Fundamental Journal of Mathematics and Applications, 1 (2) (2018) 212-219
Research Article

FUJMA

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma
ISSN: 2645-8845

$\alpha_\kappa-$Implicit Contraction in non-AMMS with Some Applications

Ekber Girgin[a] and Mahpeyker Öztürk[a*]

[a]*Department of Mathematics, Faculty of Science and Arts, Sakarya University, Sakarya, Turkey*
[*]*Corresponding author*

Article Info

Keywords: *Implicit contraction, non-Archimedean modular metric space, Ulam-Hyers stability, Well-posedness.*
2010 AMS: *25A12, 34G10, 34H15*
Received: *5 December 2018*
Accepted: *23 December 2018*
Available online: *25 December 2018*

Abstract

In this article, we establish $\alpha_\kappa-$implicit contraction and provide some fixed point results in non-AMMS. Our results progress and generalize some famous consequences in a suitable resource. As an implementation, we study stability in the sense of Ulam-Hyers and a fixed point problem's well-posedness. In addition, some examples are given for new concepts. Also, an application to integral equations is discussed.

1. Some basic concepts and definitions

In this work, we will write MMS to modular metric space and non-AMMS to non-Archimedean modular metric space. In 2010, Chistyakov [1], [2] defined a new generalized space which is a modular metric space and introduced basic concepts and topological properties.
Let M be a nonempty set, a function $\kappa : (0,\infty)\times M\times M\to[0,\infty]$ be defined

$$\kappa_\lambda(\xi,\eta)=\kappa(\lambda,\xi,\eta)$$

for all $\lambda>0$ and $\xi,\eta\in M$.

Definition 1.1. *A function $\kappa : (0,\infty)\times M\times M\to[0,\infty]$ is named a modular metric if the following conditions are supplied:*

(i) $\xi=\eta\Leftrightarrow\kappa_\lambda(\xi,\eta)=0$*, for all* $\lambda>0$*;*
(ii) $\kappa_\lambda(\xi,\eta)=\kappa_\lambda(\eta,\xi)$*, for all* $\lambda>0$ *and* $\xi,\eta\in M$*;*
(iii) $\kappa_{\lambda+\mu}(\xi,\eta)\le\kappa_\lambda(\xi,\nu)+\kappa_\mu(\nu,\eta)$*, for all* $\lambda,\ \mu>0$ *and* $\xi,\eta,\nu\in M$*.*

Then, M_κ *is named an MMS.*

In the above definition, if we make use of the condition:

(i_1) $\kappa_\lambda(\xi,\xi)=0$ for all $\lambda>0$ and $\xi\in M$,

instead of (i), then M_κ is a pseudomodular metric space. M_κ is called regular if the condition (i) is supplied as:

$$\xi=\eta \quad \text{if and only if} \quad \kappa_\lambda(\xi,\eta)=0 \quad \text{for some} \quad \lambda>0.$$

The space M_κ is named convex if for $\lambda,\ \mu>0$ and $\xi,\eta,\nu\in M$, the condition supplies:

$$\kappa_{\lambda+\mu}(\xi,\eta)\le\frac{\lambda}{\lambda+\mu}\kappa_\lambda(\xi,\nu)+\frac{\mu}{\lambda+\mu}\kappa_\mu(\nu,\eta).$$

Definition 1.2. *[1], [2] recognised that κ be a pseudomodular on M and $\xi_0\in M$ and fixed. The sets:*

$$M_\kappa=M_\kappa(\xi_0)=\{\xi\in M:\kappa_\lambda(\xi,\xi_0) \quad as \quad \lambda\to\infty\}$$

and

$$\mathrm{M}^*_\kappa=\mathrm{M}^*_\kappa(\xi_0)=\{\xi\in M:\exists\lambda=\lambda(\xi)>0 \text{ such that } \kappa_\lambda(\xi,\xi_0)<\infty\}$$

are identified modular spaces (around ξ_0*).*

Email addresses and ORCID numbers: **girginekber@gmail.com, 0000-0002-8913-5416 (E. Girgin), mahpeykero@sakarya.edu.tr, 0000-0003-2946-6114 (M. Öztürk)**

It is trivial that $M_\kappa \subset M_\kappa^*$. Suppose that κ is a modular on M; from [1], [2], it can be obtained that the modular space M_κ can be settled with a (nontrivial) metric, induced by κ and given by:

$$d_\kappa(\xi,\eta) = \inf\{\lambda > 0 : \kappa_\lambda(\xi,\eta) < \lambda\},$$

for all $\xi,\eta \in M_\kappa$.
Consider that if κ is a convex modular on M, then specify [1], [2], the two modular space coincide, i.e., $M_\kappa = M_\kappa^*$, and this common set can be defined with the metric d_κ^* given by:

$$d_\kappa^*(\xi,\eta) = \inf\{\lambda > 0 : \kappa_\lambda(\xi,\eta) < 1\},$$

for all $\xi,\eta \in M_\kappa$. These distances are named Luxemburg distances.

Definition 1.3. *[3] Let M_κ be a MMS, A be a subset and $(s_n)_{n\in N}$ be a sequence in M_κ. Therefore:*

(1) $(s_n)_{n\in N}$ is named $\kappa-$convergent to $\xi \in M_\kappa$ if and only if $\kappa_\lambda(s_n,\xi) \to 0$ as $n \to \infty$ for all $\lambda > 0$. ξ will be called the $\kappa-$limit of (s_n).
(2) If for all $\lambda > 0$, $\kappa_\lambda(s_n,s_m) \to 0$, as $m,n \to \infty$, $(s_n)_{n\in N}$ is called $\kappa-$Cauchy.
(3) A is called $\kappa-$closed if the $\kappa-$limit of $\kappa-$convergent of A always belong to A.
(4) If any $\kappa-$Cauchy sequence in A is $\kappa-$convergent, then A is named $\kappa-$complete.
(5) A is called $\kappa-$bounded if for all $\lambda > 0$, we have

$$\delta_\omega(A) = \sup\{\kappa_\lambda(\xi,\eta); \xi,\eta \in A\} < \infty.$$

Paknazar et al. [4] modified the third condition of MMS.

