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Canonical Type First Order Boundedly Solvable Differential Operators

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Abstract. The main goal of this work is to describe of all boundedly solvable extensions of the minimal operator generated by first-order linear canonical type differential-operator expression in the weighted Hilbert space of vector-functions at finite interval in terms of boundary conditions by using the methods of operator theory. Later on, the structure of spectrum of this type extension will be investigated.

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1. Description of Boundedly Solvable Extensions

The general information on the degenerate differential equations in Banach spaces can be found in book of A. Favini and A. Yagi [2]. The fundamental interest to such equations are motivated by applications in different fields of life sciences. The solvability of the considered problems may be seen as boundedly solvability of linear differential operators in corresponding functional Banach spaces. Note that the theory of boundedly solvable extensions of a linear densely defined closed operator in Hilbert spaces was presented in the important works of M. I. Vishik in [9, 10]. Generalization of these results to the nonlinear and complete additive Hausdorff topological spaces in abstract terms of abstract boundary conditions have been done by B. K. Kokebaev, M. O. Otelbaev and A. N. Synybekov in [5–7]. Another approach to the description of regular extensions for some classes of linear differential operators in Hilbert spaces of vector-functions at finite interval has been offered by A. A. Dezin [1] and N. I. Pivtorak [8]. Remember that a linear closed densely defined operator on any Hilbert space is called boundedly solvable, if it is one-to-one and onto and its inverse is bounded.

Let H be a separable Hilbert space and $\alpha : (0,1) \to (0,\infty)$, $\alpha \in C(0,1)$ and $\int_0^1 \frac{dt}{\alpha(t)} < \infty$. In the weighted Hilbert space $L^2_\alpha(H,(0,1))$ of $H-$ valued vector-functions defined at the interval $(0,1)$ consider the following degenerate type differential expression with operator coefficient for first order in a form

$$l(u) = J(\alpha u)'(t) + A(t)u(t),$$

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where:
(1) operator-function $A(\cdot):(0,1)\to L(H)$ is continuous on the uniform operator topology;
(2) $A\in L(H)$
(3) $J\in L(H),\ J^*=J,\ J^2=E$ and $JA=AJ$.

By the standard way the minimal L_0 and maximal L operators corresponding differential expression $l(\cdot)$ in $L^2_\alpha(H,(0,1))$ can be defined [4]. In this case $KerL_0=\{0\}$ and $\overline{Im(L_0)}\neq L^2_\alpha(H,(0,1))$ (see Sec.3).

In this work, firstly all boundedly solvable extensions of the minimal operator generated by first order linear degenerate type differential-operator expression in the weighted Hilbert space of vector-functions in (0,1) in terms of boundary conditions are described. Later on, the structure of spectrum of these type extensions will be investigated.
In this section using the Vishik's methods all boundedly solvable extensions of the minimal operator generated by linear degenerate type differential-operator expression $l(\cdot)$ in weighted Hilbert space $L^2_\alpha(H,(0,1))$ are represented.

Before of all note that using the knowing standard way the minimal M_0 and the maximal M operators generated by differential expression

$$m(v)=J(\alpha v)'(t)$$

in Hilbert space $L^2_\alpha(H,(0,1))$ can be defined [4].

Later on, by $U(t,s),\ t,s\in[0,1)$ will be defined the family of evolution operators corresponding to the homogeneous differential-operator equation

$$J\alpha(t)\frac{\partial}{\partial t}U(t,s)f+A(t)U(t,s)f=0,\ t,s\in(0,1)$$

with boundary condition

$$U(s,s)f=f,\ f\in H.$$

The operator $U(t,s),\ t,s\in(0,1)$ is linear continuous and boundedly solvable in H. And also for any $t,s\in(0,1)$ there is the following equation:

$$U^{-1}(t,s)=U(s,t)$$

(for detail analysis see [3]).

