Celal Bayar University Journal of Science
Volume 14, Issue 4, 2018, p 357-361

Some Null Quaternionic Curves in Minkowski Spaces

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Received: 01 January 2018
Accepted: 08 October 2018
DOI: 10.18466/cbayarfbe.376273

Abstract

In this work, we examine null quaternionic rectifying curves and null quaternionic similar curves in Minkowski space E_1^3. Also, we defined null quaternionic (1,3)-Bertrand partner curves in E_1^4. Thus, we have characterizations between curvatures of these curves in Minkowski spaces.
Keywords: Null Quaternionic Curve; rectifying curve; similar partner curve; Bertrand partner curve.

1. Introduction

The quaternions are essentially multi-dimensional complex numbers and were first defined by Hamilton in 1844 and 1845. The quaternions are both relatively simple and very effective for rotations. The quaternion algebra has played a important role recently in several areas such as differential geometry, analysis and synthesis of mechanism and machines, simulation of particle motion in molecular physics and quaternionic formulation of equation of motion in theory of relativity [10, 16].

In the fundamental theory and characterizations of space curves, special curves have very interesting and an important problem. Large part of mathematicians studied the special curves in detail. Especially, the partner curves, i.e., the curves which are related each other at the corresponding points, have drawn attention of many mathematicians so far. In the theory of curves, well-known curves are Bertrand, Mannheim, rectifying, similar and involute-evolute curves. These curves are studied on different spaces by a lot of mathematicians [12, 13].

Bertrand curves are most favourite type of partner curves. A Bertrand curve is a curve which has common principal normal vectors with another curve and characterized by speciality that $\lambda\kappa+\mu\tau=1$ where κ, τ curvatures of the Bertrand are curve and λ, μ are constants. Rectifying curves were defined by Chen in 2003 [3]. At the same year, İlarslan et al. studied rectifying curves in Minkowski 3-space [11]. Then, more mathematicians studied about rectifying curves in some spaces [2-4, 9, 15]. In 2014, Soyfidan and Güngör studied quaternionic rectifying curves in Semi-Euclidean space [14]. El-Sabbagh and Ali have defined a new curve couple called similar curves whose arc-length parameters have relationships and their tangents are the same [8].

In [1], Bharathi and Nagaraj defined Serret-Frenet formulas for a quaternionic curve in E^3 and E^4. Finally, Çöken and Tuna defined Serret-Frenet formulas for quaternionic curves and null quaternionic curves in Semi-Euclidean spaces [5-7].

In this study, we define null quaternionic rectifying curves and null quaternionic similar partner curves in Minkowski 3-space E_1^3. We show that similar results of rectifying and similar curves is almost satisfied for null quaternionic rectifying curves and null quaternionic similar partner curves. Moreover, we obtain some characterizations for these curves. Lastly, we prove definition of null quaternionic (1, 3)-Bertrand curves in Minkowski space-time E_1^4 and obtain some relations about null quaternionic (1-3)-Bertrand curves in E_1^4.

2. Material and Methods

In this section, we give basic concepts related to the semi-real quaternions. For more detailed information, we refer ref. [6, 7].

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The set of semi-real quaternions is given by

$$Q=\{q \mid \quad q=ae_1+be_2+ce_3+d\,; \quad a,b,c,d\in IR\}$$

where $e_1,e_2,e_3\in E_1^3$, $h(e_i,e_i)=\varepsilon(e_i)$, $1\le i\le 3$ and

$e_i\times e_i=-\varepsilon(e_i)$,

$e_i\times e_j=\varepsilon(e_i)\varepsilon(e_j)e_k\in E_1^3$.

The multiplication of two semi real quaternions p and q are defined by

$$p\times q=S_pS_q+S_pV_q+S_qV_p+h\left(V_p,V_q\right)+V_p\wedge V_q$$

Here in, we have inner and cross products in Semi-Euclidean space E_1^3. $q=ae_1+be_2+ce_3+d$ and $\alpha q=-ae_1-be_2-ce_3+d$ are semi real quaternion and its conjugate, respectively and inner product h are defined by

$$h(p,q)=\frac{1}{2}\left[\varepsilon(p)\varepsilon(\alpha q)\left(p\times\alpha q\right)+\varepsilon(q)\varepsilon(\alpha p)\left(q\times\alpha p\right)\right]$$

[6].