Definition 1.4. *If in Definition 1.1, we exchange (iii) by:*

(iv) $\kappa_{\max\{\lambda,\mu\}}(\xi,\eta) \le \kappa_\lambda(\xi,\nu) + \kappa_\mu(\nu,\eta),$

for all λ, $\mu > 0$ and $\xi,\eta,\nu \in M_\kappa$, then, M_κ is called non-AMMS.

Now, denote N the set of positive integers, the set of real numbers R and Ψ the set of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying:

(ψ_1) ψ is nondecreasing,
(ψ_2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each R^+, where ψ^n is the nth iterate of ψ.

Remark 1.5. *It is trivial that if $\psi \in \Psi$, then $\psi(t) < t$ for any $t > 0$.*

Definition 1.6. *[5] Let Γ be the set of all functions $\wp(t_1,...,t_6) : R_+^6 \to R$ satisfying:*

($\wp_1$) $\wp$ is nondecreasing in variable t_1 and nonincreasing in variable t_5,
($\wp_2$) there exists $\psi \in \Psi$ such that for all $u,v \ge 0$, $\wp(u,v,v,u,u+v,0) \le 0$ implies $u \le \psi(v)$, and $\wp(u,v,u,v,0,u+v) \le 0$ implies $u \le \psi(v)$.

Samet et al. [6] characterize a new notion by defining $\alpha-$admissible mapping.

Definition 1.7. *[6] Let $\alpha : M \times M \to [0,\infty)$ be a function. A mapping $\hbar : M \to M$ satisfying*

$$\alpha(\xi,\eta) \ge 1 \quad \Rightarrow \quad \alpha(\hbar\xi,\hbar\eta) \ge 1, \tag{1.1}$$

if for all $\xi,\eta \in M$, is called as $\alpha-$admissible mapping.

Example 1.8. *[6] Let $M = (0,\infty)$ and define $\hbar : M \to M$ and $\alpha : M \times M \to [0,\infty)$ by*

$$\hbar\xi = \ln\xi, \quad \text{for all } \xi \in M$$

and

$$\alpha(\xi,\eta) = \begin{cases} 2 & \text{if } \xi \ge \eta, \\ 0 & \text{if } \xi < \eta. \end{cases}$$

Then, $\hbar$ is an $\alpha-$admissible mapping.

Such papers related to above concept imagined to obtain some fixed and common fixed point results (see [7] [8], [9], [10]).

2. $\alpha_\kappa-$implicit contraction and fixed point results

In the sequel the function κ is convex and regular.

Definition 2.1. *Let M_κ be a non-AMMS. A mapping given as $\hbar : M_\kappa \to M_\kappa$ is called $\alpha_\kappa-$implicit contraction if there are two functions $\alpha : M_\kappa \times M_\kappa \to [0,\infty)$ and $\Gamma \in \wp$ in such a way that*

$$\begin{aligned} \wp(\alpha(\xi,\eta)\kappa_\lambda(\hbar\xi,\hbar\eta), \kappa_\lambda(\xi,\eta), \kappa_\lambda(\xi,\hbar\xi), \\ \kappa_\lambda(\eta,\hbar\eta), \kappa_\lambda(\xi,\hbar\eta), \kappa_\lambda(\eta,\hbar\xi)) \le 0, \end{aligned} \tag{2.1}$$

for all $\xi,\eta \in M_\kappa$.

Theorem 2.2. *Let M_κ be a complete non-AMMS and $\hbar : M_\kappa \to M_\kappa$ be a $\alpha_\kappa-$implicit contraction. Assume that:*

(i) $\hbar$ satisfies (1.1),

(ii) there is $\xi_0 \in M_\kappa$ in such a manner that $\alpha(\xi_0, \hbar\xi_0) \geq 1$,
(iii) $\hbar$ is continuous.

Then, $\hbar$ has a fixed point.

Proof. Let $\xi_0 \in M_\kappa$ be in such a way that $\alpha(\xi_0, \hbar\xi_0) \geq 1$ and let $\{\xi_n\}$ be a Picard sequence starting at ξ_0, that is $\xi_n = \hbar^n \xi_0 = \hbar\xi_{n-1}$ for all $n \in N$. First, imagine that $\kappa_\lambda(\xi_{n_0}, \xi_{n_0+1}) = 0$ for some $n_0 \in N$, since κ is regular, we get $\xi_{n_0} = \xi_{n_0+1} = \hbar\xi_{n_0}$. So, ξ_{n_0} is a fixed point of $\hbar$. Hence, we approve that $\xi_n \neq \xi_{n+1}$ such that $\kappa_\lambda(\xi_n, \xi_{n+1}) > 0$. Now, since the mapping $\hbar$ is $\alpha-$admissible and $\alpha(\xi_0, \xi_1) = \alpha(\xi_0, \hbar\xi_0) \geq 1$, we deduce that $\alpha(\hbar\xi_0, \hbar\xi_1) = \alpha(\xi_1, \xi_2) \geq 1$. Using the iterative method, we achieve