If introduce the following operator

$$\begin{aligned}&Uz(t)=U(t,0)z(t),\\&U:L^2_\alpha(H,(0,1))\to L^2_\alpha(H,(0,1)),\end{aligned}$$

then it is easily to check that

$$\begin{aligned}l(Uz)&=J\left(\alpha Uz\right)'(t)+A(t)Uz(t)\\&=JU(\alpha z)'(t)+U'_t(\alpha z)(t)+A(t)Uz(t)\\&=JU(\alpha z)'(t)+\left[J\alpha(t)U'_tz(t)+A(t)Uz(t)\right]\\&=U(\alpha z)'(t)\\&=Um(z).\end{aligned}$$

Therefore it can be obtained

$$U^{-1}l(Uz)=m(z).$$

Hence it is clear that if $\widetilde{L}$ is some extension of the minimal operator L_0, that is, $L_0\subset\widetilde{L}\subset L$, then $U^{-1}L_0U=M_0,\ M_0\subset U^{-1}\widetilde{L}U=\widetilde{M}\subset M,\ U^{-1}LU=M$.

Now we prove the following assertion.

Lemma 1.1. *$KerL_0=\{0\}$ and $\overline{Im(L_0)}\neq L^2_\alpha(H,(0,1))$.*

Proof. Consider the following boundary value problem in $L^2_\alpha(H,(0,1))$

$$\begin{aligned}&J(\alpha u)'(t)+A(t)u(t)=0,\ t\in(0,1),\\&(\alpha u)(0)=(\alpha u)(1)=0.\end{aligned}\tag{1.1}$$

Then the general solution of above differential equation is in form

$$(\alpha u)(t) = exp\left(-J\int_0^t \frac{A(s)}{\alpha(s)}ds\right)f_0,\ f_0 \in H. \tag{1.2}$$

From (1.2) and boundary condition (1.1) we have following equation

$$u(t) = 0,\ t \in (0,1).$$

Consequently, following equality $Ker(L_0) = \{0\}$ hold.

On the other hand it is clear that the general solution of following differential equation in $L^2_\alpha(H,(0,1))$

$$-J(\alpha v)'(t) + A^*(t)v(t) = 0$$

in form

$$v(t) = \frac{1}{\alpha(t)}exp\left(J\int_0^t \frac{A^*(s)}{\alpha(s)}ds\right)g,\ g \in H.$$

This means that

$$dimKerL_0^* = \infty.$$

So the following inequality is realized

$$\overline{Im(L_0)} \neq L^2_\alpha(H,(0,1)).$$

Theorem 1.2. *Each solvable extension $\widetilde{L}$ of the minimal operator L_0 in $L^2_\alpha(H,(0,1))$ is generated by the differential-operator expression $l(\cdot)$ with boundary condition*

$$(B+E)(\alpha U^{-1}u)(0) = B(\alpha U^{-1}u)(1),$$

where $B \in L(H)$, E is a identity operator in H. The operator B is determined uniquely by the extension $\widetilde{L}$, i.e $\widetilde{L} = L_B$.

On the contrary, the restriction of the maximal operator L to the manifold of vector-functions satisfy the above boundary condition for some bounded operator $B \in L(H)$ is a boundedly solvable extension of the minimal operator L_0 in $L^2_\alpha(H,(0,1))$.

Proof. Firstly, all boundedly solvable extensions $\widetilde{M}$ of the minimal operator L_0 in $L^2_\alpha(H,(0,1))$ in terms of boundary conditions will be described.