Semi-Euclidean space E_1^3 is identified clearly with null spatial quaternions $\left\{\gamma\in Q_{E_1^3} \mid \quad \gamma+\alpha\gamma=0\right\}$,

$$\gamma(s)=\sum_{i=1}^{3}\gamma_i(s)e_i,\quad 1\le i\le 3.$$

$\{l,n,u\}$ are Frenet frames of the null quaternionic curves in E_1^3 and e_2 be timelike vector. Then, Frenet formulae are

$$\begin{bmatrix} l' \\ n' \\ u' \end{bmatrix}=\begin{bmatrix} 0 & 0 & k \\ 0 & 0 & \tau \\ -\tau & -k & 0 \end{bmatrix}\begin{bmatrix} l \\ n \\ u \end{bmatrix}$$

where k and τ are curvatures of null quaternionic curve and

$h\left(l,l\right)=h\left(n,n\right)=h\left(l,u\right)=h\left(n,u\right)=0$,

$h\left(l,n\right)=h\left(u,u\right)=1$

l and n are null vectors and u is a spacelike vector. At this juncture, quaternion product is given by

$l\times n=-1-u,\quad n\times l=-1+u,\ n\times u=-n,\ u\times n=n$

$u\times l=-l,\quad l\times u=l,\quad u\times u=-1\,,\quad l\times l=n\times n=0$

[6].

Let $\gamma(s)=\gamma_1e_1+\gamma_2e_2+\gamma_3e_3$ be a quaternionic curve in E_1^3. An orthonormal basis of E_1^4 is $\{e_1,e_2,e_3,e_4=1\}$ and let e_2 be timelike vector and $\beta=\gamma_1e_1+\gamma_2e_2+\gamma_3e_3+\gamma_4e_4$ be a null quaternionic curve in E_1^4 and $\{L,N,U,W\}$ be the Frenet components of β in E_1^4. Then, Frenet formulae are given by

$$\begin{bmatrix} L' \\ N' \\ U' \\ W' \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & K \\ 0 & 0 & \tau+p & p \\ \tau+p & 0 & 0 & 0 \\ p & K & 0 & 0 \end{bmatrix}\begin{bmatrix} L \\ N \\ U \\ W \end{bmatrix}$$

where K is the first curvature of β in E_1^4. Here,

$h\left(L,L\right)=h\left(N,N\right)=h\left(L,U\right)=0$,

$h\left(N,U\right)=h\left(W,U\right)=0$

$h\left(U,U\right)=h\left(W,W\right)=1,\ h\left(L,N\right)=-1,$

$h\left(N,W\right)=h\left(L,W\right)=0$

L and N are null vectors, U and W are spacelike vectors for which the quaternion product is given by

$L\times N=1-U\ ,\ N\times L=1+U\ ,\ N\times U=N$

$U\times N=-N,\ \ U\times L=L\ ,\ L\times U=-L$.

$U\times U=-1\,,\ \ L\times L=N\times N=0$

3. Results and Discussion

3.1. Null Quaternionic Rectifying Curves

Now, we give attribution of null quaternionic rectifying curves in E_1^3.

Definition 3.1. Let $\gamma(s)$ be null quaternionic curve in E_1^3. If γ is null quaternionic rectifying curve. Then, γ is defined by

$$\gamma(s)=\lambda(s)l(s)+\mu(s)n(s)$$

where $\lambda(s)$ and $\mu(s)$ are some functions by arclength parameter s of the curve.

Theorem 3.1. *Let $\gamma(s)$ be null quaternionic rectifying curves in E_1^3. Then;*

i) Distance function $q=|\gamma|$ satisfies $q^2=a_1s+a_2$ for some constants a_1 and a_2 and is nonconstant.

ii) Tangential component of position vector of the curve is given by $\lambda(s)=s+c_1$ for some constant c_1.

iii) Binormal component of position vector of the curve has constant length $\mu(s)=c_2$.

iv) The relationship between curvatures of the curve is given by $\mu\tau=\lambda k$.

Proof. Let $\gamma(s)$ be a null quaternionic curve in E_1^3. Suppose that γ is a null quaternionic rectifying curve. Then, by definition 3.1, we have

$$\gamma(s)=\lambda(s)l(s)+\mu(s)n(s) \tag{1}$$

for some functions λ, μ. By taking derivative of (1) with respect to s and using the Frenet formulae, we obtain that

$$l=\lambda' l+\mu' n+(\lambda k-\mu\tau)u. \tag{2}$$

We take the quaternionic inner product of (2) by itself

$$-2\lambda'\mu'=(\lambda k-\mu\tau)^2. \tag{3}$$

Using the inner product of (2) and $n(s)$, following equality holds

$$\lambda'=1. \tag{4}$$

Thus, statement *(ii)* is proved that

$$\lambda=s+c_1 \quad , \quad c_1\in IR. \tag{5}$$

From vector components of (2), we get statement *(iii)* and *(iv)*

$$\mu'=0 \text{ or } \mu=c_2 \tag{6}$$

and

$$\lambda k=\mu\tau. \tag{7}$$

Since $\rho^2=h(\gamma,\gamma)$, we obtain statement *(i)*

$$\begin{gathered}\rho^2=2\lambda\mu=a_1 s+a_2 \quad ,\\ a_1=2c_2,\ a_2=2c_1c_2 \quad c_1,c_2\in IR\end{gathered}. \tag{8}$$