$$\alpha(\xi_n, \xi_{n+1}) \geq 1, \qquad \text{for all } n \in N. \tag{2.2}$$

From (2.1) with $\xi = \xi_n$ and $\eta = \xi_{n+1}$, we have

$$\begin{gathered}\wp(\alpha(\xi_n, \xi_{n+1})\,\kappa_\lambda(\hbar\xi_n, \hbar\xi_{n+1}),\ \kappa_\lambda(\xi_n, \xi_{n+1}), \kappa_\lambda(\xi_n, \hbar\xi_n)\\ \kappa_\lambda(\xi_{n+1}, \hbar\xi_{n+1}), \hbar_\lambda(\xi_n, \hbar\xi_{n+1}), \kappa_\lambda(\xi_{n+1}, \hbar\xi_n)) \leq 0,\end{gathered}$$

that is,

$$\begin{gathered}\wp(\alpha(\xi_n, \xi_{n+1})\,\kappa_\lambda(\xi_{n+1}, \xi_{n+2}),\ \kappa_\lambda(\xi_n, \xi_{n+1}), \kappa_\lambda(\xi_n, \xi_{n+1})\\ \kappa_\lambda(\xi_{n+1}, \xi_{n+2}), \kappa_\lambda(\xi_n, \xi_{n+2}), \kappa_\lambda(\xi_{n+1}, \xi_{n+1})) \leq 0.\end{gathered}$$

By using the conditions, (iv), (2.2) and $(\wp_1)$ we get

$$\begin{aligned}&\wp(\kappa_\lambda(\xi_{n+1}, \xi_{n+2}),\ \kappa_\lambda(\xi_n, \xi_{n+1}), \kappa_\lambda(\xi_n, \xi_{n+1})\\ &\qquad \kappa_\lambda(\xi_{n+1}, \xi_{n+2}), \kappa_{\max\{\lambda,\lambda\}}(\xi_n, \xi_{n+2}), 0\Big) \leq 0\\ &= \wp(\kappa_\lambda(\xi_{n+1}, \xi_{n+2}),\ \kappa_\lambda(\xi_n, \xi_{n+1}), \kappa_\lambda(\xi_n, \xi_{n+1})\\ &\qquad \kappa_\lambda(\xi_{n+1}, \xi_{n+2}), \kappa_\lambda(\xi_n, \xi_{n+1}) + \kappa_\lambda(\xi_{n+1}, \xi_{n+2}), 0) \leq 0.\end{aligned}$$

Due to $(\wp_2)$, we obtain

$$\kappa_\lambda(\xi_{n+1}, \xi_{n+2}) \leq \psi(\kappa_\lambda(\xi_n, \xi_{n+1})), \quad \text{for all } n \in N. \tag{2.3}$$

From (2.3), it is easy to derive that

$$\kappa_\lambda(\xi_{n+1}, \xi_{n+2}) \leq \psi^{n+1}(\kappa_\lambda(\xi_0, \xi_1)), \quad \text{for all } n \in N. \tag{2.4}$$

Next, we illustrate that $\{\xi_n\}$ is a Cauchy sequence in M_κ. Take $m > n$; by the condition (iv) and (2.4), we write

$$\begin{aligned}\kappa_\lambda(\xi_n, \xi_m) &= \kappa_{\max\{\lambda,\lambda\}}(\xi_n, \xi_m)\\ &\leq \kappa_\lambda(\xi_n, \xi_{n+1}) + \kappa_\lambda(\xi_{n+1}, \xi_m)\\ &= \kappa_\lambda(\xi_n, \xi_{n+1}) + \kappa_{\max\{\lambda,\lambda\}}(\xi_{n+1}, \xi_m)\\ &\leq \kappa_\lambda(\xi_n, \xi_{n+1}) + \kappa_\lambda(\xi_{n+1}, \xi_{n+2}) + \kappa_\lambda(\xi_{n+2}, \xi_m)\\ &\ \vdots\\ &\leq \kappa_\lambda(\xi_n, \xi_{n+1}) + \kappa_\lambda(\xi_{n+1}, \xi_{n+2}) + \ldots + \kappa_\lambda(\xi_{m-1}, \xi_m)\\ &\leq \left(\psi^n + \psi^{n-1} + \ldots + \psi^{m-1}\right)\kappa_\lambda(\xi_0, \xi_1)\\ &\leq \sum_{k=n}^{\infty} \psi^k(\kappa_\lambda(\xi_0, \xi_1)).\end{aligned} \tag{2.5}$$

From (2.5) and (ψ_2) the series $\sum_{k=n}^{\infty} \psi^k(\kappa_\lambda(\xi_0, \xi_1))$ is convergent and so $\{\xi_n\}$ is a Cauchy sequence in M_κ. Because M_κ is a complete non-AMMS, then there exists a point $\nu \in M_\kappa$ such that $\kappa_\lambda(\xi_n, \nu) \to 0$ as $n \to \infty$. Thus, $\kappa_\lambda(\hbar\xi_n, \hbar\nu) \to$ as $n \to \infty$, because $\hbar$ is a $\kappa-$continuous. Then, by (iv) we obtain

$$\begin{aligned}\kappa_\lambda(\nu, \hbar\nu) &= \kappa_{\max\{\lambda,\lambda\}}(\nu, \hbar\nu)\\ &\leq \kappa_\lambda(\nu, \hbar\xi_n) + \kappa_\lambda(\hbar\xi_n, \hbar\nu)\\ &= \kappa_\lambda(\nu, \xi_{n+1}) + \kappa_\lambda(\hbar\xi_n, \hbar\nu).\end{aligned}$$

As $n \to \infty$, we get $\kappa_\lambda(\nu, \hbar\nu) = 0$. Since κ is regular, we deduce that $\hbar\nu = \nu$ and hence ν is a fixed point of $\hbar$.