Consider the following so-called Cauchy extension M_c,

$$M_c u = J(\alpha u)'(t),$$
$$M_c : D(M_c) \subset L^2_\alpha(H,(0,1)) \to L^2_\alpha(H,(0,1)),$$
$$D(M_c) = \{u \in D(L) : (\alpha u)(0) = 0\}$$

of the minimal operator M_0. It is clear that M_c is a boundedly solvable extension of minimal operator M_0 and

$$M_c^{-1}f(t) = \frac{1}{\alpha(t)}J\int_0^t f(s)ds,\ f \in L^2_\alpha(H,(0,1)),$$
$$M_c^{-1} : L^2_\alpha(H,(0,1)) \to L^2_\alpha(H,(0,1)).$$

Indeed, for any $f \in L^2_\alpha(H,(0,1))$ we have

$$\begin{aligned}
\|\frac{1}{\alpha(t)}J\int_0^t f(s)ds\|^2_{L^2_\alpha(H,(0,1))} &= \int_0^1 \alpha(t)\frac{\|J\|_H}{\alpha^2(t)}\|\int_0^t f(s)ds\|^2_H dt \\
&\leq \int_0^1 \frac{\|J\|_H}{\alpha(t)}\left(\int_0^t \frac{1}{\sqrt{\alpha(s)}}\sqrt{\alpha(s)}\|f(s)\|_H ds\right)^2 dt \\
&\leq \|J\|_H \int_0^1 \frac{dt}{\alpha(t)}\left(\int_0^1 \frac{ds}{\alpha(s)}\right)\left(\int_0^1 \|f(s)\|^2_H \alpha(s)ds\right) \\
&= \left(\int_0^1 \frac{dt}{\alpha(t)}\right)^2 \|J\|_H\|f\|^2_{L^2_\alpha(H,(0,1))}.
\end{aligned}$$

Now assumed that $\widetilde{M}$ is a solvable extension of the minimal operator M_0 in $L^2_\alpha(H,(0,1))$. In this case it is known that the domain of $\widetilde{M}$ can be written as a direct sum

$$D(\widetilde{M}) = D(M_0) \oplus \left(M_c^{-1} + K\right)V,$$

where $V = KerM_0^*,\ K \in L(H)$ (see [9, 10]).

It is easily to see that

$$KerM_0^* = \left\{\frac{1}{\alpha(t)}f : \ f \in H\right\}.$$

Therefore each function $u \in D(\widetilde{M})$ can be written in following form

$$u(t) = u_0(t) + M_c^{-1}\left(\frac{1}{\alpha(t)}f\right) + \frac{1}{\alpha(t)}Kf,\ u_0 \in D(M_0),\ f \in H.$$

And from this we have

$$(\alpha u)(t) = (\alpha u_0)(t) + J\int_0^t \frac{ds}{\alpha(s)}f + Kf,\ f \in H.$$

Hence, following equalities

$$\begin{aligned}
(\alpha u)(0) &= Kf, \\
(\alpha u)(1) &= \left(J\int_0^1 \frac{ds}{\alpha(s)} + K\right)f.
\end{aligned}$$

From these relations it is obtained that

$$\left(\int_0^1 \frac{ds}{\alpha(s)} + JK\right)J(\alpha u)(0) = JKJ(\alpha u)(1).$$

If we take $T = BJK$, then we obtained that

$$\left(J\int_0^1 \frac{ds}{\alpha(s)}J + JT\right)(\alpha u)(0) = JT(\alpha u)(1).$$

Consequently,

$$\left(E + \left(\int_0^1 \frac{ds}{\alpha(s)}\right)^{-1} JT\right)(\alpha u)(0) = \left(\int_0^1 \frac{ds}{\alpha(s)}\right)^{-1} JT(\alpha u)(1).$$

Then the last equality can be written in form

$$(B+E)(\alpha u)(0) = B(\alpha u)(1),$$

where

$$B = \left(\int_0^1 \frac{ds}{\alpha(s)}\right)^{-1} JT.$$

On the other hand note that the uniquenses of the operator $B \in L(H)$ is clear from [9, 10]. Therefore, $\widetilde{M} = M_B$. This completes of necessary part of assertion.

