x_{n+12} &= \frac{x_{n+11}x_{n+8}}{x_{n+8}-x_{n+9}} = \frac{\left(-\frac{x_{n-3}x_{n-2}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})}\right)(-x_{n-1})}{-x_{n-1}+x_n} \\
&= -\frac{x_{n-3}x_{n-2}x_{n-1}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}, \\
x_{n+13} &= \frac{x_{n+12}x_{n+9}}{x_{n+9}-x_{n+10}} = \frac{\left(-\frac{x_{n-3}x_{n-2}x_{n-1}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}\right)(-x_n)}{-x_n+\frac{x_nx_{n-3}}{x_{n-3}-x_{n-2}}} \\
&= \frac{x_{n-3}x_{n-1}x_n}{(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}, \\
x_{n+14} &= \frac{x_{n+13}x_{n+10}}{x_{n+10}-x_{n+11}} = \frac{\left(\frac{x_{n-3}x_{n-1}x_n}{(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}\right)\left(-\frac{x_nx_{n-3}}{x_{n-3}-x_{n-2}}\right)}{-\frac{x_nx_{n-3}}{x_{n-3}-x_{n-2}}+\frac{x_{n-3}x_{n-2}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})}} \\
&= -\frac{x_{n-3}x_n}{(x_{n-1}-x_n)}, \\
x_{n+15} &= \frac{x_{n+14}x_{n+11}}{x_{n+11}-x_{n+12}} \\
&= \frac{\left(-\frac{x_{n-3}x_n}{x_{n-1}-x_n}\right)\left(-\frac{x_{n-3}x_{n-2}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})}\right)}{-\frac{x_{n-3}x_{n-2}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})}+\frac{x_{n-3}x_{n-2}x_{n-1}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}} = x_{n-3},
\end{aligned}
$$

$$
\begin{aligned}
x_{n+16} &= \frac{x_{n+15}x_{n+12}}{x_{n+12}-x_{n+13}} \\
&= \frac{x_{n-3}\left(-\frac{x_{n-3}x_{n-2}x_{n-1}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}\right)}{-\frac{x_{n-3}x_{n-2}x_{n-1}x_n}{(x_{n-3}-x_{n-2})(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}-\frac{x_{n-3}x_{n-1}x_n}{(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}} = x_{n-2}, \\
x_{n+17} &= \frac{x_{n+16}x_{n+13}}{x_{n+13}-x_{n+14}} = \frac{x_{n-2}\left(\frac{x_{n-3}x_{n-1}x_n}{(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}\right)}{\frac{x_{n-3}x_{n-1}x_n}{(x_{n-2}-x_{n-1})(x_{n-1}-x_n)}+\frac{x_{n-3}x_n}{(x_{n-1}-x_n)}} = x_{n-1}, \\
x_{n+18} &= \frac{x_{n+17}x_{n+14}}{x_{n+14}-x_{n+15}} = \frac{x_{n-1}\left(-\frac{x_{n-3}x_n}{(x_{n-1}-x_n)}\right)}{-\frac{x_{n-3}x_n}{(x_{n-1}-x_n)}-x_{n-3}} = x_n.
\end{aligned}
$$

The proof has been completely done. □

2.4. Numerical confirmation

To confirm our theoretical outcomes in the previous subsections, we will provide some concrete numerical examples in this subsection.

Example 2.5. *Figure 2.1 is sketched according to the following values:* $\alpha=\gamma=1,\ \beta=6,\ x_{-3}=x_0=0.2$, *and* $x_{-1}=-x_{-2}=0.1$.