□

If we turn into the continuity of $\hbar$ with the condition (H), we attain the other result.

(H) If $\{\xi_n\}$ is a sequence in M_κ such that $\alpha(\xi_n, \xi_{n+1}) \geq 1$ for all $n \in N$ and $\xi_n \to \xi$ as $n \to \infty$, there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that $\alpha(\xi_{n_k}, \xi) \geq 1$ for all $k \in N$.

Theorem 2.3. *Let M_κ be a complete non-AMMS and $\hbar : M_\kappa \to M_\kappa$ be an $\alpha_\kappa-$implicit contraction. Granted that:*

(i) $\hbar$ satisfies (1.1),

(ii) there exists $\xi_0 \in M_\kappa$ in such a way that $\alpha(\xi_0, \hbar\xi_0) \geq 1$,
(iii) (H) is supplied.

Then, $\hbar$ has a fixed point.

Proof. Due to Theorem 2.2, we acquire that the sequence $\{\xi_n\}$, defined by $\xi_n = \hbar\xi_{n-1}$ for all $n \in N$, is a Cauchy sequence with $\alpha(\xi_n, \xi_{n+1}) \geq 1$ for all $n \in N$, which converges to some $\nu \in M_\kappa$. Next, from the condition (iii), there is a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ in such a manner that $\alpha(\xi_{n_k}, \xi) \geq 1$ for all $k \in N$. We need to show that $\hbar\nu = \nu$. Since $\hbar$ is $\alpha_\kappa-$type implicit contraction with $\xi = \xi_{n_k}$ and $\eta = \nu$ and (iv), we obtain

$$\wp(\alpha(\xi_{n_k}, \nu)\, \kappa_\lambda(\hbar\xi_{n_k}, \hbar\nu),\ \kappa_\lambda(\xi_{n_k}, \nu), \kappa_\lambda(\xi_{n_k}, \hbar\xi_{n_k})$$

$$\kappa_\lambda(\nu, \hbar\nu), \kappa_\lambda(\xi_{n_k}, \hbar\nu), \kappa_\lambda(\nu, \hbar\xi_{n_k})) \leq 0$$

$$= \wp(\alpha(\xi_{n_k}, \nu)\, \kappa_\lambda(\xi_{n_k+1}, \hbar\nu),\ \kappa_\lambda(\xi_{n_k}, \nu), \kappa_\lambda(\xi_{n_k}, \xi_{n_k+1})$$

$$\kappa_\lambda(\nu, \hbar\nu), \kappa_{\max\{\lambda,\lambda\}}(\xi_{n_k}, \hbar\nu), \omega_\lambda(\nu, \xi_{n_k+1})\Big) \leq 0$$

$$\leq \wp(\alpha(\xi_{n_k}, \nu)\, \kappa_\lambda(\xi_{n_k+1}, \hbar\nu),\ \kappa_\lambda(\xi_{n_k}, \nu), \kappa_\lambda(\xi_{n_k}, \xi_{n_k+1})$$

$$\kappa_\lambda(\nu, \hbar\nu), \kappa_\lambda(\xi_{n_k}, \nu) + \kappa_\lambda(\nu, \hbar\nu), \kappa_\lambda(\nu, \xi_{n_k+1})) \leq 0.$$

Letting k tends to infinity and using the continuity of $\wp$ and $\alpha(\xi_{n_k}, \xi) \geq 1$, we get

$$\wp(\kappa_\lambda(\nu, \hbar\nu), 0, 0,\ \kappa_\lambda(\nu, \hbar\nu), \kappa_\lambda(\nu, \hbar\nu), 0) \leq 0.$$

Finally, by condition $(\wp_2)$, it follows that $\kappa_\lambda(\nu, \hbar\nu) \leq 0$ which implies $\hbar\nu = \nu$. □

We need extra conditions to obtain uniqueness of fixed point.

(U) For all $u, v \in \mathrm{Fix}(\hbar)$, we attain $\alpha(u, v) \geq 1$, where $\mathrm{Fix}(\hbar)$ gives the set of all fixed points of $\hbar$.
$(\wp_3)$ There exists $\psi \in \Psi$ in such a way that for all $u, v > 0$,

$$\wp(\mathrm{u}, \mathrm{u}, 0, 0, \mathrm{u}, \mathrm{v}) \leq 0 \quad \text{implies} \quad \mathrm{u} \leq \psi(\mathrm{v}).$$

Theorem 2.4. *Adding conditions (U) and $(\wp_3)$ to the hypotheses of Theorem 2.2 (resp Theorem 2.3), we deduce that $\hbar$ has a unique fixed point.*

Proof. We discuss by contradiction, that is, there exist $u, v \in M_\kappa$ in such a way that $u = \hbar u$ and $v = \hbar v$ with $u \neq v$. From (1.1), we obtain

$$\wp(\alpha(u, v)\ \kappa_\lambda(\hbar u, \hbar v), \kappa_\lambda(u, v), \kappa_\lambda(u, \hbar u),$$

$$\kappa_\lambda(v, \hbar v), \kappa_\lambda(u, \hbar v), \kappa_\lambda(v, \hbar u)) \leq 0.$$

Then, by condition (U), we have

$$\wp(\kappa_\lambda(u, v),\ \kappa_\lambda(u, v), 0, 0, \kappa_\lambda(u, v), \kappa_\lambda(v, u)) \leq 0.$$

Since $\wp$ satisfies the property $(\wp_3)$, then

$$\kappa_\lambda(u, v) \leq \psi(\kappa_\lambda(u, v)) < \kappa_\lambda(u, v),$$

which is a contradiction and hence $u = v$. □

Now, we give some corollaries from above results.