On the contrary, if M_B is a operator generated by $m(\cdot)$ and boundary condition

$$(B+E)(\alpha u)(0) = B(\alpha u)(1),$$

then M_B is boundedly invertible and

$$M_B^{-1} : L^2_\alpha(H,(0,1)) \to L^2_\alpha(H,(0,1)),$$

$$M_B^{-1} f(t) = \frac{J}{\alpha(t)} \int_0^t f(s)ds + B\int_0^1 f(s)ds, \ f \in L^2_\alpha(H,(0,1)).$$

Consequently, assertion of theorem for the boundedly solvable extension of the minimal operator M_0 is true.

The extension $\widetilde{L}$ of the minimal operator L_0 is boundedly solvable in $L^2_\alpha(H,(0,1))$ if and only if the operator $\widetilde{M} = U^{-1}\widetilde{L}U$ is a boundedly solvable extension of the minimal operator M_0 in $L^2_\alpha(H,(0,1))$. Then $u \in D(\widetilde{L})$ if and only if $U^{-1}u \in D(\widetilde{M})$.

Since $\widetilde{M} = M_B$ for some $B \in L(H)$, then we have

$$(B+E)(\alpha U^{-1}u)(0) = B(\alpha U^{-1}u)(1).$$

This completes the proof of theorem.

2. Spectrum of Boundedly Solvable Extensions

In this section the structure of spectrum of boundedly solvable extensions of the minimal operator L_0 in $L^2_\alpha(H,(0,1))$ will be investigated.

Firstly, prove the following result.

Theorem 2.1. *If L_B is a boundedly solvable extension of the minimal operator L_0 and $M_B = U^{-1}L_BU$ corresponding boundedly solvable extension of the minimal operator M_0, then it is true $\sigma(L_B) = \sigma(M_B)$.*

Proof. Consider the following problem to spectrum for any boundedly solvable extension L_B in $L^2_\alpha(H,(0,1))$, that is

$$L_B u = \lambda u + f, \ \lambda \in \mathbb{C}, \ f \in L^2_\alpha(H,(0,1)).$$

From this it is obtained that

$$(L_B - \lambda E)u = f \text{ or } (UM_BU^{-1} - \lambda E)u = f.$$

Then we have

$$U(M_B - \lambda)U^{-1}(u) = f.$$

Therefore, the validity of the theorem is clear.

Now prove the main theorem on the structure of spectrum.

Theorem 2.2. *In order to $\lambda \in \sigma(L_B)$ the necessary and sufficient condition is*

$$(-1) \in \sigma\left(B\left(E - exp\left(\lambda J \int_0^1 \frac{ds}{\alpha(s)}\right)\right)\right).$$

Proof. By Theorem 2.1. for this it is sufficiently the investigate the spectrum of the corresponding boundedly solvable extension $M_B = U^{-1}L_B U$ of the minimal operator M_0 in $L^2_\alpha(H,(0,1))$.

Now consider the following problem to spectrum for the extension M_B, that is,

$$M_B u = \lambda u + f,\ \lambda \in \mathbb{C},\ f \in L^2_\alpha(H,(0,1)).$$

Then

$$J(\alpha u)'(t) = \lambda u(t) + f(t),\ t \in (0,1)$$

with boundary condition

$$(B+E)(\alpha u)(0) = B(\alpha u)(1).$$

It is clear that a general solution of the above differential equation has the form

$$u(t;\lambda) = \frac{1}{\alpha(t)} exp\left\{\lambda J \int_0^t \frac{ds}{\alpha(s)}\right\} f_0 + \frac{J}{\alpha(t)} \int_0^t exp\left\{\lambda J \int_s^t \frac{d\tau}{\alpha(\tau)}\right\} f(s)ds,\ f_0 \in H.$$

From this and boundary condition it is obtained that

$$\left(E + B\left(E - exp\left\{\lambda J \int_0^1 \frac{ds}{\alpha(s)}\right\}\right)\right) f_0 = B\left(\int_0^1 exp\left\{\lambda J \int_s^1 \frac{d\tau}{\alpha(\tau)}\right\} f(s)ds\right).$$

From last equation it is obtained the validity of claim of this theorem.

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