4.1. Null Quaternionic Similar Curves

Definition 4.1. Let $\gamma(s)$ and $\gamma^*(s^*)$ be null quaternionic curves in E_1^3 with Frenet frames $(l(s),n(s),u(s))$, $(l^*(s^*),n^*(s^*),u^*(s^*))$, respectively. Curves $\gamma^*(s^*)$ are called similar partner curves of curves $\gamma(s)$ with variable transformation $\lambda(s(s^*))$ if there exists a variable transformation

$$s^*=\int\lambda(s^*(s))\,ds \tag{9}$$

such that the tangent vectors are same for the two curves.

Theorem 4.1. Let $\gamma(s)$ and $\gamma^*(s^*)$ be null quaternionic curves in Minkowski 3-space with Frenet frames $(l(s),n(s),u(s))$, $(l^*(s^*),n^*(s^*),u^*(s^*))$ and curvatures $(\mathrm{k}(s),\tau(s))$, $(k^*(s^*),\tau^*(s^*))$, respectively. Then, curves $\gamma(s)$ and $\gamma^*(s^*)$ are null quaternionic similar curves if and only if it is hold that

$$s^*=s+\lambda+c\ ,\ c\in IR \tag{10}$$

and

$$k(s)=0. \tag{11}$$

Proof. Since $\gamma(s)$ and $\gamma^*(s^*)$ be null quaternionic similar curves in Minkowski 3-space. We can write that

$$\gamma^*(s^*)=\gamma(s)+\lambda(s)l(s) \tag{12}$$

where $\lambda=\lambda(s)$ is smooth function of parametrized by s. We obtain by taking derivation of (12),

$$l^*(s^*)\frac{ds^*}{ds}=(1+\dot{\lambda})l(s)+\lambda(s)k(s)u(s). \tag{13}$$

Using the definition 4.1, (13) can be written

$$l^*(s^*)\frac{ds^*}{ds}=(1+\dot{\lambda})l^*(s^*)+\lambda(s)k(s)u(s). \tag{14}$$

Therefore, we get

$$\frac{ds^*}{ds}=(1+\dot{\lambda})\quad ,\quad k(s)=0. \tag{15}$$

Conversely, assume that equation (15) holds for some nonzero functions $\lambda(s)$. Let define a null quaternionic curve

$$\gamma^*(s^*)=\gamma(s)+\lambda(s)l(s) \tag{16}$$

where $\lambda(s)$ is nonzero function. We will prove that null quaternionic curves $\gamma(s)$ and $\gamma^*(s^*)$ are similar curves. We get by taking derivative of (16) according to s

$$l^*(s^*)\frac{ds^*}{ds}=(1+\dot{\lambda})\,l(s)+\lambda(s)\,k(s)u(s) \qquad (17)$$

By substituting (15) in (17), we have

$$l^*(s^*)=l(s).$$

Thus, we obtain desired outcome that curves $\gamma(s)$ and $\gamma^*(s^*)$ are null quaternionic similar curves.

Corollary 4.1. Let $\gamma^*(s^*)$ be null quaternionic similar partner curve of null quaternionic curve $\gamma(s)$ in Minkowski 3-space. Then, curvature of $\gamma(s)$ is zero.

5.1. Null Quaternionic (1,3) Bertrand partner curves

Definition. Let γ_1 and γ_2 be null quaternionic Bertrand partner curves in E_1^4. $\{L_1,N_1,U_1,W_1\}$ and $\{L_2,N_2,U_2,W_2\}$ are Frenet frames at corresponding points of these curves, respectively. γ_1 and γ_2 are called null quaternionic $(1,3)$-Bertrand partner curves if there exist a bijection

$$\varphi\ :\ I_1 \to I_2$$
$$s_1 \to \varphi(s_1)=s_2 \quad ,\frac{ds_2}{ds_1}\neq 0 \qquad (18)$$

and the plane spanned by $\{N_1,W_1\}$ at each point $\gamma_1(s_1)$ of γ_1 coincides with the plane spanned by $\{N_2,W_2\}$ at corresponding point $\gamma_2(s_2)$ of γ_2.