Corollary 2.5. *Let M_κ be a complete non-AMMS and $\hbar : M_\kappa \to M_\kappa$ be a function. If there is a function $\alpha : M_\kappa \times M_\kappa \to [0, \infty)$ in such a manner that*

$$\alpha(\xi, \eta)\, \kappa_\lambda(\hbar\xi, \hbar\eta) \leq p\kappa_\lambda(\xi, \eta) + q\kappa_\lambda(\xi, \hbar\xi) + r\kappa_\lambda(\eta, \hbar\eta) + s\kappa_\lambda(\xi, \hbar\eta) + t\kappa_\lambda(\eta, \hbar\xi),$$

for all $\xi, \eta \in M_\kappa$, where $p, q, r, s, t > 0$, $p + q + r + s + t < 1$. Assume also that:

(i) $\hbar$ satisfies (1.1),
(ii) there is $\xi_0 \in M_\kappa$ in such a way that $\alpha(\xi_0, \hbar\xi_0) \geq 1$,
(iii) $\hbar$ is continuous or the condition (H) holds true.

Then, $\hbar$ has a fixed point. Additionally, if $p + r + s < 1$ and the conditions (U) and $(\wp_3)$ hold true, then $\hbar$ has a unique fixed point.

Corollary 2.6. *Let M_κ be a complete non-AMMS and $\hbar : M_\kappa \to M_\kappa$ be a function. If there is a function $\alpha : M_\kappa \times M_\kappa \to [0, \infty)$ in such a manner that*

$$\alpha(\xi, \eta)\, \kappa_\lambda(\hbar\xi, \hbar\eta) \leq k \max\{\kappa_\lambda(\xi, \eta),\ \kappa_\lambda(\xi, \hbar\xi), \kappa_\lambda(\eta, \hbar\eta), \kappa_\lambda(\xi, \hbar\eta), \kappa_\lambda(\eta, \hbar\xi)\},$$

for all $\xi, \eta \in M_\kappa$, where $k \in \left[0, \frac{1}{2}\right)$. Furthermore:

(i) $\hbar$ satisfies (1.1),
(ii) there is $\xi_0 \in M_\kappa$ such that $\alpha(\xi_0, \hbar\xi_0) \geq 1$,
(iii) $\hbar$ is continuous or the property (H) is satisfied.

Then, $\hbar$ has a fixed point. Moreover, the conditions (U) and $(\wp_3)$ hold true, then $\hbar$ has a unique fixed point.

Example 2.7. *$M_\kappa = R$ endowed with the non-Archimedean modular metric $\kappa_\lambda(\xi,\eta) = \frac{1}{\lambda}|\xi-\eta|$, for all $\xi,\eta \in M_\kappa$ and $\lambda > 0$. Obviously, M_κ is an $\kappa-$complete non-AMMS.*
Consider the self-map $\hbar : M_\kappa \to M_\kappa$ defined by $\hbar\xi = \frac{\xi}{6}$.
Also define

$$\alpha(\xi,\eta) = \begin{cases} 1, & \text{if } \xi,\eta \in [0,1] \\ 0, & \text{otherwise,} \end{cases}$$

and $\wp : R_+^6 \to R_+$ defined by

$$\wp(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \frac{3}{4}\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\right\}.$$

Let $\alpha(\xi,\eta) \geq 1$, then $\xi,\eta \in [0,1]$. Also, $\hbar\xi \in [0,1]$, for all $\xi \in [0,1]$ and so $\alpha(\hbar\xi,\hbar\eta) \geq 1$. Therefore $\hbar$ is an $\alpha-$admissible mapping. Let $\xi,\eta \in [0,1]$, we have

$$\wp\Big(\alpha(\xi,\eta)\kappa_\lambda(\hbar\xi,\hbar\eta),\kappa_\lambda(\xi,\eta),\kappa_\lambda(\xi,\hbar\xi),\kappa_\lambda(\eta,\hbar\eta),\tfrac{\kappa_\lambda(\xi,\hbar\eta)+\kappa_\lambda(\eta,\hbar\xi)}{2}\Big)$$

$$= \alpha(\xi,\eta)\kappa_\lambda(\hbar\xi,\hbar\eta) - \tfrac{3}{4}\max\{\kappa_\lambda(\xi,\eta),\kappa_\lambda(\xi,\hbar\xi),\kappa_\lambda(\eta,\hbar\eta),$$

$$\tfrac{\kappa_\lambda(\xi,\hbar\eta)+\kappa_\lambda(\eta,\hbar\xi)}{2}\Big\}$$

$$\leq \tfrac{1}{6\lambda}|\xi-\eta| - \tfrac{3}{4}\max\left\{\tfrac{1}{\lambda}|\xi-\eta|,\tfrac{6}{5\lambda}|\xi|,\tfrac{6}{5\lambda}|\eta|,\ \tfrac{1}{12\lambda}(|6\xi-\eta|+|6\eta-\xi|)\right\}$$

$$\leq 0.$$

Similarly, it is obvious that contractive condition (2.1) holds in the case ($\xi,\eta \notin [0,1]$ and ξ or η is not in $[0,1]$.) Thus, $\hbar$ is $\alpha_\kappa-$type implicit contraction. Next, it is easy to illustrate that conditions $\hbar$ is $\kappa-$continuous, (H) and (U) are satisfied.
Thus, the axioms of the Theorem 2.2, Theorem 2.3, and Theorem 2.4 are supplied and 0 is a unique fixed point.