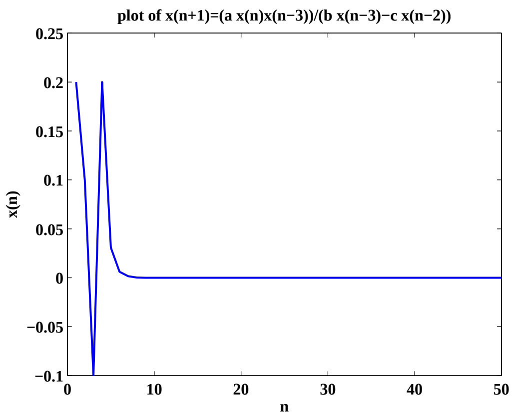

Figure 2.1

Example 2.6. *We consider* $\alpha=10,\ \ \beta=2,\ \gamma=1,\ x_{-3}=0.5,\ x_{-2}=x_0=1$ *and* $x_{-1}=-1$, *to depict the Figure 2.2.*

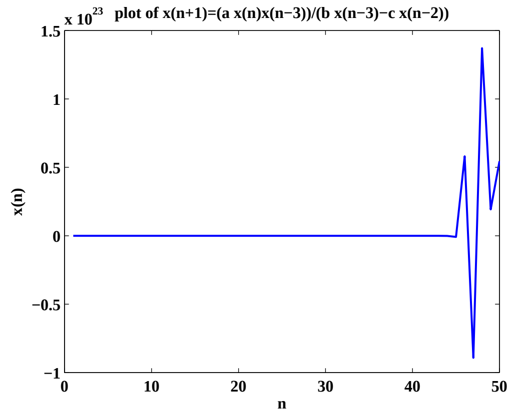

Figure 2.2

Example 2.7. *This example illustrates the periodicity of the special case equation when we take* $x_{-3}=x_{-1}=-0.1$ *and* $x_{-2}=x_0=0.1$. *See Figure 2.3.*

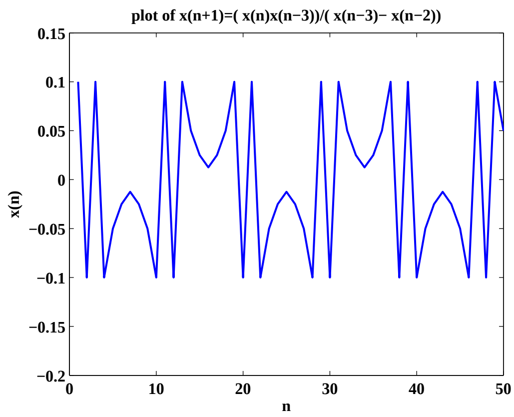

Figure 2.3

3. On the recursive relation $x_{n+1}=\frac{\alpha x_n x_{n-3}}{-\beta x_{n-3}+\gamma x_{n-2}}$

This section will offer various mathematical aspects of the following recursive form:

$$x_{n+1}=\frac{\alpha x_n x_{n-3}}{-\beta x_{n-3}+\gamma x_{n-2}}, \quad n=0,1,\ldots . \tag{3.1}$$

The initial data and the arbitrary constants are as mentioned above.

3.1. Local stability analysis

In this part, the behaviour of the solutions in the neighbourhood of the fixed point will be established via a key theorem. The fixed point of Eq.(3.1) can be simply found from the equation given by

$$\bar{x}=\frac{\alpha\bar{x}\bar{x}}{-\beta\bar{x}+\gamma\bar{x}}=\frac{\alpha\bar{x}}{-\beta+\gamma}.$$

This gives us

$$\bar{x}=0.$$

Assume that a function $h:(0,\infty)^3\longrightarrow(0,\infty)$ is described as follows:

$$h(t,s,z)=\frac{\alpha tz}{-\beta z+\gamma s}. \tag{3.2}$$

Then,

$$\begin{aligned}\frac{\partial h(t,s,z)}{\partial t} &= \frac{\alpha z}{-\beta z+\gamma s},\\ \frac{\partial h(t,s,z)}{\partial s} &= -\frac{\alpha\gamma tz}{(-\beta z+\gamma s)^2},\\ \frac{\partial h(t,s,z)}{\partial z} &= \frac{\alpha\gamma ts}{(-\beta z+\gamma s)^2}.\end{aligned} \tag{3.3}$$