Theorem 5.1. Let γ_1 and γ_2 be null quaternionic curves in E_1^4 and let $\{L_1,N_1,U_1,W_1\}$ and $\{L_2,N_2,U_2,W_2\}$ be Frenet frames of these curves, respectively. If γ_1 is null quaternionic $(1,3)$-Bertrand curve, then following equalities are hold;

i) $\tan(\theta(s_1))=-2(\lambda'+\mu K_1+\lambda(\tau_1+p_1))$

ii) $2(1+\mu p_1)(\lambda'+\mu K_1+\lambda(\tau_1+p_1))=\mu'+\lambda p_1$

Proof. We assume that γ_1 is null quaternionic $(1,3)$-Bertrand partner curves parametrized by arclenght s_1. Thus, we can write null quaternionic $(1,3)$-Bertrand partner curve γ_2

$$\gamma_2(s_2)=\gamma_1(s_1)+\lambda(s_1)N_1+\mu(s_1)W_1(s_1) \qquad (19)$$

for all $s_1\in I_1$. Here λ and μ are C^∞ functions on I_1 and s_2 is arclenght parameter of γ_2. Differentiating (19) with respect to s_1 and using the Frenet equations, we have

$$\begin{aligned}L_2\frac{ds_2}{ds_1}&=(1+\mu p_1)L_1\\&+(\lambda'+\mu K_1+\lambda(\tau_1+p_1))N_1\\&+(\mu'+\lambda p_1)W_1\end{aligned} \qquad (20)$$

From definition, we can write

$$\begin{aligned}N_2(s_2)&=\cos(\theta(s_1))\,N_1(s_1)+\sin(\theta(s_1))\,W_1(s_1)\\ W_2(s_2)&=-\sin(\theta(s_1))\,N_1(s_1)+\cos(\theta(s_1))W_1(s_1)\end{aligned} \qquad (21)$$

and $\sin(\theta(s_1))\neq 0$ for all $s_1\in I_1$. By using inner products of the equations (20) and (21), we obtain

$$\begin{aligned}-\frac{ds_2}{ds_1}&=-\cos(\theta(s_1))(1+\mu p_1)\\&+\sin(\theta(s_1))(\mu'+\lambda p_1)\end{aligned} \qquad (22)$$

and

$$0=\sin(\theta(s_1))(1+\mu p_1)+\cos(\theta(s_1))(\mu'+\lambda p_1). \qquad (23)$$

By the using (22) and (23), we get

$$\tan(\theta(s_1))=-2(\lambda'+\mu K_1+\lambda(\tau_1+p_1)). \qquad (24)$$

Taking the inner product of (20) by itself, we obtain

$$2(1+\mu p_1)(\lambda'+\mu K_1+\lambda(\tau_1+p_1))=\mu'+\lambda p_1. \qquad (25)$$

Theorem 5.2. Let γ_1 and γ_2 be null quaternionic curves in E_1^4. If γ_1 is null quaternionic $(1,3)$-Bertrand curve. Then, curvatures of curves γ_1 and γ_2 are hold that

$$K_2^2\left(\frac{ds_2}{ds_1}\right)^2=\begin{pmatrix}K_1+2\mu p_1K_1+2\lambda' p_1\\ +\lambda p_1(\tau_1+p_1)+\mu''+\lambda p_1'\end{pmatrix}^2 -\left(2\mu' p_1+\mu p_1'+\lambda p_1^2\right)\begin{pmatrix}\lambda''+2\mu'K_1+\mu K_1'\\ +(2\lambda'+\mu K_1+\lambda(\tau_1+p_1))(\tau_1+p_1)\\ +\lambda(\tau_1'+p_1')+\lambda p_1K_1\end{pmatrix}.$$

Proof. γ_1 is null quaternionic $(1,3)$-Bertrand partner curves parametrized by arclenght s_1. Thus, we have equation (20). By taking the derivation of equation (20), we have

$$L_2\frac{d^2s_2}{ds_1^2}+K_2W_2\frac{ds_2}{ds_1}=\left(2\mu' p_1+\mu p_1'+\lambda p_1^2\right)L_1 +\begin{pmatrix}\lambda''+2\mu'K_1+\mu K_1'\\ +(2\lambda'+\mu K_1+\lambda(\tau_1+p_1))(\tau_1+p_1)\\ +\lambda(\tau_1'+p_1')+\lambda p_1K_1\end{pmatrix}N_1 +\begin{pmatrix}K_1+2\mu p_1K_1+2\lambda' p_1\\ +\lambda p_1(\tau_1+p_1)+\mu''+\lambda p_1'\end{pmatrix}W_1 \quad (26)$$

Finally, taking the inner product of (26) by itself, we get desired result.

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