3. Stability problem in the sense of Ulam-Hyers

Now, we obtain the stability problem in the sense of Ulam-Hyers of fixed point. That this problem correspondences to Corollary 2.5.
Let M_κ be a non-AMMS and $\hbar : M_\kappa \to M_\kappa$ be a function. Imagine the fixed point problem

$$\xi = \hbar\xi \tag{3.1}$$

and the inequality (for $\varepsilon > 0$)

$$\kappa_\lambda(\hbar\eta,\eta) < \varepsilon. \tag{3.2}$$

We are said to be a $\hbar$ is stable in the sense of Ulam-Hyers in non-AMMS if there are $L > 0$ such that for each $\varepsilon > 0$ and a $\varepsilon-$solution $v^* \in M_\kappa$, that is, v^* supplies the condition (3.2), there is a solution $u^* \in M_\omega$ of the fixed point equation (3.1) such that

$$\kappa_\lambda(u^*,v^*) < L\varepsilon. \tag{3.3}$$

Theorem 3.1. *Let M_κ be a non-AMMS. Suppose that all the hypotheses of Corollary 2.5 hold and $\alpha(u,v) \geq 1$ for all $\varepsilon-$solution u and v, then the equation (3.1) is stable in the sense of Ulam-Hyers.*

Proof. By Corollary 2.5, we have a unique $u \in M_\kappa$ such that $u = \hbar u$, that is, $u \in M_\kappa$ is a solution of the fixed point equation (3.1). Let $\varepsilon > 0$ and $v \in M_\kappa$ be an $\varepsilon-$solution, that is,

$$\kappa_\lambda(\hbar v,v) \leq \varepsilon.$$

Since $\kappa_\lambda(u,\hbar u) = \kappa_\lambda(u,u) = 0 \leq \varepsilon$, u and v are $\varepsilon-$solutions. By hypotheses, we get $\alpha(u,v) \geq 1$ and from (3.3), so

$$\begin{aligned}
\kappa_\lambda(u,v) = \kappa_\lambda(\hbar u,v)& \\
&= \kappa_{\max\{\lambda,\lambda\}}(\hbar u,v) \\
&\leq \kappa_\lambda(\hbar u,\hbar v) + \kappa_\lambda(\hbar v,v) \\
&= \alpha(u,v)\kappa_\lambda(\hbar u,\hbar v) + \varepsilon \\
&\leq a\kappa_\lambda(u,v) + b\kappa_\lambda(u,\hbar u) + c\kappa_\lambda(v,\hbar v) \\
&+ d\kappa_\lambda(u,\hbar v) + e\kappa_\lambda(v,\hbar u) + \varepsilon \\
&= a\kappa_\lambda(u,v) + b\kappa_\lambda(u,\hbar u) + c\kappa_\lambda(v,\hbar v) \\
&+ d\kappa_{\max\{\lambda,\lambda\}}(u,\hbar v) + e\kappa_{\max\{\lambda,\lambda\}}(v,\hbar u) + \varepsilon \\
&\leq a\kappa_\lambda(u,v) + b\kappa_\lambda(u,\hbar u) + c\kappa_\lambda(v,\hbar v) \\
&+ d(\kappa_\lambda(u,v) + \kappa_\lambda(v,\hbar v)) + e(\kappa_\lambda(v,u) + \kappa_\lambda(u,\hbar u)) + \varepsilon.
\end{aligned}$$

We deduce

$$\kappa_\lambda(u,v) \leq \left(\frac{1+c+d}{1-a-d-e}\right)\varepsilon = L\varepsilon,$$

where $L = \left(\frac{1+c+d}{1-a-d-e}\right) > 0$. Thus, $\hbar$ is Ulam-Hyers stable. □

4. Well posedness of the fixed point problem

Now, we show well-posedness of a function $\hbar$ on non-AMMS.

Definition 4.1. *Let M_κ be a non-AMMS and let $\hbar : M_\kappa \to M_\kappa$, $\alpha : M_\kappa \times M_\kappa \to [0,\infty)$ be two functions. $\hbar$ is well-posedness if:*

(i) $u \in M_\kappa$ is the unique fixed point when $\alpha(u,\hbar u) \geq 1$,
(ii) there exists a sequence $\{\xi_n\}$ in such a manner that $\kappa_\lambda(\xi_n,\hbar\xi_n) \to 0$ as $n\to\infty$, then $\kappa_\lambda(\xi_n,u) \to 0$ as $n\to\infty$.

We define a new condition which needs to be the following result.

(R) If $\{\xi_n\}$ is a sequence in M_κ in such a way that $\kappa_\lambda(\xi_n,\hbar\xi_n) \to 0$ as $n\to\infty$, then $\alpha(\xi_n,\hbar\xi_n) \geq 1$ for all $n \in N$.

Theorem 4.2. *Let M_κ be a non-AMMS. If all the conditions of Corollary 2.5 and the condition (R) hold, hence (3.1) is well posed.*

Proof. By Corollary 2.5, we have a unique $u \in M_\kappa$ in such a manner that $u = \hbar u$ and $\alpha(u,\hbar u) \geq 1$. Let $\{\xi_n\}$ is a sequence in M_κ in such a way that $\kappa_\lambda(\xi_n,\hbar\xi_n) \to 0$ as $n\to\infty$. By condition (R), we get $\alpha(\xi_n,\hbar\xi_n) \geq 1$. Now, we have