Finding these partial derivatives at $\bar{x}=0$, yields

$$\begin{aligned}\frac{\partial h(\bar{x},\bar{x},\bar{x})}{\partial t} &= \frac{\alpha\bar{x}}{-\beta\bar{x}+\gamma\bar{x}} = \frac{\alpha}{\gamma-\beta} = -p_2,\\ \frac{\partial h(\bar{x},\bar{x},\bar{x})}{\partial s} &= -\frac{\alpha\gamma\bar{x}\bar{x}}{(-\beta\bar{x}+\gamma\bar{x})^2} = -\frac{\alpha\gamma}{(\gamma-\beta)^2} = -p_1,\\ \frac{\partial h(\bar{x},\bar{x},\bar{x})}{\partial z} &= \frac{\alpha\gamma\bar{x}\bar{x}}{(-\beta\bar{x}+\gamma\bar{x})^2} = \frac{\alpha\gamma}{(\gamma-\beta)^2} = -p_0.\end{aligned}$$

Following, the corresponding linearized scheme of Eq.(3.1) about $\bar{x}=0$, is

$$y_{n+1}+p_2y_n+p_1y_{n-2}+p_0y_{n-3}=0.$$

Theorem 3.1. *Assume that*

$$(\gamma-\beta)^2>\max\{\alpha(\beta+\gamma),\ \alpha(3\gamma-\beta)\}.$$

Then, the point $\bar{x}=0$, is locally asymptotically stable.

Proof. As stated by Theorem A in **[16]**, Eq.(3.1) is said to be asymptotically stable if

$$|p_0|+|p_1|+|p_2|<1,$$

which implies that

$$\left|\frac{\alpha\gamma}{(\gamma-\beta)^2}\right|+\left|-\frac{\alpha\gamma}{(\gamma-\beta)^2}\right|+\left|\frac{\alpha}{\gamma-\beta}\right|<1.$$

- If $\beta<\gamma$, then

$$\frac{2\alpha\gamma}{(\gamma-\beta)^2}+\frac{\alpha}{\gamma-\beta}<1.$$

Therefore,

$$\alpha(3\gamma-\beta)<(\gamma-\beta)^2. \tag{3.4}$$

- If $\beta>\gamma$, then

$$\frac{2\alpha\gamma}{(\gamma-\beta)^2}-\frac{\alpha}{\gamma-\beta}<1,$$

which can be easily reduced to

$$\alpha(\gamma+\beta)<(\gamma-\beta)^2. \tag{3.5}$$

Finally, combining condition (3.4) with condition (3.5) leads to

$$(\gamma-\beta)^2>\max\{\alpha(\beta+\gamma),\ \alpha(3\gamma-\beta)\},$$

which is what we require to prove. □

3.2. Global stability analysis

We now turn to analyze the global attractivity of Eq.(3.1), in which two various cases are arisen.

Theorem 3.2. *The fixed point of Eq.(3.1) is a global attractor.*

Proof. Assume that $r_1,\ r_2 \in \mathbb{R}$ and let $h : [r_1, r_2]^3 \longrightarrow [r_1, r_2]$ be a function defined by Eq.(3.2). Then, we examine the next two cases.
Case 1: Let $\beta z < \gamma s$ be true. Then, from equations (3.3) we observe that Eq.(3.2) is nondecreasing in t and z and nonincreasing in s. Now, suppose that (φ, χ) is a solution of the following rational system:

$$\begin{aligned} \varphi &= h(\varphi,\chi,\varphi) = \frac{\alpha\varphi^2}{-\beta\varphi+\gamma\chi}, \\ \chi &= h(\chi,\varphi,\chi) = \frac{\alpha\chi^2}{-\beta\chi+\gamma\varphi}. \end{aligned}$$