$$\begin{aligned}
\kappa_\lambda(\xi_n,u) &= \kappa_\lambda(\xi_n,\hbar u)\\
&= \kappa_{\max\{\lambda,\lambda\}}(\xi_n,\hbar u)\\
&\leq \kappa_\lambda(\xi_n,\hbar\xi_n) + \kappa_\lambda(\hbar\xi_n,\hbar u)\\
&\leq \alpha(\xi_n,u)\,\kappa_\lambda(\hbar\xi_n,\hbar u) + \kappa_\lambda(\xi_n,\hbar\xi_n)\\
&\leq a\kappa_\lambda(\xi_n,u) + b\kappa_\lambda(\xi_n,\hbar\xi_n) + c\kappa_\lambda(u,\hbar u) + d\kappa_\lambda(\xi_n,\hbar u)\\
&\quad + e\kappa_\lambda(u,\hbar\xi_n) + \kappa_\lambda(\xi_n,\hbar\xi_n)\\
&\leq a\kappa_\lambda(\xi_n,u) + b\kappa_\lambda(\xi_n,\hbar\xi_n) + c\kappa_\lambda(u,\hbar u) + d\kappa_{\max\{\lambda,\lambda\}}(\xi_n,\hbar u)\\
&\quad + e\kappa_{\max\{\lambda,\lambda\}}(u,\hbar\xi_n) + \kappa_\lambda(\xi_n,\hbar\xi_n)\\
&\leq a\kappa_\lambda(\xi_n,u) + b\kappa_\lambda(\xi_n,\hbar\xi_n) + c\kappa_\lambda(u,\hbar u) + d\left(\kappa_\lambda(\xi_n,u) + \kappa_\lambda(u,\hbar u)\right)\\
&\quad + e\left(\kappa_\lambda(u,\xi_n) + \kappa_\lambda(\xi_n,\hbar\xi_n)\right) + \kappa_\lambda(\xi_n,\hbar\xi_n).
\end{aligned}$$

Hence

$$\kappa_\lambda(\xi_n,u) \leq \left(\frac{1+b+e}{1-a-d-e}\right)\kappa_\lambda(\xi_n,\hbar\xi_n).$$

Since $\kappa_\lambda(\xi_n,\hbar\xi_n) \to 0$ as $n\to\infty$, it implies that $\kappa_\lambda(\xi_n,u) \to 0$ as $n\to\infty$. Thus, $\hbar$ is well posed. □

5. Consequences

Next, we will obtain non-AMMS version of some fixed point results.
In the Definition of 1.6, if we take $\psi(t) = ht$, $h \in [0,1)$, we get Berinde's results in [11].
Let Γ be the set of all continuous real functions $\wp : R_+^6 \to R_+$, for which we consider the following conditions:

$(\wp_{1a})$ F is non-increasing in the fifth variable and

$$\wp(\xi,\eta,\eta,\xi,\xi+\eta,0) \leq 0, \text{ for } \xi,\eta \geq 0 \;\Rightarrow\; \exists h \in [0,1) \text{ such that } \xi \leq h\eta;$$

$(\wp_{1b})$ $\wp$ is non-increasing in the fourth variable and

$$\wp(\xi,\eta,0,\xi+\eta,\xi,\eta) \leq 0, \text{ for } \xi,\eta \geq 0 \;\Rightarrow\; \exists h \in [0,1) \text{ such that } \xi \leq h\eta;$$

$(\wp_{1c})$ $\wp$ is non-increasing in the third variable and

$$\wp(\xi,\eta,\xi+\eta,0,\eta,\xi) \leq 0, \text{ for } \xi,\eta \geq 0 \;\Rightarrow\; \exists h \in [0,1) \text{ such that } \xi \leq h\eta;$$

$(\wp_2)$ $\wp(\xi,\xi,0,0,\xi,\xi) > 0$, for all $\xi > 0$.

Example 5.1. *The function $\wp \in \Gamma$, given by*

$$\wp(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2,$$

where $a \in [0,1)$, satisfies $(\wp_{1a})$-$(\wp_{1c})$ and $(\wp_2)$,with $h = a$.

Example 5.2. *The function $\wp \in \Gamma$, given by*

$$\wp(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - b(t_3+t_4),$$

where $b \in \left[0,\frac{1}{2}\right)$, satisfies $(\wp_{1a})$-$(\wp_{1c})$ and $(\wp_2)$,with $h = \frac{b}{1-b} < 1$.

Example 5.3. *The function $\wp \in \Gamma$, given by*

$$\wp(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - c(t_5+t_6),$$

where $c \in \left[0,\frac{1}{2}\right)$, satisfies $(\wp_{1a})$-$(\wp_{1c})$ and $(\wp_2)$,with $h = \frac{c}{1-c} < 1$.

Example 5.4. *The function $\wp \in \Gamma$, given by*

$$\wp(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a\max\left\{t_2,\frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\right\},$$

where $a \in [0,1)$, satisfies $(\wp_{1a})$-$(\wp_{1c})$ and $(\wp_2)$,with $h = a$.

Example 5.5. *The function $\wp \in \Gamma$, given by*

$$\wp(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 - b(t_3+t_4) - c(t_5+t_6),$$

where $a,b,c \geq 0$ and $a+2b+2c<1$ satisfies $(\wp_{1a})$-$(\wp_{1c})$ and $(\wp_2)$,with $h = \frac{a+b+c}{1-b-c} < 1$.

Corollary 5.6. *Let M_κ be a non-Archimedean modular metric space, $\hbar : M_\kappa \to M_\kappa$ be a self map for which $\wp \in \Gamma$ such that for all $\xi,\eta \in M_\kappa$,*

$$\wp(\kappa_\lambda(\hbar\xi,\hbar\eta), \kappa_\lambda(\xi,\eta), \kappa_\lambda(\xi,\hbar\xi), \kappa_\lambda(\eta,\hbar\eta), \kappa_\lambda(\xi,\hbar\eta), \kappa_\lambda(\eta,\hbar\xi)) \leq 0.$$

If $\wp$ satisfies $(\wp_{1a})$ and $(\wp_2)$, then $\hbar$ has a unique fixed point.

Proof. It suffices to take $\alpha(\xi,\eta)=1$ and $\psi(t)=kt$, $k\in[0,1)$ in Theorem 2.2. □

6. Application to integral equation

Next, we give implementation to show the nonlinear integral equation.

$$\xi(z) = \int_a^t K(z,p,\xi(p))\,dp, \tag{6.1}$$

where $\xi \in I = [a,b]$ and $K : I \times I \times R \to R$ is continuous. Let $M = C(I,R)$ with the usual supremum norm, that is,

$$\|\xi\| = \max_{z\in I} |\xi(z)|,$$

and the metric

$$\kappa_\lambda(\xi,\eta) = \frac{1}{\lambda}\|\xi-\eta\| = \frac{1}{\lambda}d(\xi,\eta),$$

for all $\xi,\eta \in M$. For $r>0$ and $\xi \in M$ we denote by

$$B_\lambda(\xi,r) = \{v \in M : \kappa_\lambda(\xi,\eta) \leq r\},$$

the closed ball concerned at ξ and of radius r. Note that M_κ is a $\kappa-$complete non-AMMS.
Now, imagine the mapping $\hbar : M_\kappa \to M_\kappa$

$$\hbar\xi(z) = \int_a^z K(z,p,\xi(p))\,dp. \tag{6.2}$$

Notice that (6.1) has a solution if and only if $\hbar$ has a fixed point in (6.2).

Theorem 6.1. *Let $r>0$ and we granted that the following conditions are supplied:*

(i) if $y \in B_\lambda(\xi,r)$, $\lambda>0$, then

$$|K(z,p,\xi(p)) - K(z,p,\eta(p))| \leq \frac{q(z,p)}{b-a}|\xi(p)-\eta(p)|,$$

for all $z,p \in I$, $\xi,\eta \in R$ and for some continuous function $q : I\times I \to R_+$;

(ii) $\sup\limits_{z\in I} q(z,p) = k < 1$.

Hence, (6.1) has a solution.

Proof. Since $\eta \in B_\lambda(\xi,r)$ and from (ii), we have

$$\begin{aligned}
|\hbar\xi(z) - \hbar\eta(z)| &\leq \left|\int_a^z [K(z,p,\xi(p)) - K(z,p,\eta(p))]dp\right| \\
&\leq \int_a^t |K(t,p,\xi(p)) - K(z,p,\eta(p))|dp \\
&\leq \int_a^b |K(z,p,\xi(p)) - K(z,p,\eta(p))|dp \\
&\leq \int_a^b \tfrac{q(z,p)}{b-a}|\xi(p)-\eta(p)|dp \\
&\leq \|\xi(p)-\eta(p)\| \int_a^b \tfrac{k}{b-a}dp \\
&= k\|\xi(p)-\eta(p)\|.
\end{aligned} \tag{6.3}$$

This implies that

$$\begin{aligned}
\kappa_\lambda(\hbar\xi,\hbar\eta) &= \tfrac{1}{\lambda}\|\hbar\xi - \hbar\eta\| \\
&\leq \tfrac{1}{\lambda}|\hbar\xi(z) - \hbar\eta(z)| \\
&\leq \tfrac{1}{\lambda}k\|\xi(p)-\eta(p)\| \\
&\leq k\kappa_\lambda(\xi,\eta).
\end{aligned}$$

Now, $\wp : R_+^6 \to R_+$ defined by

$$\wp(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - kt_2,$$

where $k\in[0,1)$, and so the integral operator $\hbar$ satisfies all conditions of Corollary 5.6. Thus, $\hbar$ has a fixed point, i.e., (6.1) has a solution in M_κ. □

References

[1] V. V. Chistyakov, *Modular metric spaces, I: Basic concepts*, Nonlinear Anal., **72** (2010), 1-14.
[2] V. V. Chistyakov, *Modular metric spaces, II: Application to superposition operators*, Nonlinear Anal., **72** (2010), 15-30.
[3] C. Mongkolkeha, W. Sintunavarat, P. Kumam, *Fixed point theorems for contraction mappings in modular metric spaces*, Fixed Point Theory Appl. **2011**(93) (2011) ,9 pages.
[4] M. Paknazar, M. A. Kutbi, M. Demma, P. Salimi, *On non-Archimedean Modular metric space and some nonlinear contraction mappings*, J. Nonlinear Sci. Appl., (2017), in press.
[5] V. Popa, *Fixed point theorems for implicit contractive mappings*, Stud. Cerc. St. Ser. Mat. Univ. Bacau, **7** (1997), 129-133.
[6] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for $\alpha-\psi-$contractive type mappings*, Nonlinear Analysis, **75** (2012) (2012), 2154-2165.
[7] H. Aydi, *$\alpha-$implicit contractive pair of mappings on quasi b-metric spaces and application to integral equations*, J. Nonlinear Convex Anal., **17**(12) (2015), 2417-2433.
[8] A. Hussain, T. Kanwal, *Existence and uniqueness for a neutral differential problem with unbounded delay via fixed point*, Transections of A. Razmadze Mathematical Enstitute, **172**(3) (2018), 48-490.
[9] M. Abbas, A. Hussain, B. Popovic, S. Radenovic, *Istratescu-Suzuki-Ciric type fixed point results in thee framework of G-metric spaces*, J. Nonlinear Sci. Appl., **9** (2016), 6077-6095.
[10] N. Hussain, C. Vetro, F. Vetro, *Fixed point results for $\alpha-$implicit contractions with application to integral equations,* Nonlinear Anal. Model. Control, **21**(3) (2016), 362-378.
[11] V. Berinde, *Approximating fixed points of implicit almost contractions*, Hacet. J. Math. Stat., **41** (2012), 93-102.