Obviously, this system can be written as

$$-\beta\varphi^2+\gamma\varphi\chi = \alpha\varphi^2, \tag{3.6}$$

$$-\beta\chi^2+\gamma\varphi\chi = \alpha\chi^2. \tag{3.7}$$

Subtracting Eq.(3.6) from Eq.(3.7) leads to

$$\beta(\chi^2-\varphi^2) = \alpha(\varphi^2-\chi^2).$$

Hence,

$$(\beta+\gamma)(\chi-\varphi)(\chi+\varphi) = 0.$$

This implies that

$$\varphi = \chi.$$

As claimed by Theorem B in **[17]**, the point $\bar{x} = 0$, is a global attractor.
Case 2: In this case we consider $\beta z > \gamma s$. The proof can be achieved in a similar way to the previous one. □

Remark 3.3. *Eq.(3.1) is not prime period two.*

3.3. Special case of eq.(3.1)

Now, we will formulate the solution of the recursive equation which is given as follows:

$$x_{n+1} = \frac{x_n x_{n-3}}{x_{n-2}-x_{n-3}}, \quad n = 0,1,\ldots. \tag{3.8}$$

The initial values are required to be nonzero real numbers.

Theorem 3.4. *Suppose that $\{x_n\}_{n=-3}^{\infty}$ is a solution of Eq.(3.8) and satisfying $x_{-3} = a,\ x_{-2} = b,\ x_{-1} = c$ and $x_0 = d$. Then, for $n = 0,1,\ldots$*

$$\begin{aligned} x_{3n-3} &= \frac{(-1)^{n-1}abcd}{(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)(f_{n-1}c-f_{n-2}d)}, \\ x_{3n-2} &= \frac{(-1)^{n}abcd}{(f_{n}a-f_{n-1}b)(f_{n-1}b-f_{n-2}c)(f_{n-1}c-f_{n-2}d)}, \\ x_{3n-1} &= \frac{(-1)^{n+1}abcd}{(f_{n}a-f_{n-1}b)(f_{n}b-f_{n-1}c)(f_{n-1}c-f_{n-2}d)}, \end{aligned}$$

where $\{f_n\}_{n=-2}^{\infty}$, is called Fibonacci sequence.

Proof. It can be clearly seen that the solution is confirmed for $n = 0$. Next, we assume that $n > 0$ and the above-mentioned results hold for $n-1$. This leads to that

$$\begin{aligned} x_{3n-7} &= \frac{(-1)^{n-1}abcd}{(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-3}c-f_{n-4}d)}, \\ x_{3n-6} &= \frac{(-1)^{n-2}abcd}{(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)}, \\ x_{3n-5} &= \frac{(-1)^{n-1}abcd}{(f_{n-1}a-f_{n-2}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)}, \\ x_{3n-4} &= \frac{(-1)^{n}abcd}{(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)(f_{n-2}c-f_{n-3}d)}. \end{aligned}$$

Next, from Eq. (3.8) we have

$$
\begin{aligned}
x_{3n-3} &= \frac{x_{3n-4}x_{3n-7}}{x_{3n-6}-x_{3n-7}} \\
&= \frac{\left(\frac{(-1)^{n}abcd}{(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)(f_{n-2}c-f_{n-3}d)}\right)\left(\frac{(-1)^{n-1}abcd}{(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-3}c-f_{n-4}d)}\right)}{\left[\frac{(-1)^{n-2}abcd}{(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)}-\frac{(-1)^{n-1}abcd}{(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-3}c-f_{n-4}d)}\right]} \\
&= \frac{(-1)^{2n-1}(abcd)^{2}(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-3}c-f_{n-4}d)}{(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)(f_{n-2}c-f_{n-3}d)(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-3}c-f_{n-4}d)(abcd)\left[\begin{array}{c}(-1)^{n-2}(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-3}c-f_{n-4}d)\\ -(-1)^{n-1}(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)\end{array}\right]} \\
&= \frac{(-1)^{2n-1}(abcd)(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)}{(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)\left[(-1)^{n-2}(f_{n-3}c-f_{n-4}d)-(-1)^{n-1}(f_{n-2}c-f_{n-3}d)\right]} \\
& \frac{(-1)^{2n-1}(abcd)}{(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)(-1)^{n}\left[(f_{n-3}c-f_{n-4}d)+(f_{n-2}c-f_{n-3}d)\right]} \\
&= \frac{(-1)^{n-1}(abcd)}{(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)\left[(f_{n-1}c-f_{n-2}d)\right]}.
\end{aligned}
$$

We now turn to prove the second solution of our equation. Again, from Eq. (3.8) we have

$$
\begin{aligned}
x_{3n-2} &= \frac{x_{3n-3}x_{3n-6}}{-x_{3n-6}+x_{3n-5}} \\
&= \frac{\left(\frac{(-1)^{n-1}abcd}{(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)(f_{n-1}c-f_{n-2}d)}\right)\left(\frac{(-1)^{n-2}abcd}{(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)}\right)}{\left[\left(-\frac{(-1)^{n-2}abcd}{(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)}\right)+\left(\frac{(-1)^{n-1}abcd}{(f_{n-1}a-f_{n-2}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)}\right)\right]} \\
&= \frac{(-1)^{2n-3}(abcd)^{2}(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)(f_{n-1}a-f_{n-2}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)}{(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)(f_{n-1}c-f_{n-2}d)(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)(abcd)\left[\begin{array}{c}-(-1)^{n-2}(f_{n-1}a-f_{n-2}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)\\ +(-1)^{n-1}(f_{n-2}a-f_{n-3}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)\end{array}\right]}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(-1)^{2n-3}(abcd)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)}{(f_{n-1}b-f_{n-2}c)(f_{n-1}c-f_{n-2}d)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)\left[-(-1)^{n-2}(f_{n-1}a-f_{n-2}b)+(-1)^{n-1}(f_{n-2}a-f_{n-3}b)\right]} \\
&= \frac{(-1)^{2n-3}abcd}{(f_{n-1}b-f_{n-2}c)(f_{n-1}c-f_{n-2}d)(-1)^{n-1}\left[(f_{n}a-f_{n-1}b)\right]} \\
&= \frac{(-1)^{n}abcd}{(f_{n-1}b-f_{n-2}c)(f_{n-1}c-f_{n-2}d)(f_{n}a-f_{n-1}b)}.
\end{aligned}
$$

Finally, we will show the last part of the solution. Eq.(3.8) leads to

$$
\begin{aligned}
x_{3n-1} &= \frac{x_{3n-2}x_{3n-5}}{-x_{3n-5}+x_{3n-4}} \\
&= \frac{\left(\frac{(-1)^n abcd}{(f_{n-1}b-f_{n-2}c)(f_{n-1}c-f_{n-2}d)(f_na-f_{n-1}b)}\right)\left(\frac{(-1)^{n-1}abcd}{(f_{n-1}a-f_{n-2}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)}\right)}{-\left[\frac{(-1)^{n-1}abcd}{(f_{n-1}a-f_{n-2}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)}+\frac{(-1)^{n}abcd}{(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)(f_{n-2}c-f_{n-3}d)}\right]} \\
&= \frac{\dfrac{(-1)^{2n-1}(abcd)^2(f_{n-1}a-f_{n-2}b)(f_{n-2}b-f_{n-3}c)}{(f_{n-2}c-f_{n-3}d)(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)(f_{n-2}c-f_{n-3}d)}}{\dfrac{(f_{n-1}b-f_{n-2}c)(f_{n-1}c-f_{n-2}d)(f_na-f_{n-1}b)(f_{n-1}a-f_{n-2}b)}{(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)(abcd)}\left[\begin{array}{c}-(-1)^{n-1}(f_{n-1}a-f_{n-2}b)(f_{n-1}b-f_{n-2}c)(f_{n-2}c-f_{n-3}d)\\+(-1)^n(f_{n-1}a-f_{n-2}b)(f_{n-2}b-f_{n-3}c)(f_{n-2}c-f_{n-3}d)\end{array}\right]} \\
&= \frac{(-1)^{2n-1}(abcd)(f_{n-1}a-f_{n-2}b)(f_{n-2}c-f_{n-3}d)}{\begin{array}{c}(f_na-f_{n-1}b)(f_{n-1}c-f_{n-2}d)(f_{n-1}a-f_{n-2}b)\\(f_{n-2}c-f_{n-3}d)(-1)^n[(f_{n-2}b-f_{n-3}c)+(f_{n-1}b-f_{n-2}c)]\end{array}} \\
&= \frac{(-1)^{n-1}abcd}{(f_na-f_{n-1}b)(f_{n-1}c-f_{n-2}d)[(f_nb-f_{n-1}c)]} \\
&= \frac{(-1)^{n+1-2}abcd}{(f_na-f_{n-1}b)(f_{n-1}c-f_{n-2}d)[(f_nb-f_{n-1}c)]} \\
&= \frac{(-1)^{n+1}abcd}{(f_na-f_{n-1}b)(f_{n-1}c-f_{n-2}d)(f_nb-f_{n-1}c)}.
\end{aligned}
$$

□

3.4. Numerical confirmation

This subsection is included to verify and confirm the results we obtained in this work.

Example 3.5. *This example pictured the stability of the fixed point when we take* $\alpha=\beta=1,\ \gamma=7,\ x_{-3}=-3,\ x_{-2}=3,\ x_{-1}=-5$ *and* $x_0=5$. *See Figure 3.1.*

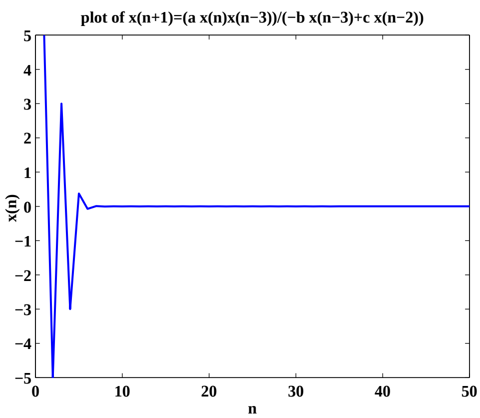

Figure 3.1

Example 3.6. *In Figure 3.2, we consider* $\alpha=15,\ \beta=1,\ \gamma=14,\ x_{-3}=0.1,\ x_{-2}=-0.5,\ x_{-1}=1$ *and* $x_0=-1$.

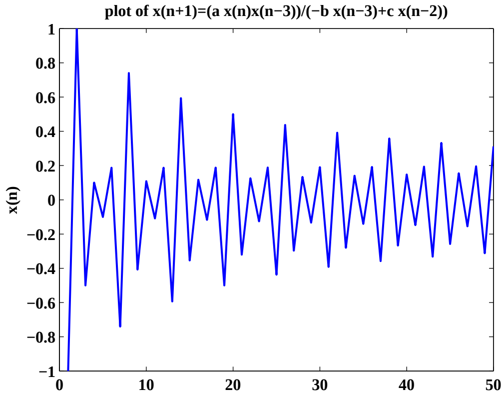

Figure 3.2

Example 3.7. *The stability of Eq.(3.8) is shown in Figure 3.3, when we let* $x_{-3}=5,\ x_{-2}=-8,\ x_{-1}=10$ *and* $x_0=-10$.

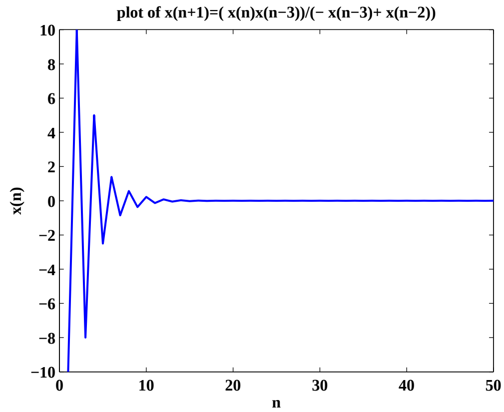

Figure 3.3

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